Matching markets with bundle discounts:
computing efficient, stable and fair solutions

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Abstract
We model a matching market in which nonstrategic vendors sell items of different types and offer discounted prices on the bundle of all items triggered by demand volumes. Each buyer acts strategically in order to maximize her utility, given by the difference between product valuations and price paid. In a market with transferable utility, buyers might be willing to cooperate by exchanging side-payments, to induce discounts. The core of the market is in general empty, therefore we consider a notion of stability that looks at unilateral deviations. We show that efficient matchings – the ones maximizing the social welfare – can be stabilized by transfers between buyers that enjoy desirable properties of rationality (according to which a buyer subsidizes only buyers who contribute to the activation of the desired discount) and fairness (according to which transfers amounts are balanced between buyers). Building on this existence results, and letting $N$, $M$ and $c$ be the numbers of buyers, vendors and product types, we propose a $O(N^2 + NM^c)$ algorithm that, given an efficient matching, computes stabilizing transfers that are rational and fair, by first determining transfers between groups of buyers with an equal product choice, and then between single buyers. Our results show that if cooperation is allowed then social efficiency and stability can coexists in a market presenting subtle externalities, and determining the right amount of cooperation is computationally tractable.

1 Introduction
We model a market in which vendors offer items of different types, and each buyer is interested in purchasing a unit of each type, possibly from different vendors.

Vendors are nonstrategic. Supplies are unlimited and each vendor has a fixed price for each item. Moreover, each vendor has a discount schedule according to which the bundle of all items is offered at discounted price if her demands exceed given thresholds. This can be seen as an incentive to loyal customers who buy from a single vendor who can sustain lower sale prices only in an economy of scale. Buyers play strategically and each selfishly tries to maximize her utility, given by the difference between the perceived value of the products
and the price paid. In order to maximize their utility, buyers might be willing to cooperate to induce vendors to activate their bundle discounts. Buyers who do not purchase any bundle (i.e., who buy from several vendors) also contribute to the activation of discounts by increasing the total demands.

The externalities present in this scenario (for which a buyer’s utility depends on the choices of others) make the core of the game empty in general [15]. That is, given a market, there might not exist a configuration such that no coalition of buyers can increase their total utility by defecting. Therefore, we consider a notion of stability that looks at deviations by single buyers rather than by coalitions of buyers. Despite it is certainly a weak notion of stability in this multi-player setup, it is suitable to model a scenario where communication and coordination between buyers is mediated by a central entity (e.g., consider an online setting in which buyers might only be able to set reserve prices).

We consider the case of transferable utility, in which utility can be transferred between buyers in the form of side-payments, to induce cooperation. To illustrate the potential benefit of side-payments, consider a buyer who desires the bundle from a certain vendor at a discounted price. In order to trigger the discount, she might be willing to pay a subsidy to other buyers to induce them to purchase from the same vendor (as they would otherwise switch to other vendors). Given a market configuration, or matching, we ask whether there exist transfers that stabilize it (i.e., side-payments such that no buyer wants to deviate).

Our pricing model combines ideas from auctions with reserve prices, typical of on-line shopping websites such as Ebay (where a buyer specifies the maximum amount she is willing to pay for a product), and “deal-of-the-day” on-line purchasing, made popular by Groupon and Living Social (which offer discounted gift certificates that become valid if enough people sign up to the deal). In particular, it could model an on-line market in which buyers indicate reserve prices for products from different vendors, who in turn offer discounts if enough people sign up. In the context of our model, reserve prices might represent buyers’ valuations, each buyer prefers the pair of products with the higher difference between her reserve price and the selling price, and buyers might be willing to pay prices that are slightly different between each other in order to trigger deals. Even if the selling price of a product is higher than her reserve price, a buyer might be willing to purchase it if somebody else bears part of her cost. Similarly, if a buyer’s reserve price for a product choice is high enough with respect to a discounted price, then she might be willing to pay a price higher than the selling price to decrease the effective price of other buyers and induce them to buy – contributing to the activation of the discount. In such a scenario, a stable assignment of buyers to vendors must be computed in a centralized fashion.

Given a matching that maximizes social welfare, it is easy to prove the existence of transfers that stabilize it. However, arbitrary transfers might be undesirable for buyers, and we look for transfers that enjoy additional properties of rationality and fairness. Rationality dictates that buyers who benefit from bundle discounts are the only who pay transfers, and each only subsidizes buyers who purchase (at least one item) from her same vendor (as they might
be necessary to trigger the discount). This is motivated by the willingness of each buyer to subsidize only buyers she benefits from. Fairness dictates that buyers pay transfers that are proportional to their surplus, that is, the difference between her current utility and the utility of their best alternative. In order to motivate this notion of fairness, observe that it might be undesirable for a buyer to pay a disproportionately large amount of the transfer needed by the buyers she benefits from if there are other buyers willing to contribute to the payment (although, from the strict point of view of stability, a buyer might be willing to transfer an amount up to her entire surplus, independently of the transfers paid by others).

**Summary of results** Our results show that if cooperation is allowed then social efficiency and stability can coexists in a market presenting complex externalities, and determining the right amount of cooperation is computationally tractable.

In Section 3, we show that, given any matching that maximizes the social welfare (or SWM matching), there exist rational transfers that stabilize it (Theorem 1 in Section 3). This means that efficient matchings are also stable up to suitable transfers (the price of stability is one, a property that is not always observed in games [9, 20]). To prove this, we partition buyers according to their choices and surplus: on the one side, groups of “rich” buyers getting the same discounted bundle and with a positive surplus (i.e., willing to pay transfers); on the other side, groups of “poor” buyers with the same product choice and negative surplus (i.e., in need of subsidy). Then, we show that there are “rational” transfers between groups of buyers such that: each rich group subsidize poor groups with at least one vendor in common; each rich group transfer at most their available surplus; and each poor group receive the necessary subsidy. Group transfers can be translated into rational and stabilizing transfers. This existence result constitutes the main contribution of this work, and its proof is based on the construction of a graph which encodes the transfers between groups of buyers and has no edges if and only if the transfers are rational.

In Section 4, we show how transfers that are rational and fair and stabilize the market can be efficiently computed given a SWM matching. First, group transfers are computed via the Ford-Fulkerson algorithm for the maximum flow on a network such that rational group transfers and feasible flows are in one-to-one correspondence (Section 4.1). Then, rational and fair transfers who stabilize the SWM matching are computed (Section 4.2). For a market with $N$ buyers, $M$ vendors and $c$ product types, group transfers are computed in time $O(M^c T)$,\(^1\) where $T$ is the total subsidy needed. If prices and buyers’ valuations do not depend on $N$ and $M$, this is $O(N M^c)$. Transfers between buyers are computed in additional $O(N^2 + N M^{c-1})$, for a cumulative time of $O(N^2 + N M^c)$. If the

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\(^1\)Consider two functions $f(x)$ and $g(x)$ of a vector $x = (x_1, \ldots, x_n)$. We say that $f(x) = \Theta(g(x))$ if there exist constants $C, M$ such that $f(x) \leq C g(x)$ for all $x$ such that $\min_{1 \leq i \leq n} x_i \geq M$. We say that $f(x) = \Omega(g(x))$ if there exist constants $C, M$ such that $f(x) \geq C g(x)$ for all $x$ such that $\min_{1 \leq i \leq n} x_i \geq M$. We say that $f(x) = \Omega(g(x))$ if both $f(x) = \Theta(g(x))$ and $f(x) = \Omega(g(x))$. 

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the number of vendors is constant (or grows as $M^c = \mathcal{O}(N)$) then the overall complexity is dominated by the $N^2$ term.

Section 5 deals with the computation of SWM matchings. A natural approach consists in a mixed integer program, see [18], requiring time exponential in $N$ and $M$, and whose relaxation is not guaranteed to have integral solutions (i.e., corresponding to valid matchings). Conditional on the number of buyers assigned to each pair of vendors, we compute a SWM matching in time $\Theta(N^2 M^c)$ via the Ford-Fulkerson algorithm for the maximum flow with minimum cost on a network such that maximum flows and feasible matchings are in one-to-one correspondence. Computing a SWM matching requires to consider a number of cases of the order of $N M^c$, and this term dominates the computational complexity. Getting rid of the exponential dependency on $M$ does not seem possible, due to the theoretical hardness of the problem. However, the overall time complexity is polynomial in $N$, and usually $M$ can be assumed much smaller than $N$ or even constant.

We conclude with a discussion in Section 6.

**Related work**  Matching models have always received considerable attention by computer scientists [18, 16, 8, 10] and economists [19, 4, 17, 6, 5, 2], as they constitute the abstraction of many real world strategic scenarios in which choice requires mutual agreement. Examples are retail markets, the labor market, college admissions, and the assignment of residents to hospitals. When externalities are present in the market, stability often becomes problematic [6, 21]. In our model, the externalities are the numbers of buyers purchasing each product from each vendor, as they determine who benefits from discounts.

Online retailing has seen a continuous growth during the last two decades [13], and recently “deal of the day” websites such as Groupon and Living Social have introduced a new form of buying, in which enough buyers must sign up for a deal to be valid. An overview of the literature on group-buying in the web is given by [1] and by [11]. Due to the interdependencies between buyers’ choices and utility, the core of the market is nonempty only under specific assumptions about preferences or discounts [7, 3].

