Chernoff Test for Strong-or-Weak Radar Models
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Abstract—The active management of the sensing stage in inference systems provides substantial gains in performance. This has been first shown by H. Chernoff in 1959 in the context of sequential hypothesis testing. This paper shows applications of these methods to cognitive radar systems. We propose a simple strong-or-weak model for the radar echoes, where the radar sequentially probes the surrounding environment using a sequence of waveforms, each tailored to a specific target. We provide an active multihypothesis sequential test in which the radar chooses the probing waveform based upon the past selections and the past radar returns. We determine how to select the sequence of waveforms, and provide simple analytical expressions for the resulting radar performance in the regime of small risks. Finally, computer simulations are used to corroborate the analytical approximations for values of risks of practical interest.

Index Terms—Chernoff test, sequential detection, strong-or-weak radar model, asymptotic performance.

I. INTRODUCTION

W ith his work on “Sequential design of experiments” [1], Chernoff laid the theoretical foundations for designing an inference system capable of learning sequentially from the environment while managing actively the sensing stage, with the goal of minimizing the average number of observations needed to achieve a desired performance. The main objective of this paper is to address the question of how this work can be relevant to modern radar applications.

At the time of Chernoff’s work radar systems were made of rather simple devices, where the sensing stage was essentially static, with no possibility of being varied in response to drifts in the environmental or operational conditions. In contrast, advances in hardware and software technologies, as well as the development of advanced signal processing tools, have made radar systems of today intelligent, integrated and environmentally adaptive: they have become cognitive devices capable of extracting a large amount of information from the sensed data.

In his milestone work [2], S. Haykin identified three constitutive features of a cognitive radar: (i) intelligent signal processing, based on learning from environment; (ii) feedback from the receiver to the transmitter; and (iii) exploitation of previous radar returns. These ingredients are also essentially contained in Chernoff’s approach [1]. As a consequence, more than fifty years later, we argue that the original ideas of Chernoff have great potential to contribute to the design modern radar systems that are active, as they control the sensing stage adaptively, and efficient, as they minimize the number of collected observations.

To cast Chernoff’s work in a radar setting, we consider a multihypothesis radar scenario, where the task is to decide about the presence of one specific target, among $m$ possible ones, or the absence of targets. It is assumed that a waveform design has been performed off-line, before the detection task and, as a consequence of such design, to each of the $m$ targets is associated a specific waveform. Using a probing waveform matched to the target produces a “strong” echo measured by the radar receiver, while using a waveform unmatched to the target results in a “weak” return. An implicit assumption for the validity of this model is that the targets are different enough from each other, and that the strong radar echoes produced by matched waveforms are of the same order of magnitude.

Although our problem setting refers to a multiple-target scenario, it should be clear that the approach pursued in this paper is naturally suited to other scenarios of practical relevance, including the case in which the alternative to the null hypothesis is the presence of a single target whose impulse response is characterized by an unknown (suitably discretized) parameter. Typical examples are the target’s angle of sight or its Doppler parameter.

We address the problem of how to select, at each probing action, the best of the $m$ waveforms, in order to minimize the expected number of probing actions required to achieve a given level of confidence of detection. This procedure, where the sensing stage is managed on-the-fly, is referred to as hypothesis testing with controlled sensing, or active hypothesis testing.

Such an operational modality bears some resemblance to the classic search-confirm radar processing, where first large radar echoes are selected by using a low threshold (rather high false alarm level), and then a more accurate investigation of the “interesting” areas is performed at lower false alarm level, to confirm the presence of a target. Here, however, the successive probing actions are adapted to the previous returns and probing signals, and in this sense we have an active operational modality.

The active hypothesis testing is naturally suited to a sequential approach in which the detection task is accomplished by illuminating the potential target with a sequence of waveforms selected from a given ensemble, and the length of the sequence is not fixed in advance but is a function of the radar echoes corresponding to these probing actions. In this sense, the system learns from the radar returns and, depending on these specific echo realizations, decides on-the-fly if, and which, further probing actions are needed, to comply with the level of performance required (e.g., the probability of taking a wrong decision).

The literature on sequential testing is extremely abundant. We list only some key references. Sequential detection was developed in [3], with the introduction of the Sequential Probability Ratio Test (SPRT). It was then proved that, for given false alarm
and detection probabilities, the expected number of samples needed to terminate the SPRT is less than that of any other sequential or fixed test [4]. In order to characterize the detection performance, in [3] the so-called Wald's approximations were introduced, basically amounting to neglect the overshots over the threshold of the test statistic. Later on, these approximations were shown to provide the exact limiting performance corresponding to different asymptotic regimes, namely, the regime of vanishing signal-to-noise ratio, and the regime (of specific interest in the present work) of vanishing risks see, e.g., [5], [6]. Generalizations and extensions of these studies have been developed in several directions, including, among others: multihypothesis SPRT [7]; higher-order refinements of the asymptotic formulas taking into account the overshots [8]; the management of the long run issue [9], [10].

