Ray propagation in nonuniform random lattices

Anna Martini
Department of Information and Communication Technology, University of Trento, via Sommarive 14, I-38050 Trento, Italy

Massimo Franceschetti
Department of Electrical and Computer Engineering, University of California at San Diego, 9500 Gilman Drive, La Jolla, California 92093-0407

Andrea Massa
Department of Information and Communication Technology, University of Trento, via Sommarive 14, I-38050 Trento, Italy

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The problem of optical ray propagation in a nonuniform random half-plane lattice is considered. An external source radiates a planar monochromatic wave impinging at an angle $\theta$ on a half-plane random grid where each cell can be independently occupied with probability $q_j = 1 - p_j$, $j$ being the row index. The wave undergoes specular reflections on the occupied cells, and the probability of penetrating up to level $k$ inside the lattice is analytically estimated. Numerical experiments validate the proposed approach and show improvement upon previous results that appeared in the literature. Applications are in the field of remote sensing and communications, where estimation of the penetration of electromagnetic waves in disordered media is of interest. © 2006 Optical Society of America

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1. INTRODUCTION

We study the penetration of a ray propagating in a nonuniform random medium. We consider the canonical scenario of an external source radiating a plane wave impinging at an angle $\theta$ on a half-plane random grid where each cell can be independently occupied with probability $q_j = 1 - p_j$, $j$ being the row index, and we ask how deep the ray can travel inside the medium before being reflected back into the empty half-plane; see Fig. 1.

Assuming grid cells to be large with respect to the wavelength, the propagation mechanism is described by means of geometrical optics and only specular reflections by occupied cells are considered. We analytically estimate the probability of penetrating up to level $k$ inside the lattice before escaping back into the empty half-plane, and validate the result with numerical experiments for different obstacles’ density profiles. We also compare our solution with the one given in Ref. 1.

Franceschetti et al. considered the same canonical problem described in the previous paragraphs in the case in which the probability $q = 1 - p$ does not depend on the row index $j$. Such a uniform two-state random grid is known as the percolation lattice (see Refs. 2 and 3). In this context, lattice cells sharing a common side are called neighbors. Neighbors of occupied sites are called occupied clusters, and similarly, neighbors of empty sites are called empty clusters. One peculiar feature of the percolation lattice is that there exists a threshold probability $p_c = 0.59275$ at which the lattice appearance suddenly changes: for $p > p_c$, an empty cluster of infinite size that spans the whole lattice forms, and we say that the model percolates, while for $p < p_c$, all empty clusters are of finite size, and the model does not percolate. Franceschetti et al. were inspired by the possibility of modeling built-up urban areas as percolating lattices with $p > p_c$, and studied the ray propagation process inside such lattices. Our present paper was motivated by their interesting results.

Our formulation improves the one in Ref. 1 in several ways: (i) it is not restricted to the uniform distribution of empty cells, but it describes propagation in random lattices with general occupation profiles $q_j$; (ii) in the special case when the occupation profile $q_j = q$ for all $j$, our solution is more accurate than that in Ref. 1 for a wide range of incidence angles and occupation probabilities; (iii) even when compared with an extension of the method in Ref. 1 to nonuniform lattices, it provides more accurate results; (iv) the proposed analytical derivation is simpler.

The formula presented in Ref. 1 for the probability $\Pr(0 \to k)$ that the propagating ray reaches a grid level $k$ inside the lattice before escaping back into the empty half-plane was obtained using Martingale theory and was given as a function of the occupation probability $q$ and of the impinging angle $\theta$. Numerical experiments showed that such a formula requires $\theta$ to be not so far from $45^\circ$ and the lattice to be not too sparse, nor too dense, to provide a good approximation of the probability distribution sought. Our simpler derivation assumes that the ray never crosses cells that it has already encountered along its path. This allows us to reduce the problem to a simple 1D random walk that does not depend on $\theta$. De-
2. MARKOV APPROACH

