Homework 2

April 2017

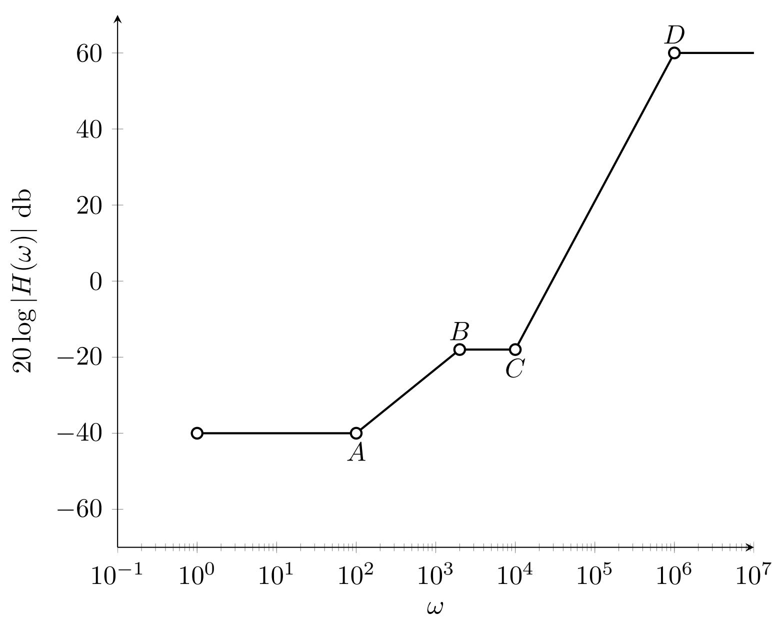
1. Draw the bode plots of

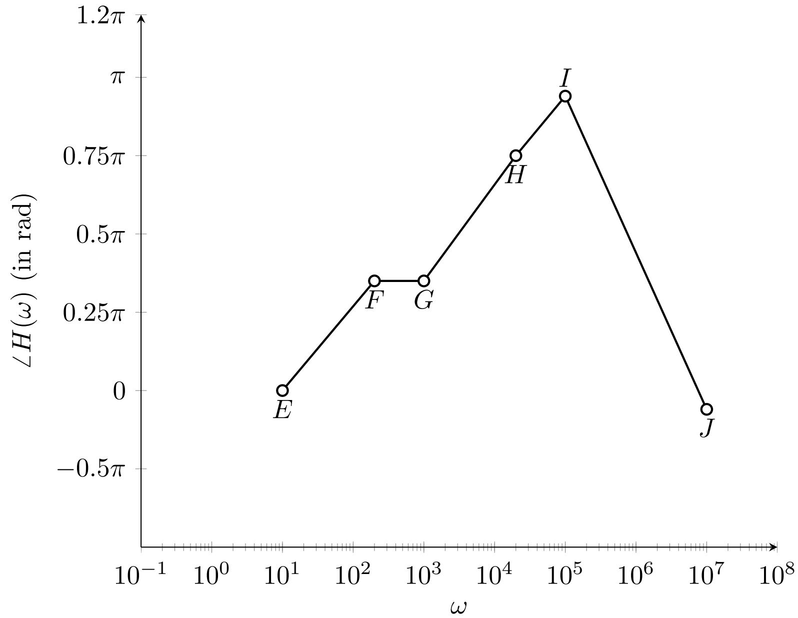
$$\frac{(100+j\omega)(1+\frac{j\omega}{10^4})^2}{(10000+5j\omega)(1+\frac{j\omega}{10^6})^2}$$

Solution:

$$H(\omega) = \frac{(100+j\omega)(1+\frac{j\omega}{10^4})^2}{(10000+5j\omega)(1+\frac{j\omega}{10^6})^2}$$

$$H(\omega) = \frac{100(1+\frac{j\omega}{100})(1+\frac{j\omega}{10^4})^2}{10000(1+\frac{j\omega}{2000})(1+\frac{j\omega}{10^6})^2}$$





