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★**Random networks for communication.**
From statistical physics to information systems.
Cambridge Series in Statistical and Probabilistic Mathematics.
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This book is about random network models and how local connectivity properties give rise to large-scale properties which emerge as the network grows in size.

The study of emergent properties of evolving, random structures, with the most prominent being random graphs, has been the focus of many researchers coming from widely diverse disciplines which include physics, mathematics, computer science as well as social sciences.

The core idea behind all these studies is that a simple local connectivity rule that defines how two elements of the structure interact with each other can give rise to more complex connectivity properties that hold globally on the structure and manifest themselves (emerge) as the structure's size increases.

Moreover, it appears that there is some critical point, or threshold value, for the local connectivity rule such that the properties emerge suddenly from non-existent to existent, when the rule crosses this point. These emergent properties are, then, aptly called threshold properties. The book is focused on the study of two elementary combinatorial structures which are rich in properties and modeling power, the random tree and the random grid. In the random tree model, we have a tree composed of an infinite number of vertices each having k children, with $k > 0$. Also, a probability value p is fixed and then each edge of the tree appears in the tree, independently of the others, with probability p. In the random grid model, the nodes are positioned on the points of $\mathbb{Z}^2$.

These models are in contrast with the classical Erdős-Rényi random graph models in that in these models adjacency between two vertices is defined by "physical proximity" while in the latter adjacency can be potentially appear between any pair of vertices (see the excellent book [B. Bollobás, *Random graphs*, Second edition, Cambridge Univ. Press, Cambridge, 2001; MR1864966 (2002j:05132)] for a comprehensive, highly rigorous and deep treatment of the properties of these random graph models).

In Chapter 1 the authors provide concise and very intuitive descriptions of the random network models that they will study in the book, hinting, at the same time, at possible application domains for each of them with regard to real network architectures. More specifically, the authors define the random tree and the discrete random grid models, describing some of their basic properties. Moreover, the authors define a very important continuous variant of the discrete random grid model based on a Poisson point process on the plane. This process first generates the grid's vertices on the plane. Now the grid is no longer a regular structure like $\mathbb{Z}^2$, on which the discrete random grid is based. This non-regularity property renders this continuous model a more suitable choice for modeling, e.g., mobile ad hoc communication networks. Now, given a placement of points on the plane according to the Poisson process, the connectivity between the points can be described in a variety of ways, based on the proximity between the points. These ways of defining connectivity give rise to various interesting, "geometric" random networks with different properties and modeling capabilities:

- Nearest Neighbor Networks: Points are first generated according to a Poisson process on the unit square. Then each point is connected to the k points that are closest to it according to Euclidean distance. Notice that changing the density parameter of the Poisson process generating the points does not change the topology of the network that results from the application of the Nearest Neighbor rule (only a scaling occurs).
- Poisson random connection model, denoted by (X, λ, g): We are given a Poisson point process X of density $\lambda > 0$ on the plane, and a connectivity function $g: \mathbb{R}^2 \to [0, 1]$ with the property $0 < \int_{\mathbb{R}^2} g(x)dx < \infty$. Then two points created by X are connected with probability $g(x - y)$, independently of all other pairs. The function g is assumed to be non-increasing and to depend only on the Euclidean norm $|x - y|$ between the points x and y. Note that, in contrast to the Nearest Neighbor Network, this model is sensitive to scale changes due to the fact that g actually *depends* on the Euclidean distance between two points.
- The Boolean model: Given a real number $r > 0$, we may define a variant of the Poisson random connection model by setting $g(x) = 1$ if $|x| \leq 2r$ and $g(x) = 0$ otherwise. In other words, we place discs of radius r centered on the Poisson process points and then draw an edge between two points if their discs have points in common.
- Interference limited network: We consider, again, a Poisson process X that generates the network points. Let x_i and x_j be two of these points and assume that x_i needs to communicate some information to x_j. Ideally, only x_i's signal would be active, while x_i is transmitting the information. However, some of the other nodes may also be active, creating a cumulative interference signal on the signal transmitted by x_i to x_j.

