Problem 1 - Queue (15 points)

Consider an ATM machine that is operative 24 hours a day, 7 days a week, where customers arrive randomly to withdraw money from their accounts. Luckily, there is enough space in front of the ATM to accommodate any number of customers waiting in line. The ATM machine processes requests in a time that is proportional to the amount to withdraw, and takes exactly 20 seconds to process USD100. However, not all customers withdraw the same amount, but each customer \( a \) withdraws an amount \( C_a \times \text{USD}100 \), where \( C_a \) is a geometric random variable with parameter \( p = 1/3 \) (independently of other customers).

Let’s model this store as a queue, and assume that the arrival rate is \( \lambda = 100 \) customers per hour. Strictly speaking, the queue is not an M/M/1 queue, but Little’s result still holds.

a) What is the average service time for a customer? (3 point)
\[ T_{\text{serv}} = E[C_a] \times 20\text{sec} = 1/p \times 20\text{sec} = 60\text{sec} \]

b) What is the service rate \( \mu \) in customers per hour? (2 point)
\[ \mu = 1/T_{\text{serv}} = 60 \text{ customers per hour} \]

c) Compute the utilization factor \( \rho \) of this ATM. What is going to happen to the number of customers waiting in line? (2 points)
\[ \rho = \lambda/\mu = 5/3 > 1 \] the system gets congested, the queue grows unbounded.

Assume now that the arrival rate is \( \lambda = 30 \) customers per hour (while the service rate remains the same).

d) Compute the utilization factor \( \rho \) in this case. What is going to happen to the number of customers waiting in line? (2 points)
\[ \rho = \lambda/\mu = 1/2 < 1 \] the system does not congested.

e) Assume that, on average, each customer spends a total of 3 minutes in front of the ATM (in line and using the ATM). What is the average number \( L \) of customers in front of the ATM? (3 points)

Note that \( \lambda = 0.5 \) customers per minute, and the average time in the system is \( T = 3 \) minutes. Little’s result \( L = \lambda T = 0.5 \times 3 = 1.5 \) customer on average.

f) How long does each customer wait in line on the average? (3 points)
\[ T_{\text{queue}} = T - T_{\text{serv}} = 3 - 1 = 2 \text{ minutes} \]
A simple channel access scheme.

The following access scheme is a simplification of the one adopted in IEEE-802.11 WLAN.

Imagine there are $n$ stations that have to share a wireless channel. Each station sends packets of fixed size and is in saturation, that is, it always has packets to transmit. All stations are synchronized, according to time slots of duration $T$, where a time slot is sufficient to sense the channel and to transmit a single packet. If a single station transmits a packet during a given time slot, then its packet is successfully delivered to the destination. Otherwise, if multiple stations transmit during the same time slot, then there is collision (involving all the transmitted packets, that have to be retransmitted).

Stations access the medium according to the following scheme. After transmitting one of its own packets (either successfully transmitted or collided), a station initializes a back-off counter, by picking a value from the set $\{0, 1, 2, \ldots, W-1\}$ uniformly at random, where $W$ is a parameter of the scheme. At the end of each time slot, the counter is decremented by one. If at the beginning of a time slot its counter is zero, then the station transmits a packet during that time slot (e.g., if the counter is initialized at 0, then the packet is immediately transmitted). After transmission, in both the cases of successful transmission and collision, the station reinitializes its counter.

Problem 2 - The back-off counter (20 points)

Let’s focus on a single station, and analyze how its back-off counter evolves over time slots. Let $t = 0, 1, 2, \ldots$ be the sequence of time slots, and let $b(t)$ be the value of the counter at the beginning of the slot $t$. The stochastic process $\{b(t)\}_{t \geq 0}$ is a Markov chain with state space $\{0, 1, \ldots, W-1\}$. For all $i, j \in \{0, 1, \ldots, W-1\}$, let $p(i, j)$ be the one-step transition probability from state $i$ to state $j$.

a) Write the transition probability of the Markov chain $\{b(t)\}_{t \geq 0}$ (recall that a packet is transmitted when the counter becomes zero, and after a transmission the counter is randomly reinitialized). In particular, you need to give the value of $p(0, i)$ for all $i = 0, \ldots, W-1$, the value of $p(i, i-1)$ for all $i = 1, \ldots, W-1$, and the value of $p(i, j)$ for all other choices of $i$ and $j$. (3 points)

$p(i, i-1) = 1$ for $1 \leq i \leq W-1$

$p(0, i) = \frac{1}{W}$ for $0 \leq i \leq W-1$

$p(i, j) = 0$ otherwise

b) Write the transition matrix $P$ of the Markov chain $\{b(t)\}_{t \geq 0}$. (2 points)

$$
P = \begin{bmatrix}
\frac{1}{W} & \frac{1}{W} & \frac{1}{W} & \cdots & \frac{1}{W} & \frac{1}{W} \\
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\cdots \\
0 & 0 & 0 & \cdots & 1 & 0
\end{bmatrix}
$$

c) Draw a graphical representation of the Markov chain $\{b(t)\}_{t \geq 0}$. (2 points)
As we discussed in class.

d) Consider the simple case of \( W = 4 \). Draw the graphical representation and write the transition matrix of the Markov chain in this case. (2 points)

As we discussed in class.