This paper is also related to and motivated by [14], that considers a more basic model with a single product on the market and where each vendor activates multiple discounts at increasing demand volumes. The novelty of our contribution with respect to [14] is twofold. On the one hand, we extend their model to the more general case of multiple products on the market and to the possibility for vendors to activate discount on bundles of items rather than single items. As discounts can be triggered by buyers who do not necessarily benefit from them, proving the existence of (rational) transfers that stabilize the market is nontrivial and necessitates an inductive argument on the market size (see Section 3). We remark that our model, results and algorithms can be extended to the case of more than two products on the market and of more complex discount schedules (see Section 6), therefore including [14] as a special case. On the other hand, we also consider the computational side of stability, by proposing a simple
and efficient algorithm to compute transfers that stabilize the market and enjoy desirable properties of rationality and fairness.

2 The model

We consider a market consisting in a set of $N$ buyers $\mathcal{B}$ and a set of $M$ vendors $\mathcal{S}$. Each vendor sells items (or products) of $c$ types denoted by $1, \ldots, c$, and we assume supplies are unlimited. Let $C = \{1, \ldots, c\}$. Each buyer is willing to purchase a single unit of each item type, possibly from two or more different vendors. As a remark, a vendor $s_\perp \in \mathcal{S}$, called the null vendor, might represent the buyer that does not buy item $k$. In what follows, prices, discounts and valuations corresponding to such vendor $s_\perp$ will be pointed out. Let $\mathcal{S}^c$ denote the cartesian product of $c$ copies of $\mathcal{S}$.

A matching is a set of tuples $\mu \in \mathcal{B} \times \mathcal{S}^c$ such that each $b \in \mathcal{B}$ appears in exactly a single tuple. A matching represents buyers’ choices and, for $\bar{s} = (s_1, s_2, \ldots, s_c) \in \mathcal{S}^c$, $(b, \bar{s}) \in \mu$ denotes that $b \in \mathcal{B}$ purchases item $k$ from vendor $s_k$ for $k = 1, \ldots, c$. We write $\mu(b) = \bar{s}$, and $\mu^k(b) = s_k$ for $k \in C$.

Given a matching $\mu$, for each $\bar{s} \in \mathcal{S}^c$, let $\hat{\mu}(\bar{s}) = \{b \in \mathcal{B} : \mu(b) = \bar{s}\}$ be the set of buyers who purchase item $k$ from vendor $s_k$ for all $k \in C$, and let $n(\bar{s}) = |\hat{\mu}(\bar{s})|$ be its cardinality. Given a matching $\mu$, for each $s \in \mathcal{S}$ and $k \in C$, let $\hat{\mu}^k(s) = \{b \in \mathcal{B} : \mu^k(b) = s\}$ be the set of buyers who purchase item $k$ from vendor $s$ and $n^k(s) = |\hat{\mu}^k(s)|$. We refer to $n(\bar{s}) = (n^1(s), \ldots, n^c(s)) \in \mathbb{N}^c$ as the demand vector of vendor $s$ (where $\mathbb{N}$ is the set of nonnegative integers).

Prices Vendors are nonstrategic. The prices offered by a vendor are determined by her demand vector, according to a price schedule defined as follows. Each vendor $s \in \mathcal{S}$ has a base price $p^k_s$ for each item $k \in C$ (we let $p^i_{s_\perp} = 0$ for the null vendor $s_\perp$). Moreover, $s$ activates discounted prices on the bundle of all items $C$ when certain thresholds are met, as we explain next. Let $p^{(i)}_s = \sum_{k \in C} p^k_s$ the base price of all items offered by $s$. We assume that $s$ has $h$ vectors $\tau_i(s) = (\tau^k_{i}(s), \ldots, \tau^c_{i}(s))$ for $i = 1, \ldots, h$, called the demand thresholds vectors of $s$, such that $\tau^k_{i+1}(s) \geq \tau^k_{i}(s)$ and $\sum_{k \in C} \tau^k_{i+1}(s) > \sum_{k \in C} \tau^k_{i}(s)$ for all $k \in C$ and $i = 0, \ldots, h - 1$. Let $\tau_0(s) = (0, \ldots, 0)$. Let $\tau_\infty(s) = (\infty, \ldots, \infty)$ for the null vendor $s_\perp$. We also assume that $s$ has $h$ prices $p^{(i)}_s$ for $i = 1, \ldots, h$ such that $p^{(i+1)}_s < p^{(i)}_s$ for all $i = 0, \ldots, h - 1$.

Given a matching $\mu$, with corresponding demand vector $n(s) = (n^1(s), \ldots, n^c(s))$ for vendor $s$, $s$ offers the bundle of all items $C$ at a cumulative price $p^{(i^*)}_s$ where

$$i^* = \max_{0 \leq i \leq h} \{i : n^k_s(s) \geq \tau^k_{i}(s), \forall k \in C\}.$$

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We do not make any assumption about the nature of the products, which are not assumed to be complements or substitutes. We only assume that buyers assign zero or negative valuation to sets of products involving multiple units of any single item.

The constraints on the demand thresholds can be written as $\tau^k_{i+1}(s) \geq \tau^k_{i}(s)$ for all $k \in C$ and $\tau^k_{i+1}(s) > \tau^k_{i}(s)$ for some $k \in C$.  

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That is, $s$ offers the bundle of all products at the price corresponding to the largest demand threshold vector that is met component-wise.

If $s$ activates one of her discounts, then a buyer $b$ such that $\mu^k(b) = s$ for all $k \in C$ (i.e., $b$ buys all items from $s$) pays a price $p^{(i^*)}_s$ instead of $p^{(0)}_s$.

Let $T(\mu) = \{ s \in S : \exists i > 0 \text{ s.t. } n^k(s) \geq \tau^k_i(s), \forall k \in C \} \subseteq S$ be the set of vendors who activate a discount under the matching $\mu$.

Utility

Each buyer $b \in B$ has a valuation for each possible product choice $\bar{s} \in S^c$. The valuation $v_b$ of buyer $b \in B$ is a mapping from $S^c$ to $\mathbb{R}^+$, such that, for $\bar{s} = (s_1, \ldots, s_c) \in S^c$, $v_b(\bar{s})$ is the valuation $b$ assigns to purchasing product $k$ from vendor $s_k$ for each $k \in C$. For each $b \in B$, let $v_b(\bar{s}) = 0$ for $\bar{s} = (s_\bot, \ldots, s_\bot)$ (i.e., the choice not to buy any item has zero valuation).

Given a matching $\mu$, each $b \in B$ has quasi-linear utility function given by

$$u_b(\mu) = v_b(\mu(b)) - p_b(\mu),$$

where $p_b(\mu)$ is the price paid by $b$ under the matching $\mu$. Given a matching $\mu$, the price $p_b(\mu)$ is computed as follows. If $\mu^k(b) = s$ for some $s \in T(\mu)$ and all $k \in C$ then $b$ pays the price $p^{(i^*)}_s$ corresponding to the largest threshold that is met. Otherwise $b$ pays $\sum_{k \in C} p^{(0)}_s$, that is, the sum of the base price for each single item.

Buyers play strategically, and each tries to maximize her utility. The social welfare $SW(\mu)$ of a matching $\mu$ is the sum of all buyers’ utilities.

$$SW(\mu) = \sum_{b \in S} u_b(\mu).$$

Definition 1 Matching $\mu$ is social welfare maximizing (SWM) if $SW(\mu) \geq SW(\mu')$ for every matching $\mu'$.

Transferable utility

We consider markets with transferable utility. That is, utility can be transferred between buyers in the form of side-payments, made in order to induce cooperation. To illustrate the potential benefit of side-payments, consider a scenario in which the best option for buyer $b$ is to buy all items from vendor $s$ at a discounted price. In order to purchase the desired products at a low price, $b$ might be willing to bear some of the cost incurred by other users purchasing one or multiple items from $s$, which would otherwise choose other vendors.

For each $b, b' \in B$, let $t_{b \rightarrow b'} \geq 0$ denote the transfer from $b$ to $b'$. Let $t$ denote the vector of transfers between all pairs of buyers. Given matching $\mu$ and transfer vector $t$, let $(\mu, t)$ be the market configuration in which buyers choose items according to $\mu$ and transfers $t$ are exchanged. The utility of $b \in B$ under $(\mu, t)$ is given by $u_b(\mu, t) = u_b(\mu) + \sum_{b' \in B} (t_{b' \rightarrow b} + t_{b \rightarrow b'})$, where the sum is the net amount of transfer received by $b$. Under the assumption of transferable
utility, given a matching $\mu$, we ask whether there exist transfers $t$ such that $(\mu, t)$ is stable, according to a suitable notion of stability.

As a remark, transfers are not equivalent to buyers becoming intermediaries. In fact, a buyer might in general subsidize only a fraction of the transfer needed by another buyer, and a buyer might receive transfer from multiple other buyers.