  The strength of this interference is proportional to $\gamma \sum_{k \neq i,j} Pl(x_k, x_j)$, with $\gamma > 0$ being a weighting factor depending on the anti-interference technology adopted by the network. Given this interference measure, we may define the Signal to Noise plus Interference Ratio (SNIR) as

  $$\text{SNIR} = \frac{Pl(x_i, x_j)}{N + \gamma \sum_{k \neq i,j} Pl(x_k, x_j)}$$

  and assume that x_i may reliably relay the information to x_j only if $\text{SNIR} > T$, for a predetermined threshold value $T > 0$. This fact is denoted by inserting an edge between x_i and x_j.
- Information-theoretic networks: The points of this model are, again, generated by a Poisson process. However, all pairs of points are connected by edges; i.e., they form a complete graph. On each edge, a weight is computed that is proportional to the quantity of information that can be transmitted along this edge. According to Information Theory, this quantity (information rate), in bits, is given by

  $$R = \log_2 \left(1 + \frac{Pl(x_i, x_j)}{N} \right) \text{ bits per second.}$$

  We can also take into account interference from other network nodes, by treating it as another noise term added to N:

$$R = \log_2\left(1 + \frac{Pl(x_i, x_j)}{N + \sum_{k \neq i,j} Pl(x_k, x_j)}\right) \text{ bits per second.}$$

Having provided the definitions and basic properties of random network models in a well-written and very readable chapter, the authors continue (in the same clear exposition style) in Chapter 2 with the study of phase transitions. The concept of a phase transition is central to many scientific disciplines, including physics, mathematics and theoretical computer science. A phase transition is a phenomenon whereby a property of a system under observation undergoes an abrupt change, manifested asymptotically as the system grows (i.e. the number of nodes in a network tends to infinity), whenever a number of conditions are satisfied (e.g., the probability of existence of an edge lies within a specific range).

In physics, for instance, we have the sudden appearance of magnetization in idealized 2-dimensional (grid) spin system models whenever the system temperature, starting from a high value (which implies spin disorder), crosses a critical temperature, called the Curie temperature, inducing, slowly, spin alignment and thus magnetization.

In the domain of mathematics, we have the transition of the probability of a randomly generated graph being colourable with three colours (i.e. vertices can be assigned one of three available colours so that adjacent vertices are coloured differently) from 1 to 0 as the edge-to-vertex ratio crosses a certain region.

In theoretical computer science, we have the abrupt transition of the probability that a randomly generated formula in Conjunctive Normal Form (CNF) with clauses of 3 literals (called the 3-SAT formula) is satisfiable from 1 to 0, if the clause-to-variable ratio crosses a certain region.

In Chapter 2 the authors investigate a similar threshold behaviour of the random network models they consider with regard to the percolation property. The authors consider infinite networks on the plane, i.e., networks with infinite number of points, which are generated by a Poisson process or are fixed to be the $\mathbb{Z}^2$ points (regular grid). The percolation property in such an infinite network refers to the formation of an infinite cluster of points such that any two pairs of points can find a communication path between them. In other words, percolation means the emergence of an infinite, connected, point structure.

It is very reasonable, of course, to assume that the probability of an edge, p, plays a crucial role in the percolation process. If, for instance, p is close to 0, then we expect the formation of numerous small clusters (i.e. of finite size). As p is allowed to increase towards 1, then we expect that these small clusters will start to coalesce and, at some value of p, form the required infinite cluster, giving rise to the emergence of percolation. In Chapter 2 the authors also provide, in a clear expository style, the mathematical analysis that establishes the conditions under which percolation emerges in the random network models defined in Chapter 1. In all cases, the authors provide conditions for the defining parameters of each model that mark a threshold point, through which the phase transition from a non-percolation to a percolation state occurs.

In Chapter 3 the authors shift their attention to finite networks and how their behaviour evolves when their defining parameters cross the threshold point. The authors focus on the random grid network, the Boolean random network and the nearest neighbor network.

They examine the networks with regard to the property of full connectivity, i.e. the existence of a component joining all points of the network.