Let us model the propagation environment by means of a half-plane infinite lattice of square cells of unitary length. Each cell is either empty, with probability $p_j$, or occupied, with probability $q_j = 1 - p_j$, $j$ being the row index of the lattice (see Fig. 1). The electromagnetic source is assumed to be external to the lattice, and it radiates a plane monochromatic wave impinging on the lattice at a prescribed angle $\theta$. Since the scatterers are assumed to be large compared to the wavelength $\lambda$, wave propagation is modeled in terms of parallel rays reflected by the obstacles according to the geometrical optics laws. Other electromagnetic interactions (i.e., refraction, absorption, diffraction at the edges, and the scattering due to the surface roughness) are neglected. As in Ref. 1, we consider the problem of determining the probability $\Pr(0 \rightarrow k)$ that a ray reaches a prescribed level $k$ inside the lattice before being reflected back and escaping in the above empty half-plane. We focus on the general case of a nonuniform random lattice, where the density $q_j$ of the occupied cells changes with the level index $j$. The homogeneous arrangement considered in Ref. 1 is a particular case with $q_j = q = 1 - p$ at every lattice level.

We proceed by transforming the problem from a 2D ray propagation problem into a simple 1D random-walk problem, where the dependence on $\theta$ is lost. We formally proceed as follows. First, we note that at each level the ray runs into one horizontal face of a square cell, independently of $\theta$, and in a number of vertical faces proportional to $\theta$ ($s = |\tan \theta|/|\tan \theta|$). Then we observe that whenever the ray hits a vertical face of an occupied cell, it does not change its vertical direction of propagation. Thus, focusing on the propagation depth it is as though reflections on vertical faces never occur. Assuming that the propagating ray never crosses cells that it has already encountered along its path, we consider propagation in the vertical direction occurring with steps that are independent of each other.

Focusing on reflections on horizontal faces, we have that a ray proceeding into a generic level $j$ either changes direction of propagation, remaining in the same level, or keeps the same direction of propagation, entering a new level. The former event takes place with probability $q_{j+1}$ if the ray is traveling with positive direction, or with probability $q_{j-1}$ if the ray is proceeding with negative direc-
tion. Accordingly, the ray enters a new level with probability \( p_{j+1} \) or \( p_{j-1} \), depending on its direction of propagation. Furthermore, if the ray traveling in the positive direction changes direction of propagation an even number of times before entering a new level, then the depth level is increased by one; otherwise it is decreased by one. This situation is formally described by the Markov chain\(^{14} \) depicted in Fig. 2, where states \( j^+ \) and \( j^- \) denote a ray crossing level \( j \) traveling with positive or negative direction, respectively.

We now introduce the following notation. We write \( \Pr(A \rightarrow B < C) \) to indicate the probability that a ray in state \( A \) reaches state \( B \) before going into state \( C \). According to this notation and to the Markov chain of Fig. 2, the probability that a ray reaches a grid level \( k \) inside the lattice before escaping back into the empty half-plane can be expressed as \( \Pr(0^* \rightarrow k^* < 1^*) \). As a matter of fact, when a ray reaches the state 1\(^- \) it escapes from the grid, since there are no occupied horizontal faces between level 1 and level 0. Moreover, a ray always enters a new level traveling in the positive direction; therefore a ray always reaches state \( k^* \) before state \( k^- \).

We state our main result as follows:

**Proposition 2.1**

\[
\Pr(0^* \rightarrow k^* < 1^-) = \frac{p_1 p_2}{1 + p_1 p_2 \sum_{i=0}^{k-3} q_{k-i}}, \quad k \geq 1. \tag{1}
\]

In Eq. (1) the following convention is used. Consider a generic summation \( \sum_{i=0}^{m} f(i) \). When \( m = n + 1 \) the value returned is 0, while for \( m > n + 1 \) the value returned is \( -\sum_{i=0}^{m-1} f(i) \). Accordingly, in Proposition 2.1, the summation returns 0 for \( k = 2 \), and \( -q_j / (p_j p_{j-1}) \) for \( k = 1 \).