**Stability** The strongest notion of stability for a market configuration is to exhibit the core property [15]. A matching-transfer pair $(\mu, t)$, has the core property if no coalition of buyers can increase their total utility by deviating from $\mu$. Maximizing the social welfare is necessary condition for the core property (otherwise all buyers can increase their social welfare by deviating to a SWM matching). However, the core of a market $M$ (the set of matching-transfer pairs with the core property) can be empty (refer to the example in Appendix A). We therefore turn our attention to a notion of stability which looks at deviations by single users rather than groups of users. Given a matching-transfer pair $(\mu, t)$, there are two ways a buyer $b$ can deviate from it. First, $b$ might deviate by changing her product choice (resulting in a matching $\mu'$ such that $\mu'(b') \neq \mu(b)$ and $\mu'(b') = \mu(b')$ for each $b' \neq b$). In this case $b$'s utility would be given by the difference between her valuation of the newly chosen product pair and the price paid. We assume that, after defection, $b$ will not be involved in any transfer (as this would not constitute an unilateral action by $b$), and that she cannot enjoy any discount (as other buyers might not allow $b$ to enjoy discounts without paying transfers). Therefore, we assume that after deviation, $b$ pays the base prices of the chosen products. Second, $b$ might deviate by dropping her transfers in full or in part. A buyer $b$ who enjoys a discount from $s \in T(\mu)$ can benefit from other buyers purchasing from vendor $s$ as they can trigger a lower price for $b$. In this case, $b$’s payoff after defection assumes that buyers loose incentive to buy from vendor in $s$, resulting in a price increase. That is, we assume that $b$ dropping her transfers results in the deviation by both subsidized and non-subsidized buyers purchasing from $s$. This assumption does not affect the validity of our results, as we will look at stabilizing SWM matchings: any SWM matching minimizes the number of buyers that need to be subsidized in order to trigger a given price, and the deviation of each of these buyers would result in a price increase. Letting $\mu$ and $\mu'$ be respectively the matching before and after defection by $b$, we have that $u_b(\mu') = v_b(\mu'(b)) - \sum_{k \in C} p_{\mu'k(b)}$, in both cases of $\mu'(b) \neq \mu(b)$ and $\mu'(b) = \mu(b)$. The following definition formalizes the notion of stability just presented.

**Definition 2** A matching-transfer pair $(\mu, t)$ is stable if no buyer can unilaterally and profitably deviate from it. That is, for all $b \in B$, $u_b(\mu) + \sum_{b' \in B} (t_{b' \rightarrow b} + t_{b \rightarrow b'}) \geq u_b(\mu')$ for each $\mu'$ such that $\mu'(b') = \mu(b')$ for each $b' \neq b$.

Given a matching $\mu$, let $u^*_b(\mu)$ be the maximum utility $b$ can achieve by deviating from $\mu$, and let $\sigma_b(\mu) = u_b(\mu) - u^*_b(\mu)$ be the surplus of $b$ under $\mu$. If $\sigma_b(\mu) < 0$ then $b$ needs to receive a positive transfer in order not to deviate.
from \( \mu(b) \) to her best alternative. If \( \sigma_{b}(\mu) > 0 \) then \( b \) might be willing to pay a subsidy to certain buyers to induce them not to deviate from \( \mu \).

**Definition 3** Given a market \( \mathcal{M} \) and a matching \( \mu \), a transfer vector \( t \) is stabilizing if the matching-transfer pair \((\mu, t)\) is stable.

Given a SWM matching \( \mu \), the existence of a stabilizing transfer is trivial to prove.

**Observation 1** For any market \( \mathcal{M} \) and any SWM matching \( \mu \), there exist stabilizing transfers \( t \).

We prove Observation 1 by contradiction. Let \( x = \sum_{b} \sigma_{b}(\mu) > 0 \) be the total transfer available under \( \mu \), and let \( y = \sum_{b} \sigma_{b}(\mu) < 0 \) be the total transfer needed. Assume there are no stabilizing transfers, that is, \( x < y \). A matching in which each \( b \) such that \( \sigma_{b}(\mu) < 0 \) switches to her best alternatives has utility at least \( SW(\mu) + y - x > SW(\mu) \), generating a contradiction. Observe that, maximizing the social welfare is sufficient but not necessary for the existence of stabilizing transfers (see counterexample in Appendix B).

**Rational and fair transfers** We are not interested in arbitrary transfers, as they could be undesirable for certain buyers. Observe that not all buyers are willing to pay transfers. Under a matching \( \mu \), a buyer \( b \) is willing to pay transfers to other buyers only if the price paid by \( b \) under \( \mu \) is strictly smaller than the sum of the base prices of the chosen items (i.e., \( b \) buys all products from a single \( s \in T(\mu) \)) and \( b \) has positive surplus.

Consider buyers \( b \) and \( b' \) such that \( b \) buys all products from a single \( s \in T(\mu) \), \( \sigma_{b}(\mu) > 0 \) and \( \sigma_{b'}(\mu) < 0 \). If \( s \notin \mu(b') = \emptyset \) then \( b \) is not willing to pay any transfer to \( b' \) as her choice does not affect the price \( p_{b}(\mu) \). If \( s \in \mu(b') \neq \emptyset \) then \( b \) is willing to pay a transfer to buyer \( b' \). In particular, \( b \) is willing to pay a cumulative transfer of at most \( \sigma_{b}(\mu) \) to all such buyers \( b' \). The reason is that these buyers might be necessary to trigger the discount \( b \) currently benefits of, and they might defect if they do not receive any transfer. Moreover, if two buyers purchase the same items from the same vendors and have the same surplus, it would be undesirable for one of them to pay a higher transfer than the other.

We consider the following definitions of rationality and fairness.

**Definition 4** A transfer vector \( t \) is rational if, for all \( b, b' \in \mathcal{B} \), \( t_{b \rightarrow b'} > 0 \) only if \( \sigma_{b}(\mu) > 0 \), \( \sigma_{b'}(\mu) < 0 \), \( \mu^{s}(b) = s \) for all \( k \in \mathcal{C} \) and \( s \in \mu(b') \) for some \( s \in T(\mu) \).

**Definition 5** A rational transfer vector \( t \) is fair if for each \( b, b' \in \mathcal{B} \) such that \( \mu(b) = \mu(b') \), \( \sigma_{b}(\mu) > 0 \) and \( \sigma_{b'}(\mu) > 0 \), the transfers paid by \( b \) and \( b' \) are proportional to \( \sigma_{b}(\mu) \) and \( \sigma_{b'}(\mu) \).

### 3 Existence of rational and stabilizing transfers

Our main result states that, maximizing the social welfare is sufficient condition for the existence of rational and stabilizing transfers.
**Theorem 1** For any market $\mathcal{M}$ and SWM matching $\mu$, there exist rational and stabilizing transfers.

For each $s \in \mathcal{T}(\mu)$, let

$$\mathcal{P}(s) = \{ b \in \mathcal{B} : \mu^b(b) = s \ \forall k \in C, \sigma_b(\mu) > 0 \}$$

be the set of buyers who purchase all items from vendor $s$ (at a discounted price) and have positive surplus. For $s \notin \mathcal{T}(\mu)$ let $\mathcal{P}(s) = \emptyset$. Each $b \in \mathcal{P}(s)$ is willing to pay transfers up to $\sigma_b(\mu)$ to buyers who have negative surplus and purchase at least a product $k \in C$ from $s$, for a total of

$$P(s) = \sum_{b \in \mathcal{P}(s)} \sigma_b(\mu).$$

For $s \notin \mathcal{T}(\mu)$, let $P(s) = 0$.

For each subset of vendors $x \subseteq \mathcal{S}$, let

$$\mathcal{N}(x) = \{ b \in \mathcal{B} : \mu^b(b) \in x \ \forall k \in C, \sigma_b(\mu) < 0 \}$$

be the set of buyers who purchase items from all and only the vendors in $x$ and have negative surplus. Observe that $\mathcal{N}(x) = \emptyset$ for all $|x| > c$, so we will implicitly assume $|x| \leq c$. In order not to deviate from $\mu$ by switching to her best alternative, each $b \in \mathcal{N}(x)$ must receive a transfer of $-\sigma_b(\mu)$, for a total of

$$N(x) = -\sum_{b \in \mathcal{N}(x)} \sigma_b(\mu).$$

According to Definition 4, given rational transfers $t$, if $b \in \mathcal{P}(s)$ and $b' \in \mathcal{N}(x)$ for some $s \subseteq \mathcal{S}$ such that $s \notin x$ then $t_{b \to b'} = 0$.

As a remark, given a SWM matching $\mu$, if $s \notin \mathcal{T}(\mu)$ for all $s \in x \subseteq \mathcal{S}$ then $\mathcal{N}(x) = \emptyset$, otherwise, a matching with higher social welfare is obtained if buyers $\mathcal{N}(x)$ switch to their best alternatives.\(^4\)

**Group transfers** In the proof of Theorem 1, we will consider transfers between groups of buyers rather than transfers between single buyers. This is enough as transfers between single buyers can be computed from group transfers in arbitrary ways (we provide a computationally efficient way which also guarantees fairness in Section 4.2). In particular, for $s \in \mathcal{S}$ and $x \subseteq \mathcal{S}$, let

$$\tilde{t}_{s \to x} = \sum_{b \in \mathcal{P}(s)} \sum_{b' \in \mathcal{N}(x)} t_{b \to b'}$$

be the total transfer from buyers $\mathcal{P}(s)$ to buyers $\mathcal{N}(\{s\})$. If transfers $t$ are rational then $\tilde{t}_{s \to x} = 0$ whenever $s \notin x$ (and the group transfers are said to be

\(^4\)In particular, if $s \notin \mathcal{T}(\mu)$, then $\mathcal{N}(\{s\}) = \emptyset$. Buyers $\mathcal{N}(\{s\})$ are the ones who purchase all items from $s$ and have negative surplus. When we restrict our attention to rational transfers, buyers $\mathcal{N}(\{s\})$ can only receive transfer from buyers $\mathcal{P}(s)$.
rational). To prove Theorem 1, we need to show that there exist group transfers \( \bar{t} \) such that

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\begin{align*}
P(s) &\geq \sum_{x, s \in x} \bar{t}_{s \rightarrow x} \quad \forall s \in S \\
N(x) &= \sum_{s \notin x} \bar{t}_{s \rightarrow x} \quad \forall x \subseteq S \\
\bar{t}_{s \rightarrow x} &= 0 \quad s \notin x.
\end{align*}
\]

(1)

The first two constraints require that the matching \( \mu \) can be stabilized by group transfers \( \bar{t} \), while the third constraint requires \( \bar{t} \) to be rational. Group transfers \( \bar{t} \) satisfying (1) are said rational and stabilizing. We consider the following definition of cross-transfer.