Critical $\omega = 100, 2000, 10^4, 10^6$ rad.
At A, $20\log(H(100)) = -40db$
At B, $20\log(H(2000)) \approx -20db$ (slightly greater than $-20db$
At C, $20\log\big(H(10^4)\big) \approx -20db$ (same as point B)
At D, $20\log\big(H(10^6)\big) \approx 60db$
At E, $\angle H(10) = 0$ rad
At F, $\angle H(200) = 0.35\pi$ rad
At G, $\angle H(1000) = 0.35\pi$ rad
At H, $\angle H(20000) = 0.75\pi$ rad
At I, $\angle H(10^5) = 0.94\pi$ rad
At J, $\angle H(10^7) \approx -0.06\pi$ rad

2. Are the following signals periodic? If yes, find the fundamental frequency and fundamental period

(a)
$$x(t) = (cos(2\pi t))^2 + cos(4\pi t)cos(6\pi t)$$

Solution:
Yes,

$$x(t) = 0.5cos(4\pi t) + 0.5 + 0.5cos(10\pi t) + 0.5cos(2\pi t)$$

fundamental frequency, $\omega_0 = 2\pi$ rad
fundamental period, $T = \frac{2\pi}{\omega_0} = 1$ sec

(b)

$$x(t) = 4sin(\frac{4\pi}{3}t + \frac{\pi}{3}) + 2cos(2\pi t + \frac{\pi}{6}) + sin(4\pi t + \frac{\pi}{12})$$

Solution:
Yes,

$$x(t) = 4sin(\frac{4\pi}{3}t + \frac{\pi}{3}) + 2cos(2\pi t + \frac{\pi}{6}) + sin(4\pi t + \frac{\pi}{12})$$

fundamental frequency $\omega_0 = \frac{2\pi}{3}$ rad
fundamental period $T = \frac{2\pi}{\omega_0} = 3$ sec

(c)

$$x(t) = 12cos(\frac{\pi}{3}t + \pi) + 3sin(\frac{5\pi t}{9} + \frac{\pi}{6}) + cos(\pi t + \frac{\pi}{3})$$

Solution:
Yes
Fundamental frequency $\omega_0 = \frac{\pi}{9}$ rad
Fundamental period $T = 4.5$ sec

(d)

$$x(t) = cos(\pi t) + cos(t)$$

Solution:
Not periodic.
$\cos(t)$ has fundamental period 2π sec and $cos(\pi t)$ has fundamental period 2 sec. There is no value of t where both $\cos(t)$ and $\cos(\pi t)$ start a new cycle as $2n\pi$ is never an integer (π is a irrational number).

(e)

$$x(t) = (sin(t))^4$$

Solution:
Yes

$$\begin{aligned} x(t) = sin^4(t) &= \Big(\frac{e^{jt} - e^{-jt}}{2j}\Big)^4 \\ &= \frac{1}{16}\Big(e^{2jt} + e^{-2jt} - 2\Big)^2 \\ &= \frac{1}{16}\Big(2\cos(4t) - 8\cos(2t) + 6\Big) \end{aligned} \tag{1}$$

Fundamental frequency $\omega_0 = 2$ rad
Fundamental period $T = \pi$ sec

3. Determine the coefficients of the Fourier series of the periodic signals in question 2.

(a)

$$x(t) = (cos(2\pi t))^2 + cos(4\pi t)cos(6\pi t)$$

Solution:

$$x(t) = 0.5cos(4\pi t) + 0.5 + 0.5cos(10\pi t) + 0.5cos(2\pi t)$$

fundamental frequency $\omega_0 = 2\pi$ rad

$$x(t) = 0.5\Big(\frac{e^{-j4\pi t} + e^{j4\pi t}}{2}\Big) + 0.5\Big(\frac{e^{-j10\pi t} + e^{j10\pi t}}{2}\Big) + 0.5\Big(\frac{e^{-j2\pi t} + e^{j2\pi t}}{2}\Big) + 0.5e^{j0}$$

Coefficient of the Fourier Series of $x(t)$ are

$$c_k = \begin{cases} 0.25 & k = 1, -1, 2, -2, 5, -5 \\ 0.5 & k = 0 \\ 0 & otherwise \end{cases}$$

(b)

$$x(t) = 4sin(\frac{4\pi}{3}t + \frac{\pi}{3}) + 2cos(2\pi t + \frac{\pi}{6}) + sin(4\pi t + \frac{\pi}{12})$$