In order to study its asymptotic behaviour, they start by defining a finite network along with a number of random variables of interest that depend on a network size parameter n. Then n is allowed to tend to infinity and various asymptotic results are derived for the defined random variables. One of the most important contributions of this chapter to the reader is the excellent introduction to the important Chen-Stein method for showing that, asymptotically, a sequence of random variables depending on an asymptotic parameter (n in our case) has Poisson behaviour. If the random variable of interest can be written as a sum of n independent low probability indicator random variables, then convergence to a Poisson random variable is easy to obtain.

If, however, the indicator random variables are dependent, but not to a high extent, then the Chen-Stein method can still be used to show convergence to a Poisson random variable in the limit. The authors apply the Chen-Stein method in a detailed way, with easy-to-understand steps, so its applicability to other, similar settings becomes evident in the end. The authors conclude this chapter with an interesting extension of their results for networks whose nodes have lifetime variation, i.e., they cease to be functional after some time. This variation is modeled as a random variable with known distribution. This model is important for real-life networks, especially mobile ad hoc ones, since nodes tend to fail after some time and one needs to ascertain that the network still has the desired full connectivity property, under some appropriate conditions on the behaviour of the network as well as the random variable that models a node's lifetime.

In Chapter 4 the authors continue along the lines set in Chapter 3, i.e., they study connectivity properties of finite graph models and investigate the behaviour of random graph models with regard to the regions where the defining parameter (i.e. probability that an edge exists) is above the critical value, called the supercritical region, as well as below this value, called the subcritical region. More specifically, the authors show, rather surprisingly, that in the supercritical region there is, asymptotically, almost certainly a single infinite cluster of vertices. Moreover, as the probability of the existence between two nodes increases above the critical value, many disjoint paths appear between pairs of nodes. On the contrary, in the subcritical region there is no infinite cluster, almost certainly, and the network consists of numerous components of finite size.

The authors also consider the property of having many paths that connect pairs of nodes from opposite sides of a square centered on the origin of the grid.

One of the most important contributions of the chapter, however, lies near its end. The authors cite an important result, not usually cited in classical random graph books and technical expositions: Russo's approximate zero-one law. Zero-one, or 0-1, laws are a widely studied subject in Mathematics and Theoretical Computer Science. A landmark result in this area is the important theorem of R. Fagin [see J. Symbolic Logic **41** (1976), no. 1, 50–58; MR0476480 (57 #16042)], who showed that in the classical Erdős-Rényi random graph model with edge probability 0.5 (actually, any constant probability would do), properties expressible in the first-order language of graphs obey a 0-1 law.

The first-order language of graph uses first-order logic statements to describe graph properties. It uses the existential and universal quantifiers, the logical connectives and one predicate symbol to indicate the existence of an edge. It can describe a rich set of graph properties, but some properties escape its expressive power (see [J. H. Spencer, *The strange logic of random graphs*, Springer, Berlin, 2001; MR1847951 (2003d:05196)] for a good exposition of the details).

If a property is 0-1, then as the graph grows to infinity the property holds either with probability tending to 1 or with probability tending to 0. No other value is possible. Fagin's proof was later generalized to the so-called extension theorem, which provides a general condition under which a random graph model has a 0-1 law behaviour for properties expressible in the first-order language of graphs (see [J. H. Spencer, op. cit.] for an excellent introduction to 0-1 laws and conditions for their existence).

Furthermore, properties that are not expressible in the first order language and require the use of a higher order logic fragment do not, necessarily, display a 0-1 behaviour. This depends on the logic fragment into which the property is cast (see [J.-M. Le Bars, Bull. Symbolic Logic **6** (2000), no. 1, 67–82; MR1791876 (2001h:03054)] for the interesting technical details).

All these results are asymptotic in nature, which means that they can be used to show 0-1 behaviour in the limit. Russo's result, on the other hand, differs from the approach outlined above in two important respects: (a) it is applicable to finite graphs, and (b) it does not involve a formal language of graph. A property, in this context, needs to be written as a sum of a large but finite number of independent random variables, so, as the property is not significantly affected by the behaviour of a single one of these random variables, are almost always predictable, i.e., the probability of observing the property is arbitrarily close either to 1 or close to 0, depending on whether the value of the edge probability is above or below the critical value respectively.