Before proving Proposition 2.1, some observations are appropriate. First of all, we note that when propagation in uniform random lattices is considered our solution reduces to

\[
\Pr(0^* \rightarrow k^* < 1^-) = \frac{p^2}{(k-2)q + 1}, \quad k \geq 1, \tag{2}
\]

which simplifies the previously proposed formula of Ref. 1, being independent of the incident angle \( \theta \). We also note that for very sparse or very dense lattices we have, as expected,

\[
\lim_{q \to 0} \Pr(0^* \rightarrow k^* < 1^-) = 1, \quad j \geq 2, \tag{3}
\]

\[
\lim_{q \to 1} \Pr(0^* \rightarrow k^* < 1^-) = 0. \tag{4}
\]

Finally, we note that our solution is derived assuming that the propagating ray never crosses cells that it has already encountered along its path. Clearly this assumption does not hold whatever the value of \( \theta \) and for all occupation profiles. When \( \theta \) is far from 45\(^\circ \), the ray is more likely to travel back through the same cells whenever a reflection occurs (see the left-hand side of Fig. 3). On the other hand, when the obstacle density increases, the ray tends to travel over and over on the same sequence of cells (see the right-hand side of Fig. 3). Accordingly, we expect the proposed solution to be more accurate as the obstacles are more sparse and the incidence angle \( \theta \) is closer to 45\(^\circ \). This is confirmed by the numerical experiments reported in Section 4.

To prove Proposition 2.1, we now state and prove some preliminary lemmas.

**Lemma 2.2**

\[
\Pr((j-1)^* \rightarrow j^* < 1^-) = \frac{p_j}{p_j + q_j \Pr((j-1)^* \rightarrow (j-1)^* < 1^-)}, \quad j \geq 2. \tag{5}
\]

**Proof of Lemma 2.2.** According to the Markov chain depicted in Fig. 2, we can write

\[
\Pr((j-1)^* \rightarrow j^* < 1^-) = p_j + q_j \Pr((j-1)^* \rightarrow (j-1)^* < 1^-) \times \Pr((j-1)^* \rightarrow j^* < 1^-), \quad j \geq 2, \tag{6}
\]

thus

\[
\Pr((j-1)^* \rightarrow j^* < 1^-) = \frac{p_j}{1 - q_j \Pr((j-1)^* \rightarrow (j-1)^* < 1^-)}, \quad j \geq 2. \tag{7}
\]

Now, since the events \( \{ (j-1)^* \rightarrow (j-1)^* < 1^- \} \) and \( \{ (j-1)^* \rightarrow 1^- < (j-1)^* \} \) are mutually exclusive, Eq. (7) can be written as

\[
\lim_{q \to 0} \Pr(0^* \rightarrow k^* < 1^-) = 1, \quad j \geq 2, \tag{3}
\]

\[
\lim_{q \to 1} \Pr(0^* \rightarrow k^* < 1^-) = 0. \tag{4}
\]

Fig. 2. Markov chain. The ray propagation in the nonuniform random half-plane lattice is modeled as a Markov process.
Let $A$ be the event that, starting from $j^-$, the ray reaches state $1^-$ before reaching state $(j-1)^+$. Right-hand side, high density of scatterers; the ray tends to travel over and over on the same sequence of cells.

$$\Pr[(j-1)^+ \rightarrow j^- < 1^-] = \frac{p_j}{p_j + q_j \Pr[(j-1)^+ \rightarrow 1^- < (j-1)^+]} \quad j \geq 2. \quad (8)$$

**Lemma 2.3**

$$\Pr[j^- \rightarrow 1^- < j^+] = \frac{p_{j-1} \Pr[(j-1)^- \rightarrow 1^- < (j-1)^+]}{p_j + q_j \Pr[(j-1)^+ \rightarrow 1^- < (j-1)^+]} \quad j \geq 2. \quad (9)$$

**Proof of Lemma 2.3.** We consider the two following disjoint events. Let $A$ be the event that, starting from $j^-$, the ray reaches state $1^-$ before reaching state $(j-1)^+$; and let $B$ be the event that, starting from $j^-$, the ray reaches $(j-1)^+$ first, and then $1^-$. We have

$$\Pr[j^- \rightarrow 1^- < j^+] = \Pr[A] + \Pr[B], \quad j \geq 2. \quad (10)$$

According to the Markov chain depicted in Fig. 2, we then write the two terms of the sum as

$$\Pr[A] = p_{j-1} \Pr[(j-1)^- \rightarrow 1^- < (j-1)^+], \quad (11)$$

$$\Pr[B] = p_{j-1} \Pr[(j-1)^- \rightarrow (j-1)^+ < 1^-] \times \Pr[(j-1)^+ \rightarrow 1^- < j^+]. \quad (12)$$