Solution:

$$x(t) = 4sin(\frac{4\pi}{3}t + \frac{\pi}{3}) + 2cos(2\pi t + \frac{\pi}{6}) + sin(4\pi t + \frac{\pi}{12})$$

fundamental frequency $\omega_0 = \frac{2\pi}{3}$ rad

$$x(t) = 4\Big(\frac{e^{j\pi/3}e^{j4\pi t/3} - e^{-j\pi/3}e^{-j4\pi t/3}}{2j}\Big) + 2\Big(\frac{e^{j\pi/6}e^{j2\pi t} + e^{-j\pi/6}e^{-j2\pi t}}{2}\Big)$$
$$+ \Big(\frac{e^{j\pi/12}e^{j4\pi t} - e^{-j\pi/12}e^{-j4\pi t}}{2j}\Big) \quad (2)$$

Coefficient of the Fourier Series of $x(t)$ are

$$c_k = \begin{cases} -2je^{j\pi/3} & k = 2 \\ 2je^{-j\pi/3} & k = 2 \\ e^{j\pi/6} & k = 3 \\ e^{-j\pi/6} & k = -3 \\ -0.5je^{j\pi/12} & k = 6 \\ 0.5je^{j\pi/3} & k = -6 \\ 0 & otherwise \end{cases}$$

(c)

$$x(t) = 12cos(\frac{\pi}{3}t + \pi) + 3sin(\frac{5\pi t}{9} + \frac{\pi}{6}) + cos(\pi t + \frac{\pi}{3})$$

Solution:

$$x(t) = 12cos(\frac{\pi}{3}t + \pi) + 3sin(\frac{5\pi t}{9} + \frac{\pi}{6}) + cos(\pi t + \frac{\pi}{3})$$

$\omega_0 = \frac{\pi}{9}$ rad $T = 4.5$ sec

$$x(t) = 12\Big(\frac{e^{j\pi}e^{j\pi t/3} + e^{-j\pi}e^{-j\pi t/3}}{2}\Big) + 3\Big(\frac{e^{j\pi/6}e^{j5\pi t/9} - e^{-j\pi/6}e^{-j5\pi t/9}}{2j}\Big) + \Big(\frac{e^{j\pi/3}e^{j\pi t} + e^{-j\pi/3}e^{-j\pi t}}{2}\Big) \tag{3}$$

Coefficient of the Fourier Series of $x(t)$ are

$$c_k = \begin{cases} -6 & k = 3, -3 \\ -1.5je^{j\pi/6} & k = 5 \\ 1.5je^{-j\pi/6} & k = -5 \\ 0.5e^{j\pi/3} & k = 9 \\ 0.5e^{-j\pi/3} & k = -9 \\ 0 & otherwise \end{cases}$$

(d)

$$x(t) = cos(\pi t) + cos(t)$$

Solution:
Not periodic. So, Fourier series representation does not exist.

(e)

$$x(t) = (sin(t))^4$$

Solution:

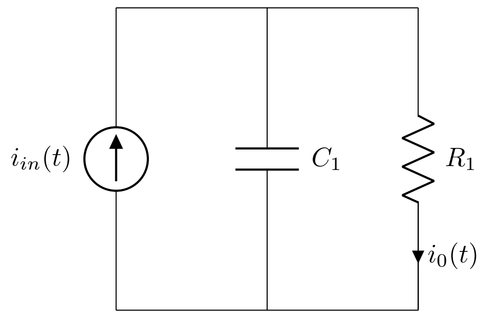
$$\begin{aligned} x(t) &= sin^4(t) \\ &= \frac{1}{16}\big(2cos(4t) - 8cos(2t) + 6\big) \end{aligned} \tag{4}$$

$$x(t) = \Big(\frac{e^{j4t} + e^{-j4t} - 4e^{j2t} - 4e^{-j2t} + 6}{16}\Big)$$

$\omega_0 = 2$ rad
Coefficient of the Fourier Series of $x(t)$ are

$$c_k = \begin{cases} -0.25 & k = 1, -1 \\ 1/16 & k = 2, -2 \\ 3/8 & k = 0 \\ 0 & otherwise \end{cases}$$

4. (a) What type of filters are the following circuits? Justify your answer.
(i)