In Chapter 5, the authors present an important aspect of random networks, that of information transmission capacity in terms of the network node positions and their transmission strategies. This chapter expands on the information-theoretic random network model defined in Chapter 1. With regard to the placement of nodes on the plane, this is defined by a Poisson point process, as dictated by the model definition. With regard to the transmission strategies, the authors consider the following: (a) whenever two nodes wish to establish communication, other nodes behave cooperatively, i.e. they help the two nodes by routing information item exchanged between them, and (b) all the nodes attempt, aggressively, to establish communication simultaneously with no consideration of the other nodes' communication needs.

The authors first provide a nice exposition of some elementary facts from information theory, with emphasis on channel capacity with noise, and they provide upper bounds on the information communication rate achievable as a function of the noise characteristics as well as the information item distribution. The noise is assumed to be additive, Gaussian distributed. The authors consider the transmission of both discrete and continuous information signals.

In the last chapter of the book, Chapter 6, the authors conclude with some algorithmic aspects of random communication network models. More specifically, they consider the problem of locating in a network an existing combinatorial structure (e.g. a path between two nodes). In many random graph problems there is a vast difference between proving the existence of and actually locating a structure within a random graph (the "needle in a haystack" problem). This problem often arises in classical random graph theory: one can prove that a certain object exists in a random graph but finding the object may be computationally intractable. There may be, actually, cases where such an object can be located efficiently using advanced algorithmic techniques, as exemplified by the important result of Beck in [J. Beck, Random Structures Algorithms **2** (1991), no. 4, 343–365; MR1125954 (92i:05219)] for the problem of hypergraph colouring (see, e.g., [N. Alon and J. H. Spencer, *The probabilistic method*, Second edition, Wiley-Intersci., New York, 2000; MR1885388 (2003f:60003)] for more on this important technique).

The authors prove that, for a random network created with edge probability in the supercritical region, there are numerous node disjoint paths linking them in the network that, in addition, are easy to construct following adjacent nodes until the destination node is reached. The authors distinguish between short-range and long-range models. In short-range models (e.g., a random grid model), edges exist between nodes that are rather close to each other. In long-range nodes, edges may exist between nodes that are very far apart (e.g., a geometric random graph model).

The authors provide a clear and easy to follow analysis of what happens in both types of models with regard to establishing routing paths between pairs of nodes. The core observation is that in short-range (large world) models, as the authors rigorously show, establishing such a path can be easier than in long-range (small world) random models.

However, in long-range models, there are classes of edge probability functions, defined by a power-law decaying (with physical distance) probability distribution, that allow efficient information routing whereas different distribution functions, differing only in the exponent of the distribution function, do not allow an easy discovery of routing paths. This is another very interesting instance of threshold behaviour, with the exponent value being the critical parameter.

In addition to the theoretical exposition, each chapter is aptly complemented by exercises that, most often, encourage the reader to finish sketched or incomplete proofs given in the text. The exercises are carefully designed so as to be tractable, with some effort, and to increase, at the same time, the intuition and understanding of the reader of the similarities and differences between the various random graph models. Also, at the end of the book, the authors provide an appendix with some useful background material on basic probability theory.

In summary, this book is a clear, readable and highly intuitive introduction to the properties and applications of random network models that also provides all the rigorous details or invites the reader to fill them in, in the exercises section. The models tackled by the authors are characterized by the important property that the geometry of the nodes has a pivotal role in the formation of the network connections, as opposed to classical Erdős-Rényi random graph models in which there is no notion of geometry and edges can be inserted (with some probability) between any pair of nodes.

The balance between intuition and rigor is ideal, in my opinion, and reading the book is an enjoyable and highly rewarding endeavor. I believe this book will be useful to physicists, mathematicians, and computer scientists who look at random graph models in which point locations affect the shape and properties of the resulting network: physicists will acquaint themselves with complex networks having rich modeling capabilities (e.g., models for random interaction particle systems such as spin glasses), mathematicians may discover connections of the networks with formal systems (much like the connection of the classical Erdős-Rényi random graph properties with first- and second-order logic), and computer scientists will greatly appreciate the applicability of the theory given in the book to the study of realistic, ad hoc mobile networks in which network node connections change rapidly and unpredictably as a function of the geometry of the current node positions.

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