**Definition 6** For \( s \in S \) and \( x \subseteq S \), group transfers \( \bar{t}_{s \rightarrow x} \) is a cross-transfer if \( s \notin x \).

Group transfer \( \bar{t} \) are rational if all cross-transfers are zero. Transfers \( t \) (between buyers) are rational if and only if all cross-transfers (between groups) are zero.

Group transfers \( \bar{t} \) and \( \bar{t}' \) are equivalent if buyers \( P(s) \) pay the same transfer and buyers \( N(x) \) receive the same transfer under \( \bar{t} \) and \( \bar{t}' \).

**Definition 7** Group transfers \( \bar{t} \) and \( \bar{t}' \) are equivalent if

\[
\begin{align*}
\sum_{x \subseteq S} \bar{t}_{s \rightarrow x} &= \sum_{x \subseteq S} \bar{t}'_{s \rightarrow x} \quad \forall s \in S \\
\sum_{s \in S} \bar{t}_{s \rightarrow x} &= \sum_{s \in S} \bar{t}'_{s \rightarrow x} \quad \forall x \subseteq S.
\end{align*}
\]

**Proof of Theorem 1** We assume \( \mu \) is a SWM matching. We proceed by contradiction, making the following assumption.

**Assumption 1** There are no stabilizing and rational group transfers \( \bar{t} \). That is, for any stabilizing group transfer \( \bar{t} \), there are no equivalent and rational group transfers \( \bar{t}' \).

Given a SWM matching \( \mu \) and stabilizing group transfers \( \bar{t} \), we construct a graph \( G(\bar{t}) \) (called the cross-transfer graph) which encodes all cross-transfers in \( \bar{t} \) and has no edges if and only if group transfers \( \bar{t} \) are rational. We then show that, given group transfers \( \bar{t} \), there exist equivalent group transfers \( \bar{t}' \) such that the corresponding graph \( G(\bar{t}') \) is directed and acyclic. Assumption 1 implies that any such \( G(\bar{t}') \) has edges, and we complete the proof by showing that a matching \( \mu' \) with \( SW(\mu') > SW(\mu) \) can be obtained, generating a contradiction with the assumption that \( \mu \) is a SWM matching.

**Definition 8** Given group transfers \( \bar{t} \), the cross-transfer graph \( G(\bar{t}) \) is the directed graph with node set equal to \( S \), and directed edge \( (s, s') \) if and only if there exist \( x \subseteq S \) such that \( s \notin x, s' \in x \), \( \bar{t}_{s \rightarrow x} > 0 \).

In words, in \( G(\bar{t}) \) there is an edge from \( s \in S \) to \( s' \in S \) if buyers \( P(s) \) pay a cross-transfer to buyers \( N(x) \) for some \( x \subseteq S \) such that \( s \notin x, s' \in x \). An example of cross-transfer graph is given in Figure 1.

The following results state that rational group transfers correspond to cross-transfer graphs with no edges, and that we can restrict our attention to directed acyclic graphs.

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Figure 1: Example of a cross-transfer graph. Assume that \( S = \{s_1, s_2, s_3, s_4\} \), and that the only nonzero cross-transfers are \( \bar{t}_{s_1 \rightarrow x'} > 0 \) for \( x' = \{s_3, s_4\} \) and \( \bar{t}_{s_2 \rightarrow x''} > 0 \) for \( x'' = \{s_4\} \). According to Definition 8, \( G(\bar{t}) \) has nodes \( \{s_1, s_2, s_3, s_4\} \) and directed edges \( \{(s_1, s_4), (s_1, s_4), (s_2, s_4)\} \).

**Lemma 1** Group transfers \( \bar{t} \) are rational if and only if \( G(\bar{t}) \) has no edge.

**Lemma 2** Given group transfers \( \bar{t} \), there exist equivalent group transfers \( \bar{t}' \) such that the corresponding cross-transfer graph \( G(\bar{t}') \) is acyclic.

The proof of Lemma 1 follows by the definition of cross-transfer graph and is therefore omitted. The proof of Lemma 2 is given in Appendix C.

Without loss of generality, consider stabilizing group transfers \( \bar{t} \) and assume that \( G(\bar{t}) \) is a directed acyclic graph. By Assumption 1, there are no equivalent group transfers \( \bar{t}' \) such that \( G(\bar{t}') \) has no edge. A vendor \( s \in S \) is called a source node if there is no edge \( (s' \rightarrow s) \) in \( G(\bar{t}') \), and an internal node otherwise. Let \( S_{\text{SRC}} \subseteq S \) be the set of vendors corresponding to source nodes in \( G(\bar{t}') \). Let \( S_{\text{IN}} \subseteq S \) be the set of vendors corresponding to internal nodes in \( G(\bar{t}') \). Let

\[
N_{\text{IN}} = \bigcup\{ N(x) \text{ s.t. } x \subseteq S_{\text{IN}} \}
\]

be the set of buyers who purchase products only from vendors \( S_{\text{IN}} \) (and possibly some product from the null vendor \( s_\bot \)) and have negative surplus. By Lemma 2, we can assume without loss of generality that all buyers who receive transfer purchase products only from vendors \( S_{\text{IN}} \). Let

\[
P_{\text{IN}} = \bigcup\{ P(s) \text{ s.t. } s \in S_{\text{IN}} \}
\]

be the set of buyers who buy all items \( C \) from a single vendor in \( S_{\text{IN}} \) and have positive surplus. Similarly, let

\[
P_{\text{SRC}} = \bigcup\{ P(s) \text{ s.t. } s \in S_{\text{SRC}} \}
\]

be the set of buyers who buy all items \( C \) from a single vendor in \( S_{\text{SRC}} \) and have positive surplus. According to \( G(\bar{t}') \), buyers \( P_{\text{IN}} \) are not able to pay the total amount of transfer needed by buyers \( N_{\text{IN}} \), and additional transfer from \( P_{\text{SRC}} \) is needed (observe...
that the latter buyers get no benefit from the product choice of buyers $\mathcal{N}^{IN}$.

Under Assumption 1, letting

$$X = \sum_{b \in \mathcal{P}^{IN}} \sigma_b(\mu) \quad \text{and} \quad Y = -\sum_{b \in \mathcal{N}^{IN}} \sigma_b(\mu),$$

be the amounts of transfer made available by $\mathcal{P}^{IN}$ and needed by $\mathcal{N}^{IN}$ respectively, we have that $X < Y$. Consider the matching $\mu'$ in which all buyers $\mathcal{N}^{IN}$ and $\mathcal{P}^{IN}$ deviate to their best alternatives $^5$. Buyers $\mathcal{P}^{IN}$ can either gain or loose utility after deviation, but each cannot incur a loss larger than $\sigma_b(\mu)$, resulting in an upper bound of $X$ on the cumulative loss.$^7$ Buyers $\mathcal{N}^{IN}$ incur a cumulative gain of at least $Y$ (the gain would be strictly greater than $Y$ if some new threshold is activated for these buyers, after deviation$^6$). Buyers $\mathcal{P}^{IN}$ cannot loose utility, as no buyer deviates from sellers $\mathcal{S}^{SRC}$ as we consider deviations by buyers $\mathcal{N}^{IN}$. All remaining buyers are the ones in $\mathcal{N}(x)$ for $x \subseteq \mathcal{S}$ such that $x \cap \mathcal{S}^{SRC} \neq \emptyset$ (i.e., buyers with negative surplus who do not buy all items from $\mathcal{S}^{IN}$) and all buyers with nonnegative surplus who are not enjoying any discount. Since these buyers do not enjoy the discounts by vendors $\mathcal{S}^{IN}$, they cannot loose utility in $\mu'$ with respect to $\mu$. We have that $SW(\mu') \geq SW(\mu) + Y - X > SW(\mu)$, generating a contradiction with the assumption that $\mu$ is a SWM matching.

4 Computation of fair transfers

Given a market $\mathcal{M}$ and a SWM matching $\mu$, Theorem 1 guarantees the existence of rational and stabilizing transfers. In this section we present an efficient procedure to compute rational and stabilizing transfers that are also fair according to Definition 5. Recall that $N$, $M$ and $c$ are the numbers of buyers, sellers and product types, respectively. We assume that a SWM matching $\mu$ is given, and we proceed as follows. In Section 4.1 we show how to compute rational and stabilizing group transfers $\bar{t}_{s \rightarrow x}$ from $\mathcal{P}(s)$ to $\mathcal{N}(x)$ for all $s \in \mathcal{S}$, $x \subseteq \mathcal{S}$ ($|x| \leq c$), via the max-flow Ford-Fulkerson algorithm on a flow network such that feasible flows are in one-to-one correspondence with rational group transfers. Let $T = \sum_x N(x)$ be the total transfer needed by buyers with negative surplus (that is, all $b$ such that $\sigma_b(\mu) < 0$) and who are not purchasing both items from the same vendor. Assuming that prices and valuations are constant in $\mathcal{N}$ and $\mathcal{M}$, and observing that $|\cup_x \mathcal{N}(x)| \leq N$, we have that $T = \mathcal{O}(N)$, and rational and stabilizing group transfers can be computed in time $\mathcal{O}(TM^c) = \mathcal{O}(NM^c)$.