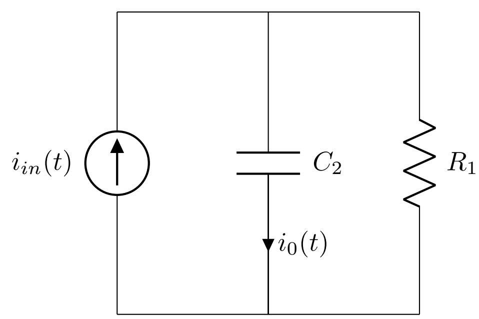


Solution:

$$I_0 = I_{in}\frac{\frac{1}{Z_R}}{\frac{1}{Z_R}+\frac{1}{Z_C}}$$

$$H_1(\omega) = \frac{I_0}{I_{in}} = \frac{1}{1+\frac{Z_R}{Z_C}} = \frac{1}{1+j\omega R_1 C_1}$$

When $\omega \approx 0$, $|H_1(\omega)| \approx 1$ and as $\omega \to \infty$, $|H_1(\omega)| \approx 0$
This is a low pass filter.
(ii)



Solution:

$$I_0 = I_{in}\frac{\frac{1}{Z_C}}{\frac{1}{Z_R}+\frac{1}{Z_C}}$$

$$H_2(\omega) = \frac{I_0}{I_{in}} = \frac{1}{1+\frac{Z_C}{Z_R}} = \frac{1}{1+\frac{1}{j\omega R_2 C_2}}$$

When $\omega \approx 0$, $|H_2(\omega)| \approx 0$ and as $\omega \to \infty$, $|H_2(\omega)| \approx 1$
This is a high pass filter

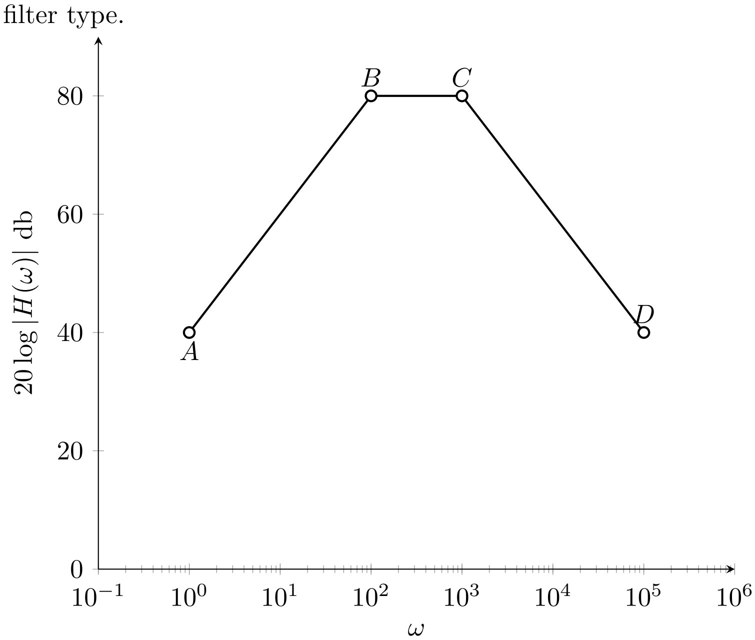
(b) The cut off frequency ω_0 in (i) is 1000 rad and the cut off frequency ω_0 in (ii) is 100 rad.
Determine the relationship between R_1 and C_1 and the one between R_2 and C_2

Solution:
(i)$\omega_C = 1000$ rad $\implies R_1C_1 = 1/1000$
(ii)$\omega_C = 100rad \implies R_2C_2 = 1/100$

(c) Let $H(\omega) = H_1(\omega)H_2(\omega)$ where $H_1(\omega)$ and $H_2(\omega)$ are the transfer function of the first and second filter. What kind of filter does $H(\omega)$ represent? Verify by drawing the bode plot.
Solution:

$$H(\omega) = H(\omega_1)H(\omega_2) = \frac{1}{1 + \frac{j\omega}{1000}} \cdot \frac{j\omega/100}{1 + \frac{j\omega}{100}}$$

This is a band pass filter. Plot of $|H(\omega)|$ can be seen in the figure.
NOTE: It is not required to draw the plot of $\angle H(\omega)$ to deduce the filter type.