The second term can be further expanded as follows:

$$\Pr[B] = p_{j-1} \left[ (1 - \Pr[(j-1)^- \rightarrow 1^- < (j-1)^+]) \times (1 - \Pr[(j-1)^+ \rightarrow j^- < 1^-]) \right]$$

$$= p_{j-1} \left[ \frac{q_j \Pr[(j-1)^- \rightarrow 1^- < (j-1)^+](1 - \Pr[(j-1)^+ \rightarrow j^- < 1^-])}{p_j + q_j \Pr[(j-1)^+ \rightarrow 1^- < (j-1)^+]} \right], \quad (13)$$

where the last equality follows by applying Lemma 2.2. Now, combining Eqs. (10), (11), and (13), after some algebra we get

$$\Pr[j^- \rightarrow 1^- < j^+] = \frac{p_{j-1} \Pr[(j-1)^- \rightarrow 1^- < (j-1)^+]}{p_j + q_j \Pr[(j-1)^+ \rightarrow 1^- < (j-1)^+]} \quad j \geq 2. \quad (14)$$

We are now ready to give a proof of the main result.

**Proof of Proposition 2.1.** The proof is by induction. The base case $k=1$ trivially gives $\Pr[0^+ \rightarrow 1^- < 1^-] = p_1$. Let us now assume that Eq. (1) holds for $k-1$, and let us show that this implies Eq. (1) holds for $k$. By expressing the unknown as

$$\Pr[0^+ \rightarrow k^+ < 1^-] = \Pr[0^+ \rightarrow (k-1)^+ < 1^-] \times \Pr[(k-1)^+ \rightarrow k^+ < 1^-], \quad k \geq 2, \quad (15)$$

the result of Eq. (1) follows immediately after some algebra (see Appendix A) if we can show that
Pr\((k-1)^+ \rightarrow k^+ < 1^-\)

\[= \frac{p_{k-1} \cdot p_k}{p_{k-1} + q_k \cdot \Pr(0^+ \rightarrow (k-1)^+ < 1^-)}, \quad k \geq 2. \]  \hspace{1cm} (16)

According to Lemma 2.2 we have

\[\Pr((k-1)^+ \rightarrow k^+ < 1^-) = \frac{p_k}{p_k + q_k \cdot \Pr((k-1)^+ \rightarrow 1^- < (k-1)^+)}, \quad k \geq 2. \]  \hspace{1cm} (17)

Thus Eq. (16) follows if we can show that

\[\Pr((k-1)^+ \rightarrow 1^- < (k-1)^+) = \frac{\Pr(0^+ \rightarrow (k-1)^+ < 1^-)}{p_{k-1}}, \quad k \geq 2. \]  \hspace{1cm} (18)

To prove that Eq. (18) holds, we need an additional induction argument. The base case \(k=2\) trivially gives \(\Pr(1^- \rightarrow 1^- < 1^-) = 1\). Let us now assume that Eq. (18) holds, and let us compute \(\Pr(k^- \rightarrow 1^- < k^-)\) for \(k \geq 2\). We apply Lemma 2.3, which in this case is stated as

\[\Pr(k^- \rightarrow 1^- < k^-) = \frac{p_{k-1} \cdot \Pr((k-1)^+ \rightarrow 1^- < (k-1)^+)}{p_k + q_k \cdot \Pr((k-1)^+ \rightarrow 1^- < (k-1)^+)}, \quad k \geq 2. \]  \hspace{1cm} (19)

Substituting Eq. (18) in the numerator of Eq. (19) we obtain

\[\Pr(k^- \rightarrow 1^- < k^-) = \frac{\Pr(0^+ \rightarrow (k-1)^+ < 1^-)}{p_k + q_k \cdot \Pr((k-1)^+ \rightarrow 1^- < (k-1)^+)}, \quad k \geq 2. \]  \hspace{1cm} (20)

Now, we note that by Eqs. (15) and (17),

\[\Pr(0^+ \rightarrow k^- < 1^-) = \frac{p_k \cdot \Pr(0^+ \rightarrow (k-1)^+ < 1^-)}{p_k + q_k \cdot \Pr((k-1)^+ \rightarrow 1^- < (k-1)^+)}, \quad k \geq 2, \]  \hspace{1cm} (21)

and thus, by comparing Eq. (20) with Eq. (21), we can argue that

\[\Pr(k^- \rightarrow 1^- | k^-) = \frac{\Pr(0^+ \rightarrow k^- < 1^-)}{p_k}, \quad k \geq 2, \]  \hspace{1cm} (22)

which concludes the proof.