$^5$Matching $\mu'$ is the results of a deviation from $\mu$ by multiple buyers. We do not directly use this deviation to proof the stability of a matching-transfer pair (whose definition looks at unilateral deviations). We use $\mu'$ to derive a contradiction on the assumption that $\mu$ is a SWM matching.

$^6$Even if for the sake of stability buyers cannot enjoy discounts after deviation, here we consider that discount thresholds might be triggered as we are interested in computing the social welfare of $\mu'$.

$^7$It is necessary to assume that also buyers $\mathcal{P}^{IN}$ deviate to their best alternatives, as their surplus $\sigma_b(\mu)$ depends on their best alternatives given the matching-transfer pair $(\mu, t)$. 

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Given rational and stabilizing group transfers, in Section 4.2 we show how to compute rational and stabilizing transfers between buyers that are fair according to Definition 5. This requires time $\mathcal{O}(N^2 + NM^{e-1})$, for an overall time $\mathcal{O}(N^2 + NM^e)$.

4.1 Step 1: rational and stabilizing group transfers

We consider the following flow network $G$ (refer to Figure 2). Nodes are as follows.
- A single source node $r$, and a single sink node $t$.
- A node $v_x$ for each $x \subseteq S$, $|x| \leq c$, corresponding to $N(x)$. There are $\mathcal{O}(M^e)$ such nodes.
- A node $u_s$ for each $s \in S$, corresponding $\mathcal{P}(s)$. There are $M$ such nodes.

Edges and capacities are as follows.
- For each node $v_x$, an edge from $r$ to $v_x$ with capacity $N(x)$. Flow from $s$ to $v_x$ represents the total transfer to $N(x)$. There are $\mathcal{O}(M^e)$ such edges.
- For each node $v_x$, and edge from $v_x$ to $u_s$ for all $s \in x$, each with capacity $N(x)$. Flow from $v_x$ to $u_s$ represents the group transfer from $\mathcal{P}(s)$ to $N(x)$. There are $\mathcal{O}(M^e)$ such edges (as each node $v_x$ has at most a constant number $c$ of outgoing edges).
- For each node $u_s$, an edge from $u_s$ to $t$ with capacity $P(s)$. Flow from $u_s$ to $t$ represents the total transfer given by $\mathcal{P}(s)$. There are $M$ such edges.

Given a flow $f$ on the network $G$, $f(x,y)$ represents the flow from node $x$ to node $y$. Let $\mathcal{F}(N)$ be the set of all feasible flows on $G$ and $T(M)$ be the set of all rational group transfers in the market $\mathcal{M}$ (given the SWM matching $\mu$). Consider the mapping $\omega : \mathcal{F}(N) \rightarrow T(M)$ such that a feasible flow $f \in \mathcal{F}(N)$ is mapped to group transfers $\tilde{t} = \omega(f)$ such that:

$$\begin{align*}
\{ & \tilde{t}_{s \rightarrow x} = f(v_x, u_s) & x \subseteq S, |x| \leq c, s \in S \text{ such that there is edge } (v_x, u_s) \text{ in } G \\
& \tilde{t}_{s \rightarrow x} = 0 & \text{otherwise.}
\end{align*}$$

Observe that the capacity constraints on edges $(u_s, t), s \in S$ imply that $\sum_x \tilde{t}_{s \rightarrow x} \leq P(s)$ for all $s \in S$. $\tilde{t}$ is rational as in $G$ there is no edge $(v_x, u_s)$ for $s \notin x$.

**Proposition 1** The mapping $\omega : \mathcal{F}(N) \rightarrow T(M)$ is a bijection. Let $f^*$ be a maximum flow on $G$. Then, $\omega(f^*)$ defines rational and stabilizing group transfers.

The proof is given in Appendix D. We can therefore compute rational and stabilizing group transfers via the Ford-Fulkerson algorithm for the maximum flow (see for example [12]). To bound the running time of the algorithm, we assume that the capacities of all edges in $G$ are integer, that is, all terms $P(s)$ and $N(x)$ are integer. This is the case if valuations and prices are multiples of the same unit (e.g., dollars or cents). For a network with $n$ nodes, $e$ edges, integer capacities, and the total capacity of the edges exiting the source equal to $T$, the running time of the algorithm is $\mathcal{O}((m + n)T)$. In $G$, we have that $n = \mathcal{O}(M^c), e = \mathcal{O}(M^c)$, and $T = \sum_x N(x)$. Therefore, stabilizing group
Figure 2: Scheme of the flow network $\mathcal{G}$. A single node $v_x$ for a set $x = \{s_1, s_2, s_3\} \subseteq \mathcal{S}$ is represented. There is an edge from the source $r$ to $v_x$ with capacity $N(x)$, to accommodate the total transfer needed by $\mathcal{N}(x)$. For $i = 1, 2, 3$, there is an edge from $v_x$ to $u_s$ with capacity $N(x)$, to accommodate the transfer from $\mathcal{P}(s_i)$ to $N(x)$. For $i = 1, 2, 3$, there is an edge from $u_s$ to the sink $t$ with capacity $P(s_i)$, to accommodate the total transfer from $\mathcal{P}(s_i)$ (transfer not only to $N(x)$).

Transfers can be computed in time $O(TM^c)$. If we assume that prices and valuations are $O(1)$ (that is, constant in the market size $N, M$), we have that $T = O(N)$ (as $|\bigcup_x \mathcal{N}(x)| \leq N$) and that $O(TM^c) = O(NM^c)$.

4.2 Step 2: transfer between buyers

In this section we show how rational, fair and stabilizing transfers between buyers can be computed from rational and stabilizing group transfers. Observe that each buyer $b \in \mathcal{B}$ belongs at most to a single set $\mathcal{P}(s)$ for some $s \in \mathcal{S}$ or to a single set $\mathcal{N}(x)$ for some $x \subseteq \mathcal{S}$, $|x| \leq c$. We consider the following definition of fairness, equivalent to Definition 5 when we restrict our attention to stabilizing transfers.

Definition 9 Given a market $\mathcal{M}$ and a SWM matching $\mu$, rational and stabilizing transfers $t$ (with corresponding group transfers $\bar{t}$) are fair if, for each $s \in \mathcal{S}$ such that $\mathcal{P}(s) \neq \emptyset$ and each $b \in \mathcal{P}(s)$, the total transfer paid by $b$ is

$$\sum_{b' \in \mathcal{B}} t_{b \rightarrow b'} = \sigma_b(\mu) \sum_{k \in \mathcal{S}} \bar{t}_{s \rightarrow k}/P(s).$$

Observe that all buyers $\mathcal{P}(s)$ are required to pay a cumulative transfer of $\sum_x \bar{t}_{s \rightarrow k}$ to buyers $\bigcup_x \mathcal{N}(x)$, out of an available cumulative surplus of $P(s) = \sum_{b \in \mathcal{P}(s)} \sigma_b(\mu)$. Under rational, fair and stabilizing group transfers $\bar{t}$, Condition (1) guarantees that no buyer with $\sigma_b(\mu) > 0$ pays more than $\sigma_b(\mu)$, and that each buyer with $\sigma_b(\mu) < 0$ can receive the required side-payment.

We now present our algorithm to compute rational and fair stabilizing transfers from rational and stabilizing group transfers. First, $t_{b \rightarrow b'}$ is initialized at
ALGORITHM 1: Algorithm $A_1$, transfers from buyers in $P(s)$

**Input:** $\ell_{j \rightarrow jk}$ for all $k = 0, \ldots, M$, $\sigma_b(\mu)$ for all $b \in B$.

**Initialize:** $\tilde{\sigma}_b = \sigma_b(\mu)$ for each $b \in P(s)$;

for $k = 0, \ldots, M$ do
    if $\ell_{j \rightarrow jk} > 0$ then
        $s \leftarrow \sum_{b \in P(s)} \tilde{\sigma}_b$;
        $\alpha \leftarrow \ell_{j \rightarrow jk}/s$;
        $\beta \leftarrow \ell_{j \rightarrow jk}/(\ell_{j \rightarrow jk} + \ell_{k \rightarrow jk})$;
        Algorithm $A_2$ with input $\{a\tilde{\sigma}_b : b \in P(s)\}$, $\{-\beta \tilde{\sigma}_{b'} : b' \in N(j, k)\}$;
        for $b \in P(s)$ do
            $\tilde{\sigma}_b \leftarrow (1 - \alpha)\tilde{\sigma}_b$;
        end
    end
end

zero for each $b, b' \in B$. Fair transfers from buyers $P(s)$ (for a fixed $s \in S$ such that $P(s) \neq \emptyset$) are computed by algorithm $A_1$ (in Table 1), as follows.

Assume that $\ell_{s \rightarrow x} > 0$ for $x = x_1, \ldots, x_h$ (with $s \in x_k$ for all $k = 1, \ldots, h$), as output by the algorithm in Section 4.1. Observe that $h = O(M^{c-1})$ as we are considering sets $x$ such that $|x| \leq c$ and $s \in x$.