5. Are the following systems linear? Are the following systems time invariant?
Let $x(t)$ and $y(t)$ be the input-output pairs

(a)
$$(x(t), y(t) = 5x(t-10) + 7x(t-3))$$

Solution:
Yes, both linear and time invariant
For input $ax(t)$, the output is $ay(t)$. Also, for input $x(t-k)$, the output is $y(t-k)$

(b)

$$(x(t), y(t) = cos(x(t)))$$

Solution:
Not linear but time invariant.
If input $ax(t)$, output is $y^{'}(t) = cos(a(x(t)) \neq ay(t)$ For input $x(t-k)$, the output is $y(t-k)$

(c)

$$(\cos(4t), y(t) = \sin(4t) + \cos^2(2t))$$

Solution:
Not linear but time invariant.
If input is $ax(t)$, output is

$$y^{'}(t) = a\sin(4t) + a^2\cos^2(2t) \neq ay(t)$$

For input $x(t-k)$, the output is $y(t-k)$.

(d)

$$(x(t), y(t) = \frac{dx(t)}{dt})$$

Solution:
Yes, both linear and time invariant.

6. Find the Fourier Series representation of the following periodic signals of fundamental period $T_0 = 4sec$.

(a)

$$x(t) = \begin{cases} 1 & 2 \leq t \leq 4 \\ 0 & 0 \leq t \leq 2 \end{cases}$$

Solution:

$$\omega_0 = \frac{2\pi}{T_0} = \frac{\pi}{2}$$

For $n \neq 0$,

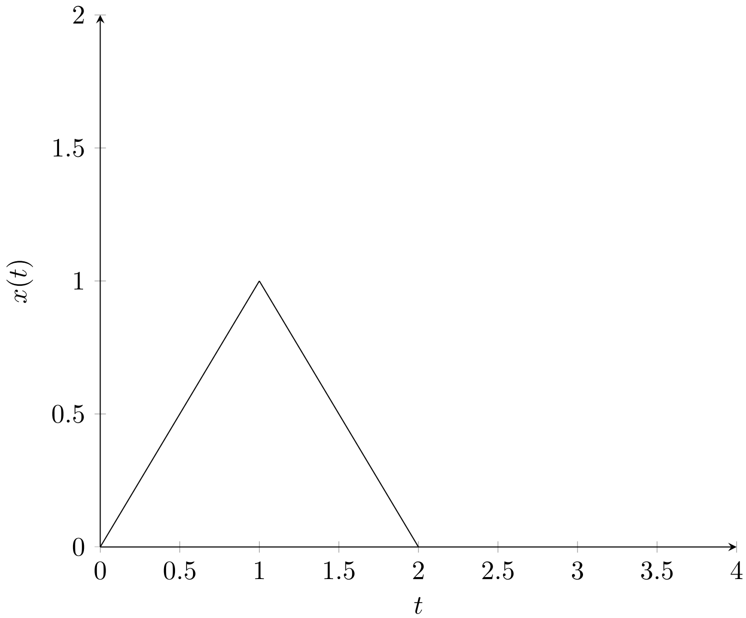
$$\begin{aligned} C_n &= \frac{1}{T}\int_0^T x(t)e^{-nj\omega_0 t}dt \\ &= \frac{1}{4}\int_0^4 x(t)e^{-nj\frac{\pi}{2}t}dt \\ &= \frac{1}{4}\int_2^4 1e^{-nj\frac{\pi}{2}t}dt \\ &= \frac{1}{4}\frac{e^{-nj2\pi} - e^{-nj\pi}}{-nj\frac{\pi}{2}} \\ &= \frac{e^{-jn\pi}}{j2n\pi} - \frac{e^{-j2n\pi}}{j2n\pi} \\ &= \frac{(-1)^n}{2nj\pi} - \frac{1}{j2n\pi} = \frac{(-1)^n - 1}{j2n\pi} \end{aligned} \tag{5}$$

$$C_0 = \frac{1}{T}\int_0^T x(t)dt$$

$$C_0 = 0.5$$

Note: C_n is FS coefficient
$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j\omega_0 nt}$

(b) $x(t) =$



Solution:
$T = 4, \omega_0 = \pi/2$

$$x(t) = \begin{cases} t & 0 \le t \le 1 \\ 2 - t & 1 < t \le 2 \\ 0 & otherwise \end{cases}$$