The concept of active hypothesis testing was pioneered in [1] for two hypotheses, and extended to the multihypothesis scenario in [11]. These studies have been recently reprinted and extended by [12], [13], and, from a Bayesian perspective, by [14], [15]. The usage of the Chernoff test in radar applications has been rather limited. One relevant example is represented by [16], where a waveform design is proposed for a binary Gaussian scenario, exploiting the reflectivity or scattering function of the target. In this work we consider a different problem where: (i) the system is only allowed to pick a waveform from a prescribed fixed dictionary; (ii) the dictionary has been designed beforehand, based on a particular model for the radar returns, which we call strong-or-weak model; (iii) we deal with multiple hypotheses and general non Gaussian observation distributions.

A. Contribution

We consider the problem of how a radar detector should choose among the available probing signals, at each step of the sequential procedure, basing this choice on the previous radar echoes and on the probing signals used in the previous steps. In general, the signal selection procedure can only be expressed in algorithmic form, as the result of a linear program, and the system performance is a function of the output of this program. In a radar setting, it is important to give physical readability to these algorithmic results. Our main contribution is to provide, under the strong-or-weak model, simple, closed-form expressions of the signal selection and of the performance figures. We study the asymptotic performance of the hypothesis test, in the limit of vanishing risks, and show how much is gained by a clever choice of the probing signals. We also show a numerical analysis of the performance that is compared to the theoretical findings for values of risks of practical interest. We consider both the classic Gaussian shift-in-mean model for the radar returns, as well as the general case of arbitrary echo distributions and of an arbitrary number of targets.

The remainder of this paper is organized as follows. The next section introduces the problem and presents the Chernoff test. The strong-or-weak model for the radar returns is presented in Section III. Section IV addresses the general strong-or-weak model, and in Section V we derive analytical approximations of the system performance. These approximations are compared to the results of computer experiments in Section VI. Conclusions are drawn in Section VII, while two appendices contain ancillary material.

II. PROBLEM FORMULATION

A. Notation

$\mathcal{M}$ is the set of the integers $\{0, 1, \ldots, m\}$, and $\mathcal{M}_i := \mathcal{M} \setminus \{i\}$ is the set obtained by removing the value $i$ from $\mathcal{M}$. $\mathcal{Q}_i$ denotes the set of all the probability mass functions defined over the alphabet $\mathcal{M}_i$. The symbol $y^n$ stems for the vector $(y_1, \ldots, y_n)^T$, where $T$ denotes the transpose. Random variables are denoted by capital letters, and their realizations are denoted by corresponding lowercase letters. $\mathbb{P}$ is the probability measure, $\mathbb{P}_i$ denotes the probability measure given that the true hypothesis is $\mathcal{H}_i$, and $\mathbb{E}_i$ and $\text{VAR}_i$ are the expectation and the variance operators under measure $\mathbb{P}_i$. Let $\alpha$ and $\beta$ be arbitrary quantities functions of an independent variable $x$; the symbolism $\alpha \sim \beta$ means that $\lim_{x \to 0} \alpha/\beta = 1$ (typically, the independent variable will be the maximal risk $R$, defined after eq. (2)). The notation $\alpha \approx \beta$ denotes instead an approximation. The Kullback-Leibler (KL) divergence between two probability density functions $\alpha$ and $\beta$ is denoted by $D(\alpha || \beta)$. Finally, all the logarithms are taken to base $e$, $1$ denotes an $m \times 1$ vector of all ones, and $\mathbb{R}$ denotes the set of the real numbers.

B. General Form of the Radar Problem

The core of any radar system is the basic detection algorithm, and we focus here on the physical layer part of its design. We take the viewpoint by which the detector device consists of an antenna emitting a waveform whose echo is measured at the receiver and used to decide about the presence of a target, by implementing some appropriate statistical hypothesis test. No target parameter estimation is considered, even in the most basic form of range or doppler computation.

We consider the case of extended targets for which the speed of light divided by the target spatial extension is comparable to the bandwidth of the probing waveform. In this case, the radar echoes can be modeled as the superposition of several target points in an extended region of space, leading to the concept of target impulse response [17]–[21]. Recent contributions to the detection of range-spread targets can be found in [22]–[24].

The probing waveform can be designed to match a specific target in order to maximize its response. We assume that a signal design phase is performed off-line, so that a specific probing waveform can be associated to each of the $m$ targets and a target probed by a signal matched to it produces a strong return while a target probed by an unmatched signal produces a weak return, as it is also the case for any probing signal when no target is present. For any given waveform-target pair, the radar return is modeled as a random variable, and we translate the physical meaning of “strong” and “weak” in terms of divergence between probability distributions governing these random variables.