3. MARTINGALE APPROACH

In Ref. 1 Franceschetti et al. presented an analytical derivation based on Martingale theory, obtaining a solution for \(\Pr(0 \rightarrow k)\) that depends on the ray incident angle \(\theta\) on the lattice. Their method was restricted to the case of uniform random lattices; however, it can also be generalized to nonuniform random lattices. The detailed derivation and discussion of the range of validity of the approach in this case is the subject of a companion paper. Next, we briefly summarize the main steps required for this generalization, and then we compare the results with our approach presented in the previous section.

With reference to Fig. 4, we define the following stochastic process:

\[r_n = r_0 + \sum_{m=1}^{n} x_m, \quad n \geq 0, \]  \hspace{1cm} (23)

where \(r_n\) is the horizontal coordinate of the point \(r_0\) and \(r_1\) takes place (i.e., it is the horizontal component of vector \(\hat{r}_n\)) and

\[x_m = r_m - r_{m-1}, \quad m \geq 1. \]  \hspace{1cm} (24)

We now express the probability of reaching level \(k\) inside the lattice as

\[\Pr(0 \rightarrow k) = \sum_{i} \Pr(0 \rightarrow k | r_0 = i) \cdot \Pr(r_0 = i), \]  \hspace{1cm} (25)

where \(\Pr(r_0=i)\) is the probability mass function of the first jump \(r_0\) and \(\Pr(0 \rightarrow k | r_0 = i)\) is the probability that a ray goes beyond level \(k\) conditioned to the level where the first reflection occurs.

As far as \(\Pr(r_0=i)\) is concerned, proceeding along the same lines of Ref. 1, yields

\[\Pr(r_0 = i) = \begin{cases} q_1, & i = 0, \\ q_{e_1} \prod_{j=1}^{i-1} p_{e_j}, & i \geq 0, \end{cases} \]  \hspace{1cm} (26)

where \(p_{e_j} = p_j^+ = p_j^\tan \theta_{j+1}\) is the effective probability that a ray, traveling with positive direction and angle \(\theta\) through level \(j\), reaches level \(j+1\).

We now consider the second term of Eq. (25), i.e., \(\Pr(0 \rightarrow k | r_0 = i)\). Following the same procedure as in Ref. 1, it can be shown that

![Fig. 4. (Color online) Martingale approach. The propagation process is modeled as the sum of many vectorial variables. The \(n\)th component of the stochastic process \(r_n\) is the horizontal component of the vector \(\hat{r}_n\). Under some assumptions, the process \(\sum_{n=0}^{\infty} x_n\) behaves as a martingale with respect to the sequence \(\{x_m\}\) (Ref. 1).](image-url)
4. NUMERICAL COMPARISON

We now validate our proposed approach with numerical experiments, and we provide a comparison with the method in Ref. 1 and its generalization presented in Section 3.

In the following we refer to our proposed approach as the Markov approach (MKV), while we refer to the approach in Ref. 1 and its generalization as the Martingale approach (MTG).

As a reference, the propagation depth has been evaluated by means of computer-based ray tracing experiments. $N = 100$ random lattices with the same scatterers’ density have been generated and $M = 500$ rays have been launched from different entry positions for every grid. By using the same numerical procedure described in Ref. 1, the probability $P_{r}(0 \rightarrow k)$ has been estimated from the collection of paths in the first $K_{max} = 32$ levels of the lattice.

We define the following error figures:

$$\delta_k = \frac{|P_{r}(0 \rightarrow k) - P_{R}(0 \rightarrow k)|}{\max_{k}[P_{r}(0 \rightarrow k)]} \times 100, \quad (\text{Prediction Error}),$$  \hspace{1cm} (29)

$$\langle \delta \rangle = \frac{1}{K_{max}} \sum_{k=1}^{K_{max}} \delta_k, \quad (\text{Mean Error}),$$  \hspace{1cm} (30)

$$\delta_{max} = \max_{k} \{\delta_k\}, \quad (\text{Maximum Error}),$$  \hspace{1cm} (31)

where the subscript $R$ indicates the value estimated with the reference approach, and the subscript $P$ stands for the same value computed by means of either Eq. (1) or Eq. (28).