For $n \neq 0$,

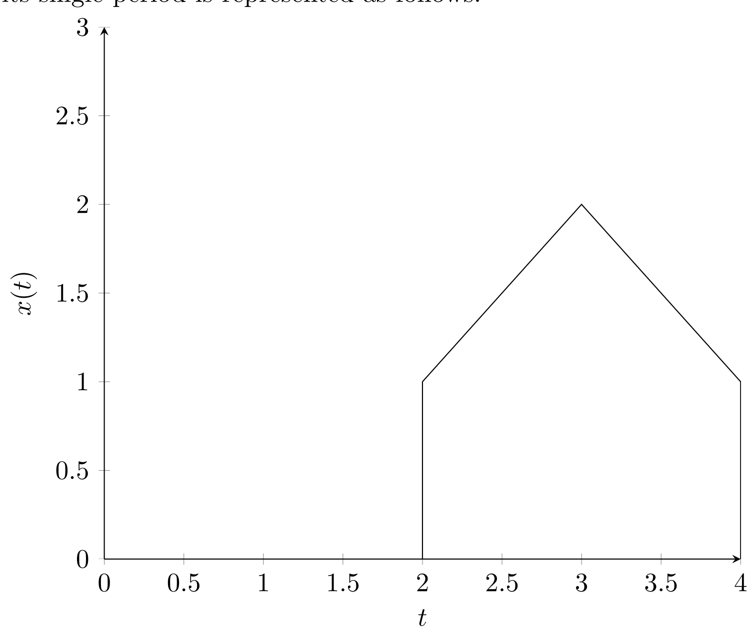
$$
\begin{aligned}
C_n &= \frac{1}{4}\int_0^4 x(t)e^{-nj\frac{\pi}{2}t}dt \\
&= \frac{1}{4}\Big(\int_0^1 te^{-nj\frac{\pi}{2}t}dt + \int_1^2 (2-t)e^{-nj\frac{\pi}{2}t}dt\Big) \\
&= \frac{1}{4}\Big(\frac{e^{-nj\frac{\pi}{2}t}(jn\frac{\pi}{2}t-1)}{(n\frac{\pi}{2})^2}\Big|_0^1 - \frac{e^{-nj\frac{\pi}{2}t}(jn\frac{\pi}{2}t-1)}{(n\frac{\pi}{2})^2}\Big|_1^2 + 2\frac{e^{-nj\pi}-e^{-nj\frac{\pi}{2}}}{-nj\frac{\pi}{2}} \\
&= -\frac{1}{4}\Big(\frac{e^{-jn\pi}-2e^{-jn\pi/2}+1}{(j\pi n/2)^2}\Big) \\
&= \frac{(e^{-jn\pi/2}-1)^2}{\pi^2 n^2}
\end{aligned}
\tag{6}
$$

$$C_0 = \frac{1}{T}\int_0^T x(t)dt$$

$$C_0 = 0.25$$

Note: C_n is FS coefficient
$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j\omega_0 nt}$

7. Let $H(\omega) = cos(\omega) + jsin(\omega)$ be the transfer function of an LTI system. If the input signal $x(t)$ is periodic of fundamental period $T_0 = 4$ sec and its single period is represented as follows:



Find the output of the system y(t).

Hint: Use the results from question 6 to find the FS coefficient.
Solution:

$$x_1(t) = \begin{cases} 1 & 2 \le t \le 4 \\ 0 & 0 \le t \le 2 \end{cases}$$

$$x_2(t) = \begin{cases} t & 0 \le t \le 1 \\ 2-t & 1 < t \le 2 \\ 0 & otherwise \end{cases}$$