C. Sequential Test

We consider a hypothesis testing problem involving $m + 1$ hypotheses: $\mathcal{H}_0$, in which no target is present and the radar measurement is made of only noise, and $\mathcal{H}_i$, $i \in \mathcal{M}_0$ representing the physical situation in which the $i$-th target is present on the scene and contributes to the radar echo. Thus, we assume that one among $m$ possible targets is on the scene, or the radar is working under the “only-noise” scenario. The characteristics of
the $m$ possible targets are known in advance so that a signal design can be performed beforehand: the radar system can illuminate the region under surveillance by emitting a probing signal, which is chosen among $m$ possible waveforms, each designed to elicit a “strong” response from a specific target.

Each of the $m$ waveforms used as probing signals is identified by an index $u \in \mathcal{M}_0$ and the $u$-th probing signal is designed to detect the $u$-th target, i.e., it is matched to the $u$-th hypothesis $\mathcal{H}_u$. Accordingly, we have $m+1$ exhaustive and mutually exclusive hypotheses:

$$
\begin{align*}
\mathcal{H}_0 & : \text{no target is present,} \\
\mathcal{H}_i & : \text{target $i$ is present,} \\
\vdots & \\
\mathcal{H}_m & : \text{target $m$ is present,}
\end{align*}
$$

and $m$ possible probing signals. To decide which $\mathcal{H}_i$, $i \in \mathcal{M}$, is actually in force the system emits, at successive instant times $k = 1, 2, \ldots$, a sequence of probing signals $u_k \in \mathcal{M}_0$ and collects the corresponding radar echoes $y_k \in \mathbb{R}$. Using a sequential detection approach, the number $N$ of such measurements is not fixed in advance but depends upon the specific values taken by these measurements.

The sequential test used to decide among the $m+1$ hypotheses (1) is a triplet $(\phi, N, \delta)$, described as follows,

- The strategy $\phi$, employed to select the sequence of probing signals $\{u_k\}_{k=1.2\ldots}$, is a causal control policy, namely, the signal selected for the $k$-th probing action is allowed to depend only on the past observations and controls $(y_k^{k-1}, u_k^{k-1})$.

- The sequential test requires some stopping rule for ending the procedure, and this implicitly defines the stopping time random variable $N$, representing how many samples have been used to take the final decision; our stopping rule is in eq. (5).

$$
\delta(y^n, u^n) \in \mathcal{M} \text{ is the decision rule providing the index of the chosen hypothesis at the stopping time } N, \text{ see (6).}
$$

Given the two sequences $(y_k^{k-1}, u_k^{k-1})$, the choice of the probing signal $u_k$ at time $k$ is not necessarily deterministic: to allow full generality, such choice is made at random according to some prescribed probability mass function (PMF)

$$
q(u_k; k | y_k^{k-1}, u_k^{k-1}) := \mathbb{P}\{ \text{select } u_k \text{ at step } k | y_k^{k-1}, u_k^{k-1} \},
$$

which we wish to design according to some optimality criterion. In the above equation, the possible dependence on the time index $k$ has been emphasized, and for $k = 1$ the equation becomes $q(u_1; 1) = \mathbb{P}\{ \text{select } u_1 \text{ at the first step} \}$, which does not entail any conditioning.

Under hypothesis $\mathcal{H}_i$, $i \in \mathcal{M}$, by using the probing signal $u_k$, the radar measurement $Y_k$ is a random variable whose conditional probability density function (PDF) $p_{i}(y_k | u_k)$ is denoted by $p_{u_k}^{yi}(y_k)$. For simplicity, we consider only continuous observations, namely it is throughout assumed that all these PDFs exist as Radon-Nikodym derivative of the corresponding probability measure, with respect to the usual Lebesgue measure on $\mathbb{R}$.

The radar echo $Y_k$ at step $k$ depends upon the probing signal $u_k$ but, given $u_k$, $Y_k$ is conditionally independent of all the past $(y_l^{k-1}, u_l^{k-1})$. Accordingly, by the chain rule, the joint distribution under $\mathcal{H}_i$ of the observation vector $y^n$ and control sequence $u^n$ can be factorized as follows:

$$
p_i(y^n, u^n) = q(u_1; 1) p_{u_1}^{yi}(y_1) \prod_{k=2}^{n} p_i(y_k | y_k^{k-1}, u_k^{k-1})
= q(u_1; 1) \prod_{k=2}^{n} q(u_k; k | y_k^{k-1}, u_k^{k-1}) \prod_{k=1}^{n} p_{u_k}^{yi}(y_k),
$$

from which we see that the likelihood ratio between two different hypotheses, say $\mathcal{H}_i$ and $\mathcal{H}_j$, is only function of $\prod_{k=1}^{n} [p_{u_k}^{yi}(y_k) / p_{u_k}^{yj}(y_k)]$.