In the remainder of this section, we first consider the case of a homogeneous grid, providing a comparison with the result in Ref. 1. Then we consider the nonuniform grid case.

A. Uniform Random Lattices

As a first test case, a sparse grid is considered with $q = 0.05$. In Fig. 5, we report the estimated $P_{r}(0 \rightarrow k)$ as a function of the penetration index $k$, for different values of $\theta$. It is evident that the MKV approach describes very well the propagation in the random medium in this case. The range of $\langle \delta \rangle$ is from 0.23% (for $\theta = 45^\circ$) to 1.11% (for $\theta = 15^\circ$), while 0.74% $\leq \delta_{max} \leq 1.42%$. On the other hand, the MTG approach does not perform very well, resulting in values 2.16% $\leq \langle \delta \rangle \leq 20.37%$ and 3.87% $\leq \delta_{max} \leq 25.14%$. This is not surprising since the MTG approach is not expected to work well for low-density media.

A similar behavior is observed when $q$ is increased to 0.15. In Fig. 6 it is shown that the MKV approach again gives the best prediction of the propagation depth in this case when $\theta = 45^\circ$ ($\langle \delta \rangle = 0.46%$ and $\delta_{max} = 0.68%$). As expected from the theory, results become worse as $\theta$ diverges from $45^\circ$. Nevertheless, the MKV approach outperforms the MTG approach for all considered incident angles, and the error is also more stable for different values of the penetration index $k$ and of the angle $\theta$.

When $q$ increases even further and the grid becomes more dense (i.e., $q = 0.25$ and $q = 0.35$), prediction results of the MKV approach become worse (see Fig. 7). In fact, the assumptions behind the method fail: The ray tends to travel over and over through the same sequence of cells and independence is lost. Nevertheless, the MKV approach is more stable than the MTG approach with respect to both the incidence angle $\theta$ and the lattice depth $k$, and prediction results are still good for a wide range of incident angles.

B. Nonuniform Random Lattices

We now consider the nonuniform grid case with various obstacles’ density profiles. The profiles depicted in the left-hand side of Fig. 8 are increasing linear profiles of the type

$$q(x) = q + a(x - 1),$$  \hspace{1cm} (32)

while the profiles depicted in the right-hand side of the figure are double exponential profiles of the type

$$q(x) = \begin{cases} a \exp[(x - L)\tau], & x \leq L, \\ a \exp[(L - x)\tau], & x > L, \end{cases}$$  \hspace{1cm} (33)

$x$ being the lattice depth. The parameters’ values corresponding to the plots in Fig. 8 are $q = 0.05, L = K_{max}/2 = 16$, and $a$ and $\tau$ as described in Table 1.

We first consider the case $\theta = 45^\circ$ for different density profiles. Results for the linear profiles are depicted in Fig. 9. It is evident that the MKV approach outperforms the MTG approach in all the considered cases. For this method, the values of $\langle \delta \rangle$ range from 0.29% (profile L1) to 0.71% (profile L4). On the other hand, performance of the MTG approach is very sensitive to the considered profile with $\langle \delta \rangle$ increasing with the slope $a$ of the occupation profile from 1.11% (profile L1) to 7.82% (profile L4).

Results for the double exponential profiles are depicted in Fig. 10. Similar observations hold in this case. For the MKV approach, the values of $\langle \delta \rangle$ range from 0.31% (profile DB1) to 1.09% (profile DE4), while for the MTG approach, we have 1.44% $\leq \langle \delta \rangle \leq 10.25%$. It is worth noting that MKV satisfactorily performs even in correspondence with level $L = 16$ where the discontinuity in the occupancy profile of Eq. (33) occurs. On the contrary, when the MTG approach is used, we can observe nonnegligible errors around the level $L = 16$ (see Fig. 10). This is because in the
MTG approach, ray jumps following the first one are considered as a single mathematical entity, i.e., they are governed only by \( \Pr(0 \rightarrow k | r_0 = i) \) [see Eq. (25)]. On the contrary, in the MKV approach, each single jump is considered. As a consequence, abrupt changes in the slope of \( \Pr(0 \rightarrow k) \) due to discontinuities in the obstacles’ density profile are correctly detected and reconstructed.