For each $b \in P(s)$, at any given point in the execution of the algorithm, $\tilde{\sigma}_b$ denotes $b$’s residual surplus, that is, the amount $b$ has still available to make side-payments. At initialization, let $\tilde{\sigma}_b = \sigma_b(\mu) > 0$. Transfers to buyers $N(x_k)$ are computed in phases, in increasing order of $k$. At each phase $k = 0, \ldots, h$, let $\alpha = \ell_{s \rightarrow x_k}/\sum_{b \in P(s)} \tilde{\sigma}_b(\mu)$ be the ratio between the group transfer from $P(s)$ to $N(x_k)$ and the residual surplus of $P(s)$, and let $\beta = \ell_{s \rightarrow x_k}/N(x_k)$ be the fraction of transfer that $N(x_k)$ receives from $P(s)$, out of the total transfer from $\cup_{b \in P(s)} P(s)$. Algorithm $A_2$ in Table 2 computes transfers between buyers $P(s)$ to buyers $N(x_k)$ such that each $b \in P(s)$ transfers $\alpha \tilde{\sigma}_b(\mu)$ and each $b' \in N(x_k)$ receives $-\beta \tilde{\sigma}_{b'}(\mu)$. Before increasing the value of $k$, each $b \in P(s)$ updates her residual surplus to $(1 - \alpha)\tilde{\sigma}_b(\mu)$.

The correctness of algorithm $A_2$ is straightforward. Given this, the correctness of algorithm $A_1$ follows by observing that, for each $s \in S$ and $b \in P(s)$, $b$’s transfer in each instance of algorithm $A_2$ never exceed $\tilde{\sigma}_b$, and that for each $x \subseteq S$, $|x| \leq c$ and $b' \in N(x)$, $b'$ receives a total of $-\sigma_b(\mu)$ in the (at most) $c$ instances of algorithm $A_2$ she is involved in.

**Time complexity** Let $T_{A_1}(s)$ and $T_{A_2}(s, x)$ be the number of operations required, respectively, by algorithm $A_1$ for buyers in $P(s)$, and by algorithm $A_2$ to compute transfers from $P(s)$ to $N(x)$. The total time to compute fair, rational and stabilizing transfers is $T(M, N) = O(N^2) + \sum_{s \in S} T_{A_1}(s)$, where the first terms accounts for the initialization of $t$.

We have that $T_{A_2}(s, x) = O(|P(s)| + |N(x)|)$, as during each iteration of
Algorithm 2: Algorithm $A_2$, transfers from buyers in $\mathcal{P}(s)$ to buyers in $\mathcal{N}(j, k)$

**Input:** amounts offered $\{x_1, \ldots, x_n\}$ by $\{b_{h_1}, \ldots, b_{h_n}\}$; requested $\{y_1, \ldots, y_m\}$ by $\{b_{k_1}, \ldots, b_{k_m}\}$.

**Output:** transfers $b_{h_i \rightarrow k_\ell}$ for $i = 1, \ldots, n$ and $\ell = 1, \ldots, m$

**Initialize:** $i = 1$, $\ell = 1$.

while $(\ell \leq m)$ and $(y_\ell > 0)$ do

if $x_i \geq y_\ell$ then

$b_{h_i \rightarrow k_\ell} \leftarrow y_\ell$;

$x_i \leftarrow x_i - y_\ell$;

$\ell \leftarrow \ell + 1$;

else

$b_{h_i \rightarrow k_\ell} \leftarrow x_i$;

$y_\ell \leftarrow y_\ell - x_i$;

$i \leftarrow i + 1$;

end

end

the while loop, one of the indexes $i$ and $\ell$ increases by one, and each iteration requires a constant number of operations.

To upper bound $T_{A_1}(s)$, each iteration of the for loop requires $\mathcal{O}(|\mathcal{P}(s)|)$ operations to compute $s$, and $T_{A_2}(s, x)$ operations for the execution of algorithm $A_2$. Therefore, the cumulative running time is upper bounded by

$$T(M, N) = \mathcal{O}(N^2) + \sum_{s \in \mathcal{S}} T_{A_1}(s)$$

$$= \mathcal{O}(N^2) + \sum_{s \in \mathcal{S}} \sum_{|x| \leq c : x \in \mathcal{E}} (T_{A_2}(s, x) + \mathcal{O}(|\mathcal{P}(s)|))$$

$$= \mathcal{O}(N^2) + \sum_{s \in \mathcal{S}} \sum_{|x| \leq c : x \in \mathcal{E}} \mathcal{O}(|\mathcal{P}(s)| + |\mathcal{N}(x)|)$$

$$= \mathcal{O}(N^2) + \sum_{s \in \mathcal{S}} \mathcal{O}(M^{c-1}) \mathcal{O}(|\mathcal{P}(s)|) + \sum_{s \in \mathcal{S}} \sum_{|x| \leq c : x \in \mathcal{E}} \mathcal{O}(|\mathcal{N}(x)|)$$

$$= \mathcal{O}(N^2) + \mathcal{O}(M^{c-1}N) + \mathcal{O}(N) = \mathcal{O}(N^2 + M^{c-1}N)$$

as $\sum_{s \in \mathcal{S}} |\mathcal{P}(s)| \leq N$, $\sum_{s \in \mathcal{S}} \sum_{|x| \leq c : x \in \mathcal{E}} |\mathcal{N}(x)| \leq cN$, and $\{|\mathcal{S} : s \in \mathcal{x}| = \mathcal{O}(M^{c-1})$.

Combining with the result in Section 4.1, fair, rational and stabilizing transfers between buyers can be computed in time $\mathcal{O}(N^2 + NM^c)$ given a SWM matching.
5 Computation of social welfare maximizing matching

A natural approach to compute a SWM matching is to formulate a mixed integer program (see [18]) in which, for each \( b \in B \) and \( \bar{s} \in S^c \), a binary assignment variable \( x_{b,\bar{s}} \) indicates whether \( \mu(b) = \bar{s} \), and, for each \( s \in S \), a binary variable \( z_{s,i} \) indicates whether the demand of vendor \( s \) meets the threshold \( \tau_i(s) \) (and the corresponding discount is triggered). These would account to \( NM^c + H \) integer variables, where \( H \geq M \) is the total number of discount thresholds of all vendors, and a running time exponential in this quantity. A relaxation of the problem by letting each assignment variable lay in the interval \([0, 1]\) would leave only \( H \) integer variables (and the computational complexity exponential in \( M \)). However, the existence of an integral solution (corresponding to a valid matching) is an open question.

Instead, we follow a different approach, similar to [14]. Conditional on the number of buyers \( n(\bar{s}) = |\hat{\mu}(\bar{s})| \) for each \( \bar{s} \in S^c \) (which we refer to as a partition of the buyers), we compute a SWM matching via the Ford-Fulkerson algorithm for the max-flow with min-cost in time \( O(N^2M^c) \). Then by considering all feasible allocations \( \{n(\bar{s}) : \bar{s} \in S^c\} \) (that are however exponential in \( M^c \)), we determine the SWM matching.

Let \( \Pi = \{\{n(\bar{s}) : \bar{s} \in S^c\} : \sum_{\bar{s} \in S^c} n(\bar{s}) = N\} \) be the set of all feasible partitions, that is, partitions such that each of the \( N \) buyers can be assigned to a single pair of vendors.

Fix \( \pi \in \Pi \), and define a flow network \( G(\pi) \) as follows (see Figure 3). Nodes are the following.

- A single source node \( r \), and a single sink node \( t \).
- For each \( b \in B \), a node \( b \). There are \( N \) such nodes.
- For each \( \bar{s} \in S^c \), a node \( \bar{s} \). There are \( M^c \) such nodes.

The edges, with corresponding capacities and costs, are as follows.

- For each \( b \in B \) and \( \bar{s} \in S^c \), an edge from \( b \) to \( \bar{s} \) with capacity 1 and cost \(-v_b(\bar{s})\), that is, the opposite of the valuation buyer \( b \) gives to product choice \( \bar{s} \). A unit flow on this edge represents buyer \( \mu(b) = \bar{s} \). There are \( NM^c \) such edges.
- For each \( \bar{s} \in S^c \), an edge from \( \bar{s} \) to \( t \) with capacity \( n(\bar{s}) \) and cost 0. An integral flow on this edge represents the total number of buyers choosing \( \bar{s} \). There are \((M + 1)^c\) such edges.

Feasible integral flows on \( G(\pi) \) are in one-to-one correspondence with matchings conditional on \( \pi \). Let \( \mu(f) \) be the matching corresponding to flow \( f \). Given an integral flow \( f \) on \( G(\pi) \), its value equals the number of buyers that are matched to vendor pairs in \( \mu(f) \), and its cost equals the negative of the total valuation by buyers under \( \mu(f) \). Every max-flow \( f \) on \( G(\pi) \) has value \( N \), that is, each buyer is matched to a vendor pair under \( \mu(f) \). Given \( \pi \in \Pi \), the total price paid by buyers is constant for each max-flow \( f \) on \( G(\pi) \). Therefore, maximizing
the social welfare of a matching conditional on \( \pi \) corresponds to minimizing the cost of an integral max-flow on \( G(\pi) \). The total numbers of nodes and edges in \( G(\pi) \) are respectively \( n = \Theta(N + M^c) \) and \( e = \Theta(NM^c) \), and the total capacity of the edges exiting the source is \( T = N \). Since all capacities are integer, the Ford-Fulkerson algorithm finds an integral max-flow with minimum cost in time \( \Theta(T(n + m)) = \Theta(N^2 M^c) \).