D. Chernoff Detector

Let us define the risk $R_i$ as the probability of wrongly deciding for $\mathcal{H}_i$:

$$
R_i := \sum_{j \in \mathcal{M}_i} \pi(j) \mathbb{P}_{j}[\delta = i],
$$

where $\pi(j), j \in \mathcal{M}$ is any prior distribution with full support on the $m+1$ hypotheses, and let $\mathcal{R} := \max_{i \in \mathcal{M}} R_i$ be the maximal risk.\footnote{An alternative formulation would involve the conditional error probabilities $\mathcal{E}_i := \mathbb{P}_{j}[\delta \neq j], j \in \mathcal{M}$, and their maximum $\max_{i \in \mathcal{M}} \mathcal{E}_i$. However, as pointed out in [12], the formulation in terms of the risks in (2) has some advantages and the asymptotic results for $\mathcal{R} = \max_{i \in \mathcal{M}} R_i \to 0$ are stronger than those obtained in the limit $\max_{i \in \mathcal{M}} \mathcal{E}_i \to 0$.}

A basic result in sequential hypothesis testing [11], [13] states that for all $i \in \mathcal{M}$, the best sequential test $(\phi, N, \delta)$ to decide among the $m+1$ hypotheses (1) achieves

$$
\lim_{N \to \infty} -\log R_i 
= E_i[N] = \min_{q \in \mathcal{Q}_0, j \in \mathcal{M}_i} \sum_{u \in \mathcal{M}_0} q(u) D(p_i^n || p_j^n),
$$

where $D(p_i^n || p_j^n) := \int p_i^n(y) \log \frac{p_i^n(y)}{p_j^n(y)} dy$ is the KL divergence between the two PDFs $p_i^n(y)$ and $p_j^n(y)$ [25]. Unless otherwise specified it is throughout assumed that all the divergences are well-defined, finite, and strictly positive. We also assume that $\int p_i^n(y) \log \frac{p_i^n(y)}{p_j^n(y)} dy < \infty$.

The vector $q = (q(1) \ldots q(m))^T \in \mathcal{Q}_0$ appearing in the right-hand side (RHS) of (3) is the “control PMF.” For a given hypothesis $\mathcal{H}_i$, we denote by $q_i^* = (q_1^* \ldots q_m^*)^T$ the PMF achieving the maximum in (3):

$$
q_i^* := \arg \min_{q \in \mathcal{Q}_0, j \in \mathcal{M}_i} \sum_{u \in \mathcal{M}_0} q(u) D(p_i^n || p_j^n).
$$

Note that the optimal control PMF $q_i^*$ is independent of the time index, while it is hypothesis-dependent.

Equation (3) shows that in the regime of vanishingly small risks $R_i$’s, the stopping time $E_i[N]$ grows unbounded, and the best scaling law of $R_i$ is exponential in $E_i[N]$, namely $R_i \approx \exp(-c E_i[N])$, with $c = \min_{j \in \mathcal{M}} \sum_{u \in \mathcal{M}_0} q_i^*(u) D(p_i^n || p_j^n)$.

Equation (3) bounds the performance of any sequential test. A natural question is whether there exists a test achieving such bound. Well, Chernoff’s test [1] achieves the limiting performance (3) and is therefore asymptotically optimal. This is defined as follows:
at step $k - 1$, a Maximum Likelihood (ML) temporary decision among the $m + 1$ hypotheses is made using all the available information up to that time: the chosen index is $i_{k-1} = \arg\max_{i \in M} p_i(y^{k-1}, u^{k-1})$;

- at step $k$, the probing signal $U_k$ is randomly chosen according to the PMF $q_{i_{k-1}}$ corresponding to the most credited hypothesis $H_{i_{k-1}}$;

- the stopping time $N$ is the first index $n$ such that the worst-case likelihood ratio crosses a certain threshold level $\gamma$, i.e., the test stops when

$$\log \frac{p_{i_n}(y^n, u^n)}{\max_{j \neq i_n} p_j(y^n, u^n)} \geq \gamma.$$  

(5)

Whenever the threshold is crossed, the final decision is

$$\delta(y^n, u^n) = \hat{i}_N,$$  

(6)

namely the ML decision at the stopping time $N$.

The PMF $q_{i_n}$ in (4) is the solution to an optimization problem and using this PMF in the Chernoff test leads to a sequential procedure that is optimal in the asymptotic sense specified by (3). The performance of the Chernoff test that employs some control PMF $q_{i_n}$ different from the optimal $q_{i_n}$ is instead [1]:

$$\lim_{R \to 0} - \log R_{i_n} = \frac{1}{\mu_i} \min_{u \in M_0} \sum_{j \in M} q_{i_n}(u) D(p_{i_n}^j || p_{j}^n).$$  

(7)