Let’s consider the case of a general value of \( W \). For \( 0 \leq k \leq W - 1 \), let \( \pi_k = \lim_{t \to \infty} \Pr(b(t) = k) \). In our case, the stationary distribution of \( \{b(t)\}_{t \geq 0} \) exists and is denoted by

\[
\pi = (\pi_0, \pi_1, \ldots, \pi_{W-1})
\]

e) A way to uniquely determine \( \pi \) consists in solving a system of equations. Which are these equations? (2 points)

The equations are \( \pi P = \pi \) and \( \sum_{i=0}^{W-1} \pi_i = 1 \)

f) Find the stationary distribution in the case of \( W = 4 \) (write and solve the equations above for this simple case). (4 points)

Solving the equations above you get \( \pi = [4/10, 3/10, 2/10, 1/10] \)

g) Let’s consider the case of a general value of \( W \). Argue that \( \pi_{W-k} = \frac{k}{W} \pi_0 \) for all \( 1 \leq k \leq W \). (2 points - hard)

Observe that from the equation \( \pi P = \pi \) you have \( \pi_{W-1} = \frac{1}{W} \pi_0 \). Next \( \pi_{W-2} = \pi_{W-1} + \frac{1}{W} \pi_0 = \frac{2}{W} \pi_0 \). Then \( \pi_{W-3} = \pi_{W-2} + \frac{1}{W} \pi_0 = \frac{3}{W} \pi_0 \), and the result is proven by induction.

h) Determine \( \pi_0 \) as a function of \( W \) only. (3 points - hard)

\[
1 = \sum_{i=0}^{W-1} \pi_i = \sum_{i=1}^{W} \frac{k}{W} \pi_0 = \pi_0 \frac{W+1}{2}, \text{ thus } \pi_0 = \frac{2}{W+1}
\]
Problem 3 - The channel state (15 points)

With the access scheme described above still in mind, we now consider all the
n stations together. Let τ be the probability that each station transmits during
a given time slot. It holds that τ = π₀ (since each station transmits when its
back off counter becomes zero).

a) What is the probability that a time slot is idle? Write the expression for
general values of n (number of stations) and W (maximum back-off counter),
and compute it in the simple case of n = 3, W = 4 (in this case π₀ = 2/5).
(3 points)

\[ P_{\text{idle}} = (1 - \tau)^n = \frac{27}{125} \]

b) What is the probability that a time slot is NOT idle (that is, at least one
station transmits)? Write the expression for general n and W, and compute it
in the case of n = 3, W = 4. (2 points)

\[ P_{\text{NOT idle}} = 1 - P_{\text{idle}} = 1 - (1 - \tau)^n = \frac{98}{125} \]

c) What is the probability that in a time slot there is a successful transmission?
Write the expression for general n and W, and compute it in the case of n = 3,
W = 4. (2 points)

\[ P_{1\text{tx}} = n\tau(1 - \tau)^{n-1} = \frac{54}{125} \]

d) What is the probability that in a not idle time slot there is a successful
transmission? (Conditional probability of a successful transmission given that
the slot is not idle.) Write the expression for general n and W, and compute it
in the case of n = 3, W = 4. (3 points)

\[ P_s = \frac{P_{1\text{tx}}}{P_{\text{NOT idle}}} = \frac{27}{49} \]

e) Let Y the the random variable denoting the number of consecutive idle time
slots between two transmissions (by any station). Determine the probability
distribution of Y, that is, write Pr(Y = i) for i = 0, 1, 2 . . . (3 points)

\[ P(Y = i) = (1 - \tau)^n[1 - (1 - \tau)^n], \text{ that is, } Y \text{ is geometric of parameter} \]
\[ q = [1 - (1 - \tau)^n] = P_{\text{NOT idle}} \]

f) What is the expected value of Y? Write the expression for general n and W,
and compute it in the case of n = 3, W = 4. (2 points)

\[ E[y] = \frac{1}{q} - 1 = \frac{27}{98} \]