We now consider a second set of experiments, varying the incident angle \( \theta \). We report the results relative to the worst cases, i.e., the most variable profiles L4 and DE4. Similar considerations also hold true for the remaining profiles. First of all, by looking at Fig. 11, we observe that the MKV approach outperforms the MTG approach for all considered values of \( \theta \). As expected from theory, the per-

![Fig. 5](image_url_1)  
Fig. 5. Uniform random lattice with \( q = 0.05 \). We plot \( \Pr(0 \rightarrow k) \) versus \( k \) for different values of \( \theta \). Crosses denote reference data; solid and dashed curves represent reconstructions obtained by the MKV approach and the MTG approach, respectively.

![Fig. 6](image_url_2)  
Fig. 6. Uniform random lattice with \( q = 0.15 \). We plot the prediction error \( \delta_k \) versus \( k \) for different incidence angles \( \theta \). Left-hand side, MKV approach; right-hand side, MTG approach.
formance of the MKV approach slightly weakens when the incidence angle $\theta$ diverges from $45^\circ$. In the worst case $\theta=15^\circ$, we have $\langle \delta \rangle=3.16\%$ and $\langle \delta \rangle=3.59\%$ for the profile L4 and the profile DE4, respectively. On the contrary, the MTG approach provides reconstructions that are much more sensitive to the incident angle $\theta$. In the worst case $\theta=15^\circ$, we have $\langle \delta \rangle=17.31\%$ and $\langle \delta \rangle=19.26\%$ for the profile L4 and the profile DE4, respectively. Finally, we can compare the maximum and the average error values. The MTG approach shows a larger gap between the two values, thus showing larger variance of the error at different lattice levels.

5. CONCLUSIONS
In this paper we have statistically described the ray propagation process inside nonuniform random lattices. We have assumed a far-external source scenario and large, lossless scatterers, whose density changes with the lattice depth. Our approach is based on the key observation that in evaluating the propagation depth it is as though reflections on vertical faces of occupied cells never occur, since they do not change the vertical direction of propagation of the ray. This observation has allowed us to transform the problem from a two-dimensional ray propagation problem into a simple one-dimensional random-walk problem. By modeling propagation in terms of a Markov chain, we have derived a simple closed-form analytical formula. The solution estimates the propagation depth as a function of the obstacle distribution, and it is independent of the incident conditions.

Numerical experiments have confirmed the effectiveness of our approach, which is accurate for a wide range of incident angles and obstacle densities. They have shown improvement upon previous results as well, in particular,

Table 1. Parameters of the Density Profiles

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<th>Profile</th>
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for low-density propagation media. Our approach also outperforms generalizations of previous methods to the inhomogeneous case.

Possible extensions of the present work can be aimed at overcoming limitations that the percolation model intrinsically exhibits in describing wave propagation in disor-

Fig. 9. Linear density profiles. Estimated values of $\Pr(0\rightarrow k)$ versus $k$. Crosses denote reference data; solid and dashed line curves represent predictions obtained by the MKV approach and the MTG approach, respectively.

Fig. 10. Double-exponential density profiles. Estimated values of $\Pr(0\rightarrow k)$ versus $k$. Crosses denote reference data; solid and dashed curves represent reconstructions obtained by the MKV approach and the MTG approach, respectively.

Fig. 11. Linear and double-exponential profiles worst cases. We consider the two density profiles with the worst prediction error. We plot the mean error $\langle \delta \rangle$ and the maximum error $\delta_{\text{max}}$ versus $\theta$ for the MKV approach and the MTG approach.
dered media. With care about trading off accuracy versus simplicity, we can think about introducing in our model physical phenomena such as absorption, scattering due to surface roughness and small obstacles, and diffraction.

Finally, we stress that the percolation model can find application in a wide range of applied problems arising in the framework of wireless communications, remote sensing, and radar engineering. Our solution based on the framework of wireless communications, remote sensing, and radar engineering. Our solution based on the theory of Markov chains may be of interest in all the scenarios that are studied in percolation theory, provided that the ray approach is justified.