III. STRONG-OR-WEAK MODEL: GAUSSIAN CASE

We are now ready to cast the described hypothesis testing procedure in a more specific radar setting. We start examining a classic Gaussian shift-in-mean radar return model for which simple, exact closed-form formulas can be derived. Setting the noise variance to unity, the strong-or-weak scenario is described as follows: (i) Under $H_0$, radar returns are standard Gaussian, regardless of the control; (ii) the radar return under $H_i$, $i \in M_0$, using the probing signal matched to that hypothesis, is a unit-variance Gaussian with expectation $\mu_i$; and (iii) the radar return under $H_i$, $i \in M_0$, probed by any signal other than that matched to $H_i$, is a unit-variance Gaussian with expectation $\ll \mu_i$. Namely, for $i \in M$ and $u \in M_0$,

$$p_i^u(y) = \frac{e^{-(y - \mu_i^u)^2/2}}{\sqrt{2\pi}}.$$

(8)

Key to this model is that all the strong returns are of the same order of magnitude, namely that the $\mu_i$’s are close to each other. This can easily be accomplished in the signal design stage by appropriate normalization of the matched returns.

It is convenient to introduce the (matched) signal-to-noise ratio pertaining to the $i$-th target $\text{SNR}_i := \mu_i^u/2$, $i \in M_0$, and provide the explicit expression for the divergences: $D(p_i^u || p_j) = (\mu_i^{u} - \mu_j^{u})^2/2$.

To begin the analysis, let us assume that along with the “no target” hypothesis $H_0$ there are only two alternatives, $H_1$ and $H_2$, namely two possible targets. We let $\text{SNR}_1 = \text{SNR}_2 = \text{SNR}$ and also assume that the mean of the distribution is one and the same for both mismatched situations, i.e., $0 < \alpha_1^u = \alpha_2^u = \epsilon \ll 1$.

<table>
<thead>
<tr>
<th>TABLE I</th>
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<tbody>
<tr>
<td><strong>PERFORMANCE OF THE CHERNOFF TEST FOR THE GAUSSIAN MODEL</strong></td>
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<table>
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<tr>
<th>Hypothesis</th>
<th>Chernoff</th>
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<tbody>
<tr>
<td>$H_0$</td>
<td>$- \log_{\text{SNR}} \frac{2}{1 - \epsilon^2}$</td>
</tr>
<tr>
<td>$H_1$</td>
<td>$- \log_{\text{SNR}} \frac{1}{1 - \epsilon^2}$</td>
</tr>
<tr>
<td>$H_2$</td>
<td>$- \log_{\text{SNR}} \frac{1}{1 - \epsilon^2}$</td>
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</table>

Let $\lambda = q(1) = 1 - q(2)$. From (3) it follows that the optimal asymptotic performance of the Chernoff test under $H_i$ is

$$d_i^* := \max_{q \in Q_i} \min_{j \in M} q(u) D(p_i^u || p_j^u)$$  

(9)

Consider first $H_0$. Organizing the four divergences appearing in the previous equation in the form of a matrix $D_0$, we have:

$$D_0 = \begin{pmatrix} D(p_0^1 || p_0^1) & D(p_0^1 || p_1^1) \\ D(p_0^2 || p_0^2) & D(p_0^2 || p_1^2) \end{pmatrix} = \text{SNR} \begin{pmatrix} 1 & \epsilon^2 \\ \epsilon^2 & 1 \end{pmatrix}.$$  

(10)

and

$$d_0^* = \text{SNR} \cdot \max_{\lambda \in [0, 1]} \min \left\{ \lambda + (1 - \lambda) \epsilon^2, \lambda^2 + (1 - \lambda) \right\}.$$  

(11)

It is easily seen that $\Delta_0(\lambda)$ appearing in (11) attains its maximum at $\lambda = 1/2$, implying that the unique optimal control PMF for $H_0$ is a purely random coin flipping between the two available controls. This yields: $d_0^* = \text{SNR} (1 + \epsilon^2)/2$.

Under $H_1$, the four divergences that come into scene, organized as a matrix, are

$$D_1 = \begin{pmatrix} D(p_1^1 || p_0^1) & D(p_1^1 || p_1^1) \\ D(p_1^2 || p_0^2) & D(p_1^2 || p_1^2) \end{pmatrix} = \text{SNR} \begin{pmatrix} 1 & (1 - \epsilon^2) \\ (1 - \epsilon^2) & 1 \end{pmatrix}.$$  

(12)

From (9) we now get

$$d_1^* = \text{SNR} \cdot \max_{\lambda \in [0, 1]} \min \left\{ \lambda + (1 - \lambda) \epsilon^2, (1 - \epsilon^2) \right\},$$  

(13)

and $\Delta_1(\lambda)$ is maximum for all $\lambda \geq \frac{1}{2 + \epsilon^2}$ ($\epsilon < 1/2$). Therefore one particular optimal control is deterministic, i.e., $\lambda = q(1) = 1$, and using any optimal control yields $d_1^* = \text{SNR} (1 - \epsilon^2)$.