To determine the SWM matching of \( M \), for each \( \pi \in \Pi \) we need to determine, a SWM matching conditional on \( \pi \), for an overall time \( \Theta(N^2 M^c |\Pi|) \). However, this is dominated by a term \( N M^c \) (see Appendix E).

Getting rid of the exponential dependency in \( M \) does not seem possible, due to the theoretical hardness of the problem. In fact, fixed \( x > 0 \), deciding whether there exists a matching \( \mu \) with \( SW(\mu) \geq x \) is NP-hard, (by a reduction from the Knapsack problem, as noted by [14]). Even if computationally demanding even for small \( M \), the proposed solution requires time polynomial in the number of buyers \( N \). Our solution is significantly more efficient than both the exhaustive maximization of social welfare over all \( M^{2N} \) matchings, and solving the integer problem above (both exponential in \( N \)). Moreover, \( M \) could in general be considered much smaller than \( N \), or even constant.

6 Discussion

It is an open question whether Theorem 1 holds in the case of arbitrary price schedules, where a vendor might have several discounted prices on sets of products, as described next. Let \( C = \{ x \subseteq C \} \) be the partition of \( C \) (i.e., the set of all \( 2^c \) subsets of \( C \)). The price schedule \( p_s \) of vendor \( s \in S \) is a mapping from \( N^c \times C \) to \( \mathbb{R}^+ \) (the set of nonnegative real numbers), such that, for \( n \in N^c \) and \( x \in C \), \( p_s(n, x) \) is the price for the bundle of products \( x \) offered by \( s \) under demand \( n \). Let \( p_s(n, \emptyset) = 0 \) for each \( s \) and \( n \). We require that \( p_s(m, x) \leq p_s(n, x) \) for all \( x \in C \) if \( m \geq n \) component-wise. Letting \( e_k \) be the unit vector with the \( k \)-th component equal to one and all other components equal to zero, we refer to \( p_s^k = p_s(e_k, \{k\}) \) as the base price of item \( k \) offered by \( s \). The price paid by \( b \) under matching \( \mu \) is determined as follows. For each \( s \in S \), let \( x_b(s) = \{ k \in C : \mu^k(b) = s \} \) be the set of items \( b \) purchases from \( s \). Recalling that \( n(s) \) denotes the demand vector of \( s \) under the matching \( \mu \),

\[
p_b(\mu) = \sum_{s \in S} p_s(n(s), x_b(s)).
\]
Buyers $b$ such that $p_b(\mu) < \sum_{k \in C} p_{\mu^k(b)}$ might be willing to pay transfers.

Finally, we observe that buyers might benefit from misreporting their product valuations. For example, consider a SWM matching $\mu$ and a buyer $b$ with negative surplus $\sigma_b(\mu)$. Let $v$ be $b$'s true valuation of the products she is matched to. If $b$ reports a valuation of $v' = v - x$, for $x > 0$ such that $\mu$ remains a SWM matching under the untruthful reporting, then she can receive a higher subsidy of $-\sigma_b(\mu) + x$ (Theorem 1 guarantees the existence of rational and stabilizing transfers). We leave this issue to future research.

**APPENDIX**

**A  Emptiness of the core**

Consider a market with two product types $A$ and $B$, vendors $S = \{s_1, s_2, s_3\}$, buyers $B = \{b_1, b_2, b_3\}$, and valuations

$b_1 : \begin{align*}
   v_{b_1}(s_1, s_1) &= 8, & v_{b_1}(s_2, s_2) &= 1, & v_{b_1}(s_3, s_3) &= 1,
   v_{b_1}(s_1, s_2) &= 0, & v_{b_1}(s_1, s_3) &= 0, & v_{b_1}(s_2, s_1) &= 0,
\end{align*}$

$b_2 : \begin{align*}
   v_{b_2}(s_2, s_2) &= 8, & v_{b_2}(s_3, s_3) &= 1, & v_{b_2}(s_1, s_1) &= 1,
   v_{b_2}(s_1, s_2) &= 0, & v_{b_1}(s_2, s_1) &= 0, & v_{b_2}(s_1, s_3) &= 0,
\end{align*}$

$b_3 : \begin{align*}
   v_{b_3}(s_3, s_3) &= 8, & v_{b_3}(s_1, s_1) &= 1, & v_{b_3}(s_2, s_2) &= 1,
   v_{b_3}(s_1, s_2) &= 0, & v_{b_3}(s_2, s_1) &= 0, & v_{b_3}(s_1, s_3) &= 0,
\end{align*}$

and $v_b(s, s') = 0$ for each $b$ and $s \neq s'$. Assume that $p_s A = p_s^B = 3$ for each $s \in S$. Each $s \in S$ activates a discounted price of $p_s^{AB} = 2$ when thresholds $\tau_s^A = \tau_s^B = 2$ on the demand of $A$ and $B$ are met. There are three SWM matchings, symmetric with respect to each other. Consider one of them, for example $\mu(b_1) = \mu(b_2) = (s_1, s_1)$ and $\mu(b_3) = (s_3, s_3)$, with individual utilities $u_{b_1}(\mu) = 6$, $u_{b_2}(\mu) = -1$, $u_{b_3}(\mu) = 2$. Since $u_{b_1}(\mu) + u_{b_2} = 5$, any transfer between them would leave one with a net utility of at most 2.5. If $u_{b_1}(\mu) + t_{b_2 \rightarrow b_1} \leq 2.5$, then $b_1$ and $b_3$ can profitably deviate by agreeing on purchasing both items from $s_3$, receiving a cumulative utility of 5, and allocating for example 2.9 units to $b_1$ and 2.1 units to $b_3$. If instead $u_{b_1}(\mu) + t_{b_2 \rightarrow b_1} > 2.5$, then $b_2$ and $b_3$ can profitably deviate by agreeing on purchasing from $s_2$. The analysis for all other SWM matchings is similar.

**B  Maximizing the social welfare is not necessary for stability**

Consider a variation of the example in Appendix A above, in which $b_2$’s valuations are given by

$b_2 : \begin{align*}
   v_{b_2}(s_2, s_2) &= 8, & v_{b_2}(s_1, s_3) &= 1, & v_{b_2}(s_1, s_1) &= 0.5,
\end{align*}$

The matching $\mu$ such that $\mu(b_1) = \mu(b_2) = (s_1, s_1)$ and $\mu(b_3) = (s_3, s_3)$ has $SW(\mu) = 13/2$ and is not SWM (the matching $\mu'$ such that $\mu'(b_2) = \mu'(b_3) = (s_2, s_2)$ and $\mu'(b_1) = (s_1, s_1)$ has $SW(\mu) = 7$). However, a transfer of 15/4 from $b_1$ to $b_2$ makes $\mu$ stable.
C Proof of Lemma 2

First we assume that \( G(\bar{t}) \) contains a cycle of length two, that is edges \((s_1, s_2)\) and \((s_2, s_1)\) for \(s_1, s_2 \in S\). We show that there exist equivalent group transfers \(\bar{t}'\) such that either \(G(\bar{t}) = G(\bar{t}') - \{(s_1, s_2)\}\) or \(G(\bar{t}) = G(\bar{t}') - \{(s_2, s_1)\}\) or \(G(\bar{t}) = G(\bar{t}') - \{(s_1, s_2), (s_2, s_1)\}\).

Then, we assume that the shortest cycles in \(G(\bar{t})\) have length \(K > 2\) and let \(K = s_1, \ldots, s_K, s_{K+1}\) be such a cycle. We show that there exist equivalent group transfers \(\bar{t}'\) such that \(G(\bar{t}')\) has a cycle of length \(K - 1\) obtained by replacing two adjacent edges of \(K\) with a single edge. This completes the proof as each cycle can be reduced to a length-two cycle by iterating the argument and finally to a single edge.

Assume \(G(\bar{t})\) contains edges \((s_1, s_2)\) and \((s_2, s_1)\). Let
\[
\mathcal{X}_1 = \{x \subseteq S : s_1 \notin x, s_2 \in x, \bar{t}_{s_1 \rightarrow x} > 0\},
\]
\[
\mathcal{X}_2 = \{x \subseteq S : s_2 \notin x, s_1 \in x, \bar{t}_{s_2 \rightarrow x} > 0\}.
\]

Let
\[
t_{s_1} = \sum_{x \in \mathcal{X}_1} \bar{t}_{s_1 \rightarrow x},
\]
\[
t_{s_2} = \sum_{x \in \mathcal{X}_2} \bar{t}_{s_2 \rightarrow x}
\]
be respectively the total amount of cross-transfer that buyers \(\mathcal{P}(s_1)\) pay to all buyers \(\mathcal{N}(x), x \in \mathcal{X}_1\) and that buyers \(\mathcal{P}(s_2)\) pay to buyers \(\mathcal{N}(x), x \in \mathcal{X}_2\).