**APPENDIX A**

In this appendix we show that Eq. (1) follows if Eq. (16) holds true. By substituting Eq. (16) into Eq. (15) we get

\[
\Pr(0^* \rightarrow k^* < 1^*) = \frac{p_1 p_2}{1 + p_1 p_2 \sum_{i=0}^{k-4} \frac{q_{k-1-i}}{P_{k-1-i} P_{k-i-2}}} \left( \frac{P_{k-1} P_{k}}{1 + p_1 p_2 \sum_{i=0}^{k-4} \frac{q_{k-1-i}}{P_{k-1-i} P_{k-2-i}}} \right)
\]

Since by assumption

\[
\Pr(0^* \rightarrow (k-1)^* < 1^*) = \frac{p_1 p_2}{1 + p_1 p_2 \sum_{i=0}^{k-4} \frac{q_{k-1-i}}{P_{k-1-i} P_{k-i-2}}} \left( \frac{P_{k-1} P_{k}}{1 + p_1 p_2 \sum_{i=0}^{k-4} \frac{q_{k-1-i}}{P_{k-1-i} P_{k-2-i}}} \right)
\]

we can write

\[
\Pr(0^* \rightarrow k^* < 1^*) = \frac{p_1 p_2}{1 + p_1 p_2 \sum_{i=0}^{k-4} \frac{q_{k-1-i}}{P_{k-1-i} P_{k-i-2}}} \left( \frac{P_{k-1} P_{k}}{1 + p_1 p_2 \sum_{i=0}^{k-4} \frac{q_{k-1-i}}{P_{k-1-i} P_{k-2-i}}} \right)
\]

\[
= \frac{p_1 p_2}{1 + p_1 p_2 \sum_{i=0}^{k-4} \frac{q_{k-1-i}}{P_{k-1-i} P_{k-i-2}}} \left( \frac{P_{k-1} P_{k}}{1 + p_1 p_2 \sum_{i=0}^{k-4} \frac{q_{k-1-i}}{P_{k-1-i} P_{k-2-i}}} \right)
\]

\[
= \frac{p_1 p_2}{1 + p_1 p_2 \sum_{i=0}^{k-4} \frac{q_{k-1-i}}{P_{k-1-i} P_{k-i-2}}} \left( \frac{P_{k-1} P_{k}}{1 + p_1 p_2 \sum_{i=0}^{k-4} \frac{q_{k-1-i}}{P_{k-1-i} P_{k-2-i}}} \right)
\]

**APPENDIX B**

In this appendix we show how to obtain Eq. (28) by substituting Eqs. (26) and (27) into Eq. (25).

\[
\Pr(0 \rightarrow k) = \sum_{i=1}^{k-1} \frac{1}{k} p_i q_i^* \left( \prod_{j=1}^{i-1} p_{j+i} \right) + \sum_{i=k}^{k-1} p_i q_i^* \left( \prod_{j=1}^{i-1} p_{j+i} \right)
\]

(B1)

By expressing \(q_i^*\) in terms of \(p_i^*\), the second term of the right-hand side of Eq. (B1) can be rewritten as

\[
\sum_{j=1}^{n} p_i q_i^* \left( \prod_{j=1}^{i-1} p_{j+i} \right) = \sum_{i=1}^{k-1} p_i q_i^* \left( \prod_{j=1}^{i-1} p_{j+i} \right)
\]

\[
= p_1 \prod_{j=1}^{k-1} p_{j+i} + p_1 \prod_{j=1}^{k-1} p_{j+i} - P_1 \prod_{j=1}^{k-1} p_{j+i}
\]

\[
= p_1 \prod_{j=1}^{k-1} p_{j+i} + p_1 \prod_{j=1}^{k-1} p_{j+i} - P_1 \prod_{j=1}^{k-1} p_{j+i}
\]

\[
= p_1 \prod_{j=1}^{k-1} p_{j+i} + p_1 \prod_{j=1}^{k-1} p_{j+i} - P_1 \prod_{j=1}^{k-1} p_{j+i}
\]

\[
= p_1 \prod_{j=1}^{k-1} p_{j+i},
\]

(B2)

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Corresponding author Andrea Massa may be reached at the address on the title page, by phone at 39-0461-882057, fax at 39-0461-882093, or e-mail at andrea massa@ing.unin.it.

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