Under $H_2$ we get an expression similar to (13) with $\lambda$ replaced by $1 - \lambda$. Therefore, one optimal control is $\lambda = 0$, namely $q(2) = 1$, and $d_2^* = d_1^*$.

Using (3), the expected number of samples needed to terminate the test, in the limit of vanishing risks, can be written in very simple closed form as illustrated in Table I.

In summary, one optimal solution for the control PMF is $q_0^u = (1/2 1/2)^T$, $q_1^u = (1 0)^T$, $q_2^u = (0 1)^T$, namely, under $H_0$ the best PMF samples uniformly among the available controls, while under $H_1$ or $H_2$ the best PMF chooses deterministically the control matching the hypothesis.

We now compare the performance of the Chernoff detector, obtained with an active strategy in which the radar learns from the past and adapts to the surrounding environment, with those of a static radar system, namely one in which only one probing
signal is available. With no loss of generality, we assume that the single probing signal is matched to hypothesis \( H_1 \). For further comparison, we also consider a different active hypothesis testing, using a less clever probing signal selection. In this case, taking the approach to one extreme, we assume that the probing signals are chosen uniformly at random from the set \( M_0 \); this is referred to as the blind signal selection. The detection performance for these cases can be derived by calculations similar to those illustrated for the Chernoff detector. We report the final results in Table II.

Let \( G_i^{sta} \) be the gain, under \( H_i \), in terms of number of expected samples of our active system over its static counterpart, namely the ratio between \( E_i[N] \) pertaining to the static system and that pertaining to our active hypothesis testing. Similarly, let \( G_i^{bld} \) be the gain of the considered detector over that using a blind signal selection. We have the following asymptotic analytical relationships:

\[
\begin{align*}
G_0^{sta} &\sim \frac{1 + \epsilon^2}{2\epsilon^2}, \quad G_1^{sta} \sim 1, \quad G_2^{sta} \sim \frac{(1 - \epsilon)^2}{\epsilon^2}, \\
G_0^{bld} &\sim 1, \quad G_1^{bld} \sim G_2^{bld} \sim 2 \frac{(1 - \epsilon^2)}{1 + \epsilon^2}.
\end{align*}
\]

Equations (14) quantify in a very simple way the advantage of adopting the signal selection suggested by Chernoff’s result (3).

Some final comments are now in order. First, note that the gain \( G_0^{sta,2} \) can be very large (inversely proportional to \( \epsilon^2 \)). On the other hand \( G_1^{sta} \sim 1 \) because we have assumed that the static system selects always the probing signal matched to hypothesis \( H_1 \), meaning that the static detector acts as a clairvoyant under \( H_1 \), since the control is always adapted to the true hypothesis. The active Chernoff test, instead, must learn from the measurements what is the most appropriate probing signal, and the fact that \( G_1^{sta} \sim 1 \) confirms that, asymptotically, the length of the learning stage is negligible in terms of average number of steps. Clearly, assuming \( u = 1 \) for the static detector is paid in terms of huge losses under the other hypotheses \( H_{0,2} \).

The gain \( G_1^{bld} \) reveals that a clever choice of the probing sequence can halve the number of steps of the sequential procedure. As before, \( G_0^{bld} \sim 1 \) is obvious because the blind signal selection is optimal under \( H_0 \).

IV. STRONG-OR-WEAK MODEL: GENERAL CASE

We now consider the performance of the Chernoff detector under more general observation models. The Kullback-Leibler divergence is a non-symmetric measure of difference between statistical distributions. We formulate the strong-or-weak model in terms of this divergence. In order to simplify the notation, when it is obvious from the context that we are referring to hypothesis \( H_i \), we introduce the shorthand:

\[
d_{u,k} := D(p_u^n || p_k^n),
\]

and collect the \( d_{u,k} \)’s into an \( m \times m \) matrix \( D \). In addition, we use boldface symbols to denote the “larger” divergences. Exploiting this notation, for \( i = 0 \) we have:

\[
D_0 = \begin{pmatrix}
  d_{1,1} & d_{1,2} & \ldots & d_{1,m} \\
  d_{2,1} & d_{2,2} & \ldots & d_{2,m} \\
  \vdots & \vdots & \ddots & \vdots \\
  d_{m,1} & d_{m,2} & \ldots & d_{m,m}
\end{pmatrix}
\]

where both the row and the column indices run over \( M_0 \). We say that this matrix is “almost diagonal”, in the sense that the values on the main diagonal are much larger than the off-diagonal entries.