$x(t) = x_1(t) + x_2(t-2)$
C_n is FS coefficient of $x(t)$

$$C_n = C_n^1 + C_n^2 e^{-jn\pi}$$

$$C_n = \frac{(-1)^n - 1}{j2n\pi} + \frac{e^{-jn\pi}(e^{-jn\pi/2} - 1)^2}{\pi^2 n^2}$$

$$C_0 = 0.75$$

Let Y_n be the FS coefficient of $y(t)$

$$\begin{aligned} Y_n &= C_n H(w.n) \\ &= C_n(cos(\frac{\pi n}{2}) + jsin(\frac{\pi n}{2})) \\ &= C_n e^{jn\frac{\pi}{2}} \\ &= \frac{e^{\frac{j\pi n}{2}}((-1)^n - 1)}{j2n\pi} + \frac{e^{-jn\pi/2}(e^{-jn\pi/2} - 1)^2}{\pi^2 n^2} \end{aligned} \tag{7}$$

$$Y_0 = 0.75$$

$$y(t) = \sum_{n=-\infty}^{\infty} Y_n e^{\frac{jn\pi t}{2}}$$

8. Let $x(t) = |sin(\pi t)|$ be the input to an LTI satisfying the following differential relationship between input and output

$$\pi^2(y(t) - x(t)) = \frac{d^2x(t)}{dt^2}$$

Find the output y(t) of the system.
Solution:
$f(t) = |sin(\pi t)|$.
$|sin(\pi t)|$ repeats itself twice as often as $sin(\pi t)$
Fundamental Period $T = 1$ sec and $\omega_0 = 2\pi$ rad. Therefore,

$$
\begin{aligned}
F_n &= \frac{1}{T}\int_T sin(\pi t)e^{-j\omega_0 nt}dt \\
&= \int_0^1 sin(\pi t)e^{-j2\pi nt}dt \\
&= \frac{1}{2j}\int_0^1 (e^{j\pi t} - e^{-j\pi t})e^{-j2n\pi t}dt \\
&= \frac{1}{2j}(\frac{e^{j\pi(1-2n)}-1}{j\pi(1-2n)} - \frac{e^{-j\pi(1+2n)}-1}{-j\pi(1+2n)}) \\
&= \frac{2}{\pi(1-4n^2)}
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
\pi^2(y(t)-x(t)) &= \frac{d^2x(t)}{dt^2} \\
\pi^2(Y-X) &= (j\omega)^2 X \\
H(\omega) = \frac{Y}{X} &= 1 - \frac{\omega^2}{\pi^2}
\end{aligned} \tag{9}
$$

Y_n be the FS coefficient of y(t)

$$Y_n = H(n\omega_0)F_n = (1-\frac{n^2 4\pi^2}{\pi^2})\frac{2}{\pi(1-4n^2)} = \frac{2}{\pi}$$

$$y(t) = \sum_{n=-\infty}^{\infty}\frac{2}{\pi}e^{j\omega_0 nt} = \frac{2}{\pi}(1+\sum_{n=1}^{\infty}(e^{-2j\pi nt}+e^{2j\pi nt})) = \frac{2}{\pi}+\frac{4}{\pi}\sum_{n=1}^{\infty}cos(2\pi nt)$$

9. Find the fundamental period and the Fourier series coefficients of the following signal f(t). Then, using the properties of Fourier series, find the FS coefficient of the following signal g(t), where
$f(t) = 2 + \sum_{n=-\infty}^{\infty} x(t-4n)$ and $g(t) = \sum_{n=-\infty}^{\infty} y(t-4n)$

$$x(t) = \begin{cases} t-2 & 0 \le t \le 4 \\ 0 & otherwise \end{cases}$$

and

$$y(t) = \begin{cases} \frac{t^2}{4} & -2 \le t \le 2 \\ 0 & otherwise \end{cases}$$

Solution:

In the period [0,4], f(t)=t, so for all $n \neq 0$, we have

$$\begin{aligned} F_n &= \frac{1}{T}\int_0^T te^{-jnt\omega_0}dt \\ &= \frac{jnt\omega_0 - 1}{T\omega_0^2 n^2}e^{-jnt\omega_0}\Big|_0^T \\ &= \frac{jnT\omega_0 - 1}{T\omega_0^2 n^2}e^{-jnT\omega_0} + \frac{1}{T\omega_0^2 n^2} \\ &= \frac{j2\pi n - 1}{2\pi\omega_0 n^2}e^{-j2n\pi} + \frac{1}{2\pi\omega_0 n^2} \\ &= \frac{j2}{n\pi} \end{aligned} \tag{10}$$

and

$$F_0 = \frac{1}{4}\int_0^4 tdt = 2$$

Note that $x(t) = 2\frac{d}{dt}y(t-2)$, so we have $f(t) = 2 + 2\frac{d}{dt}g(t-2)$, and both $f(t)$ and $g(t)$ are periodic with period 4. Then by the amplitude-scaling, time-derivative, and time-shift properties, we have $F_n = 2jn\omega_0 G_n e^{-j2n\omega_0}$, for all $n \neq 0$. Thus