Suppose that \(t_{s_1} \leq t_{s_2}\). We define equivalent group transfers \(\bar{t}'\) such that
\[
t_{s_1 \rightarrow x}' = 0 \quad \text{for each } x \in \mathcal{X}_1,
\]
where buyers \(\mathcal{P}(s_1)\) switch a cumulative amount of transfer \(t_{s_1}\) from buyers \(\mathcal{N}(x), x \in \mathcal{X}_1\) to buyers \(\mathcal{N}(x), x \in \mathcal{X}_2\),
\[
\sum_{x \in \mathcal{X}_2} t_{s_1 \rightarrow x}' = t_{s_1} + \sum_{x \in \mathcal{X}_2} \bar{t}_{s_1 \rightarrow x}.
\]

Each group \(\mathcal{N}(x), x \in \mathcal{X}_1\) receives the missing amount of transfer from buyers \(\mathcal{P}(s_2)\),
\[
t_{s_2 \rightarrow x}' = \bar{t}_{s_2 \rightarrow x} + \bar{t}_{s_1 \rightarrow x} \quad \text{for each } x \in \mathcal{X}_1,
\]
for a total of \(t_{s_1}\). Buyers \(\mathcal{P}(s_2)\) decrease the cross-transfer to buyers \(\mathcal{N}(x), x \in \mathcal{X}_2\) by total amount \(t_{s_1}\),
\[
\sum_{x \in \mathcal{X}_2} t_{s_2 \rightarrow x}' = t_{s_2} - t_{s_1}.
\]

The existence of equivalent group transfers \(\bar{t}'\) such that (2)-(5) hold is straightforward. Observe that \(t_{s_1 \rightarrow x}' = 0\) for all \(x \in \mathcal{X}_1\), and therefore \((s_1, s_2) \notin G(\bar{t}')\).
If $t_{s_2} - t_{s_1} > 0$ then $\bar{t}_{s_2 \rightarrow x} > 0$ for some $x \in \mathcal{X}_2$ and $(s_1, s_2) \in G(\bar{t}')$, otherwise $(s_1, s_2) \notin G(\bar{t}')$. The proof in the case of $t_{s_1} > t_{s_2}$ similarly follows.

Assume now that the shortest cycles in $G(\bar{t})$ have length $K > 2$, and let $\mathcal{K}$ be a shortest cycle. That is, $\mathcal{K}$ is formed by edges $(s_i, s_{i+1})$ for $i = 1, \ldots, K$, with $s_{K+1} = s_1$. For each $k = 1, \ldots, K$ let

$$\mathcal{X}_k = \{x \subseteq S : s_k \notin x, s_{k+1} \in x, \bar{t}_{s_k \rightarrow x} > 0\},$$

$$t_{s_k} = \sum_{x \in \mathcal{X}_k} \bar{t}_{s_k \rightarrow x}.$$ 

Without loss of generality, assume that $s_1 \in \arg \min_{s_k \in \mathcal{K}} t_{s_k}$, that is $t_{s_1} \leq t_{s_k}$ for all $k = 2, \ldots, K$ (which is always true up to node relabeling). By the assumption that $\mathcal{K}$ is a cycle of minimum length, there is no chord in $G(\bar{t}')$, that is $(s_k, s_j) \notin G(\bar{t}')$ if $s_k, s_j \in \mathcal{K}, s_j \neq s_{k+1}$. We build group transfers $\bar{t}'$ which are equivalent to $\bar{t}$ and such that $(s_1, s_2) \notin G(\bar{t}')$ and $(s_i, s_{i+1})$ for $i = 2, \ldots, K$ with $s_{K+1} = s_2$ is a cycle of length $K - 1$ in $G(\bar{t}')$.

Group transfers $\bar{t}'$ are defined such that

$$\bar{t}'_{s_1 \rightarrow x} = 0 \quad \text{for each} \ x \in \mathcal{X}_1,$$  \hfill (6) 

and buyers $\mathcal{P}(s_1)$ switch a cumulative amount of transfer $t_{s_1}$ from buyers $\mathcal{N}(x), x \in \mathcal{X}_1$ to buyers $\mathcal{N}(x), x \in \mathcal{X}_K$, 

$$\sum_{x \in \mathcal{X}_K} \bar{t}'_{s_1 \rightarrow x} = t_{s_1} + \sum_{x \in \mathcal{X}_K} \bar{t}_{s_1 \rightarrow x}. \hfill (7)$$ 

Each group $\mathcal{N}(x), x \in \mathcal{X}_1$ receives the missing amount of transfer from buyers $\mathcal{P}(s_K)$, 

$$\bar{t}'_{s_K \rightarrow x} = \bar{t}_{s_K \rightarrow x} + \bar{t}_{s_1 \rightarrow x} \quad \text{for each} \ x \in \mathcal{X}_1,$$  \hfill (8) 

for a total of $t_{s_1}$. Buyers $\mathcal{P}(s_K)$ decrease the cross-transfer to buyers $\mathcal{N}(x), x \in \mathcal{X}_K$ by total amount $t_{s_1}$, 

$$\sum_{x \in \mathcal{X}_K} \bar{t}'_{s_K \rightarrow x} = t_{s_K} - t_{s_1}. \hfill (9)$$ 

The existence of equivalent group transfers $\bar{t}'$ such that (6)-(9) hold is straightforward. Observe that $\bar{t}'_{s_1 \rightarrow x} = 0$ for all $x \in \mathcal{X}_1$, and therefore $(s_1, s_2) \notin G(\bar{t}')$. Buyers $\mathcal{P}(s_K)$ pay a transfer of $t_{s_1}$ to groups $\mathcal{N}(x), x \in \mathcal{X}_1$. This last contribution is a cross-transfer as $s_K \notin x, s_2 \in x$ for each $x \in \mathcal{X}_1$ because $(s_1, s_K) \notin G(\bar{t})$. Therefore $(s_K, s_2) \in G(\bar{t}')$. Moreover, if $t_{s_K} - t_{s_1} > 0$ then $\bar{t}'_{s_K \rightarrow x} > 0$ for some $x \in \mathcal{X}_K$ and $(s_K, s_1) \in G(\bar{t}')$, otherwise $(s_K, s_1) \notin G(\bar{t}')$. This completes the proof.

### D  Proof of Proposition 1

It is straightforward to see that $\omega$ is a bijection, so we only prove the second part of the claim. Let $\bar{t} = \omega(f^*)$. $\bar{t}$ are rational group transfers (as $\omega$ is a bijection).
Suppose by contradiction that $\bar{t}$ is not stabilizing, that is, condition (1) does not hold for $\bar{t}$. Recall that condition (1) reads as

$$
\begin{cases}
P(s) \geq \sum_{s' : s' \in x} \bar{t}_{s \rightarrow x} & \forall s \in \mathcal{S} \\
N(x) = \sum_{s \in x} \bar{t}_{s \rightarrow x} & \forall x \subseteq \mathcal{S} \\
\bar{t}_{s \rightarrow x} = 0 & s \notin x.
\end{cases}
$$

First, suppose that $P(s) < \sum_{s' : s' \in x} \bar{t}_{s \rightarrow x}$ for some $s \in \mathcal{S}$. This would imply that the flow entering node $u_s$ is larger than the capacity of the edge $(u_s, t)$, generating a contradiction with the feasibility of the maximum flow $f^*$. Second, suppose that $N(x) > \sum_{s \in x} \bar{t}_{s \rightarrow x}$ for some $x \subseteq \mathcal{S}, |x| \leq c$. This would imply that every flow $f''$ on $\mathcal{G}$ is smaller than $\sum_{x} N(x)$, and therefore there exist no group transfers $\bar{t}'$ such that $N(x) = \sum_{s \in x} \bar{t}'_{s \rightarrow x} \forall x \subseteq \mathcal{S}$ for all $x \subseteq \mathcal{S}$, generating a contradiction with Theorem 1 (as feasible flows and rational group transfers are in one-to-one correspondence). Rationality of $\bar{t}$ implies that $\bar{t}_{s \rightarrow x} = 0$ if $s \notin x$.

**E  Computational complexity for determining SWM matchings**

We have that $|\Pi| = \binom{N+M^c-1}{M^c-1}$. To prove this, observe that computing $|\Pi|$ is equivalent to counting the number of ways in which $N$ (indistinguishable) balls can be distributed among a sorted list of $M^c$ set. Consider a line with $N+M^c-1$ empty positions. There are $\binom{N+M^c-1}{M^c-1}$ ways to place $M^c - 1$ stones on the available positions. The occupied positions (in ascending order) represent the boundaries between the $M^c$ sets, and the cardinality of each set is the number of empty positions between two successive stones (if the first position is occupied by a stone, then the first set is empty; if the $\ell$-th and $(\ell + 1)$-th positions are both occupied, then the $(\ell + 1)$-th set is empty).

Using Stirling’s approximation $n! \sim (n/e)^n (2\pi n)^{1/2}$, considering $M$ constant, we have that

$$
|\Pi| \sim \left( \frac{N}{M^c + 1} + 1 \right)^{M^c+1} \left( \frac{M^c + 1}{N} + 1 \right)^N \left( \frac{1}{2\pi N} + \frac{1}{2\pi(M^c - 1)} \right)^{1/2}.
$$

Considering $M$ as a constant, we need time $\Theta(N^2 M^c |\Pi|)$, that is,

$$
\Theta \left( N^2 M^c \left( \frac{N}{M^c - 1} \right)^{M^c-1} \right).
$$

By the upper bound $\binom{n}{k} \leq (en/k)^k$, the time to compute a SWM matching is $\Theta \left( N^2 M^c \left( \frac{eN}{M^c - 1} \right)^{M^c-1} \right)$, dominated by a term $N^{M^c}$.

*Bibliography*