Similarly, for \( i \in M_0 \), we have

\[
D_i = \begin{pmatrix}
  d_{1,0} & d_{1,1} & d_{1,2} & \ldots & d_{1,i-1} & d_{1,i+1} & \ldots & d_{1,m} \\
  d_{2,0} & d_{2,1} & d_{2,2} & \ldots & d_{2,i-1} & d_{2,i+1} & \ldots & d_{2,m} \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  d_{i-1,0} & d_{i-1,1} & d_{i-1,2} & \ldots & d_{i-1,i-1} & d_{i-1,i+1} & \ldots & d_{i-1,m} \\
  d_{i,0} & d_{i,1} & d_{i,2} & \ldots & d_{i,i-1} & d_{i,i+1} & \ldots & d_{i,m} \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  d_{i+1,0} & d_{i+1,1} & d_{i+1,2} & \ldots & d_{i+1,i-1} & d_{i+1,i+1} & \ldots & d_{i+1,m} \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  d_{m,0} & d_{m,1} & d_{m,2} & \ldots & d_{m,i-1} & d_{m,i+1} & \ldots & d_{m,m}
\end{pmatrix}
\]

where the row index runs over \( M_0 \) and the column index runs over \( M_i \). Thus, the rows are indexed as usual from 1 to \( m \), and the columns are indexed from 0 to \( m \) excluding \( i \). We see that the “large” entries in this matrix are those on the \( i \)-th row, and those with equal subindices (that does not mean necessarily that the entry lies on the main diagonal).

Key to the strong-or-weak model in this case is that all the strong divergences in (16) and (17) are approximately equal. As before, this can be easily obtained in the signal design stage by appropriate normalization of the matched returns.

We are now ready to solve problem (4), which can be rewritten as

\[
\max \sum_{q \in \mathcal{Q}_0} \sum_{u \in M_0} q(u) d_{u,j}
\]

subject to \( \sum_{u \in M_0} q(u) = 1 \) \( \sum_{u \in M_0} q(u) \geq 0 \) \( u \in M_0 \),
fictitious variable \( v \in \mathbb{R} \). It follows that the optimization in (18) is equivalent to [26]:

\[
\max_{v, q} v \quad \text{subject to} \quad \begin{align*}
& v \leq \sum_{u \in M_0} q(u) d_{u,0} \\
& v \leq \sum_{u \in M_0} q(u) d_{u,1} \\
& \vdots \\
& v \leq \sum_{u \in M_0} q(u) d_{u,m} \\
& \sum_{u \in M_0} q(u) = 1 \\
& q(u) \geq 0, \quad u \in M_0.
\end{align*}
\]

This is a linear program and therefore it can be simply solved by, e.g., the simplex algorithm, thus yielding the sought \( q_i^* \) and the corresponding

\[
d_i^* = \max_{q \in Q_0} \min_{u \in M_0} \sum_{u \in M_0} q(u) d_{u,j}.
\]

We summarize this result in the following:

**Proposition 1:** Let the hypothesis \( H_i, i \in M \), and the corresponding set of divergences \( \{d_{u,j}\} \) be given. Then a solution \( q_i^* \) to problem (18) and the associated maximum value \( d_i^* \) can be found by solving the linear program (19). 

\[ d_i^* = \max_{q \in Q_0} \min_{u \in M_0} \sum_{u \in M_0} q(u) d_{u,j}. \]

A. Analytical Solutions for Certain Cases

While Proposition 1 provides an algorithmic solution to problem (18), the specific structure of the matrices (16) and (17) suggests alternative approaches that can be easier to implement and can provide additional physical insight on the solution. Exploiting this structure, accurate approximations in simple closed analytical form can be derived for cases of practical interest.

We start by stating a well-known result from duality theory, see e.g., [27]. Let \( w \in \mathbb{R} \). First, the dual problem

\[
\min_{w, g} w \quad \text{subject to} \quad \begin{align*}
& w \geq \sum_{j \in M_1} g(j) d_{1,j} \\
& w \geq \sum_{j \in M_1} g(j) d_{2,j} \\
& \vdots \\
& w \geq \sum_{j \in M_1} g(j) d_{m,j} \\
& \sum_{j \in M_1} g(j) = 1 \\
& g(j) \geq 0, \quad j \in M_1,
\end{align*}
\]

can be associated to the optimization problem (19). Note that the constraints in (21) imply \( g \in Q \). Second, the original and the dual optimization problems have the same solution \( d_i^* \):

\[
d_i^* = \max_{q \in Q_0} \min_{u \in M_0} \sum_{u \in M_0} q(u) d_{u,j} = \min_{g \in Q} \max_{u \in M_0} \sum_{j \in M_1} g(j) d_{u,j}.
\]

Moreover, the following general result holds [26]:

**Proposition 2:** (Solution by matrix inversion.) Let the hypothesis \( H_i \) and the corresponding set of divergences \( \{d_{u,j}\} \) be given. Let us organize these divergences in a matrix \( D_i \), as done in (16) for \( i = 0 \). Suppose that:

- \( D_i \) is not singular,
- \( 1^T D_i^{-1} 1 \neq 0 \),
- \( q_{i,T} = \frac{1^T D_i^{-1} 1}{1^T \frac{1}{D_i^{-1}} 1} \) has no negative entries,
- \( g_i^* = \frac{D_i^{-1}}{1^T \frac{1}{D_i^{-1}} 1} \) has no negative entries.