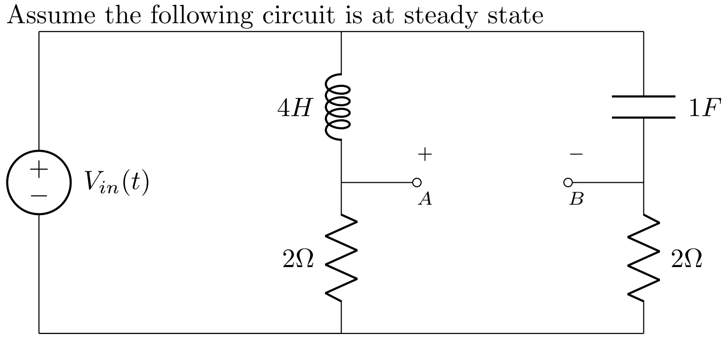
$$G_n = F_n\frac{(-1)^n}{jn\pi} = \frac{2}{\pi^2 n^2}(-1)^n$$

and $F_0 = 2G_0 + 2$, so

$$G_0 = \frac{F_0 - 2}{2} = 0$$

which is intuitive as $y(t)$ is an even function.

10. Assume the following circuit is at steady state



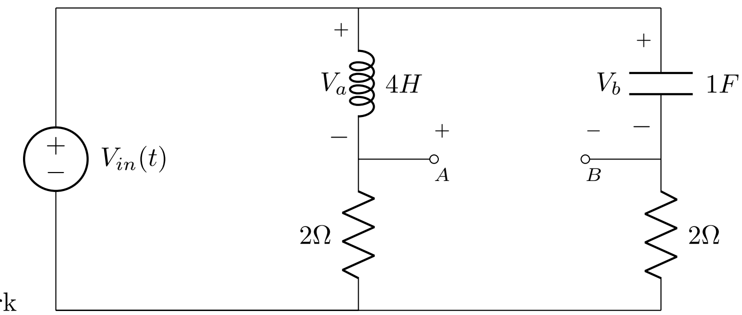
$V_0(t)$ is the output voltage at terminal A and B. If $V_{in}(t)$ is the periodic input with fundamental period 2π and the single period is defined as $x(t) = t \quad 0 \leq t \leq 2\pi$. Find the Fourier series coefficient of the output $V_0(t)$.
Solution:

$\omega_0 = 1$

$$C_n = \frac{1}{T_0}\int_0^{T_0} te^{-int\omega_0}dt = \begin{cases} 2 & n = 0 \\ \frac{i2}{n\pi} & n \neq 0 \end{cases}$$

Therefore, $V_{in}(t) = \sum_{n=-\infty}^{\infty} C_n e^{int\omega_0}$
Now given the transfer function from $V_{in}(t)$ to $V_0(t)$, we know that

$$V_0(t) = \sum_{n=-\infty}^{\infty} C_n H(in\omega_0)e^{int\omega_0}$$



Consider the phasor network

$$\begin{aligned} V_0 &= V_b - V_a \\ V_a &= V_{in}\frac{2i\omega}{1+2i\omega} \\ V_b &= V_{in}\frac{1}{1+2i\omega} \\ V_0 &= V_b - V_a = V_{in}\frac{1}{1+2i\omega} - V_{in}\frac{2i\omega}{1+2i\omega} \\ V_0 &= V_{in}\frac{1-2i\omega}{1+2i\omega} \\ H(i\omega) &= \frac{V_0}{V_{in}} = \frac{1-2i\omega}{1+2i\omega} \end{aligned} \quad (11)$$

Noting that, $\omega_0 = 1$, $H(in\omega_0) = \frac{1-2in}{1+2in}$
Therefore the output Fourier coefficients are

$$C_n H(in\omega_0) = \begin{cases} 2 & n = 0 \\ \frac{i2}{n\pi}\frac{1-2in}{1+2in} & n \neq 0 \end{cases}$$