Then, \( q_i^* \) is a solution to problem (18), and the corresponding maximum is \( d_i^* = 1/ (1^T D_i^{-1} 1) \).

Proposition 2 is useful in the case where the hypothesis believed to be in force is \( H_0 \), namely to find \( q_0^* \), because the required assumptions are typically met when the off-diagonal entries of \( D_0 \) in (16) are negligible compared to the ones over the main diagonal, i.e., when \( d_{0,k} \gg d_{u,k}, u \neq k \). Indeed, in this case the inverse matrix \( D_0^{-1} \) is almost diagonal as well, with the main-diagonal entries that will approach the inverse of the main-diagonal entries of \( D_0 \). This implies that both vectors \( 1^T \frac{1}{D_i^{-1}} 1 \) and \( 1^T \frac{1}{D_0^{-1}} 1 \) tend to have all strictly positive entries. Therefore, the assumptions in Proposition 2 are as more likely to be met as smaller are the divergences \( d_{u,k} \) with respect to the divergences \( d_{0,k} \).

The matrix \( D_0 \) cannot be exactly diagonal as long as all the divergences are strictly positive. Nevertheless it is useful, for the sake of comparison, to write the solution to problem (18) when the matrix has positive entries on the main diagonal, and zeros elsewhere. In this case Proposition 2 yields:

\[
d_0^* = \left( \sum_{u \in M_0} \frac{1}{d_{u,k}} \right)^{-1}, \quad q_0^*(u) = \frac{d_0^*}{d_{u,u}}.
\]

and it also turns out that \( q_0^* = q_0^* \). If we further assume that all the strong divergences \( d_{u,k} \) are equal, then we get \( q_0^* = 1/m \).

Typically, under \( H_i \) with \( i \in M_0 \). Proposition 2 cannot be invoked since the assumptions are not met. To find closed-form solutions in this case we then proceeds as follows. First, from the obvious inequalities \( \min_{j \in M_1} d_{u,j} \leq d_{u,j} \leq \max_{u \in M_0} d_{u,j} \), it can be easily shown that

\[
\max \min_{u \in M_0} d_{u,j} \leq d_i^* \leq \min \max_{u \in M_0} d_{u,j}.
\]

Second, for the considered strong-or-weak model, accounting for the structure of matrix \( D_i \) in (17), suppose that we find an entry \( d_i^* \) that is the maximum over its column and the minimum over its row. Such entry not necessarily exists, but if it does then it is called saddle point and it verifies [26], [27]:

\[
d_i^* = \max_{u \in M_0} \min_{j \in M_1} d_{u,j} = \min_{u \in M_0} \max_{j \in M_1} d_{u,j}.
\]

In this case, the minimax value \( d_i^* \) is an optimal solution. These considerations yield the following result.

**Proposition 3:** (Solution in the presence of a target.) Let the hypothesis \( H_i, i \in M \), be believed in force, and consider the matrix \( D_i \) in (17).

(a) The optimal \( d_i^* \) verifies:

\[
i \in M_0, \min_{j \in M_1} d_{j,i} \leq d_i^* \leq \min \max_{j \in M_1} \{d_{j,j}, d_{j,i}\},
\]

where, for \( j = 0 \), it is understood that \( \max \{d_{0,0}, d_{i,0}\} = d_{i,0} \), since \( d_{0,0} \) is not defined at all.
(b) If the minimum over the $i$-th row is attained at the first position, i.e., $d_{i,0} = \min_{j \in \mathcal{M}_{i}} d_{i,j}$, then
\[
d_{i}^{*} = d_{i,0} = \min_{j \in \mathcal{M}_{i}} \max_{q \in \Omega} \sum_{u \in \mathcal{M}_{0}} q(u) d_{u,j},
\]
and the optimizing $q_{i}^{*}$ is the degenerate PMF
\[
q_{i}^{*}(u) = \begin{cases} 1 & \text{if } u = i, \\ 0 & \text{else}. \end{cases}
\]
(27)

(c) If the minimum over the $i$-th row is attained at the column indexed by $k$, $k > 0$, i.e., $d_{i,k} = \min_{j \in \mathcal{M}_{i}} d_{i,j}$, and if $d_{i,k} \geq d_{i,k}$, then
\[
d_{i}^{*} = d_{i,k} = \min_{j \in \mathcal{M}_{i}} \max_{q \in \Omega} \sum_{u \in \mathcal{M}_{0}} q(u) d_{u,j},
\]
and the optimizing PMF $q_{i}^{*}$ is again that in (28).

Proposition 3 can be used to obtain approximate solutions to the signal selection problem even if neither condition in (b) or (c) is verified. Indeed, the performance achieved with the deterministic rule (28) is $\min_{i,j \in \mathcal{M}} d_{i,j}$. However, part (a) of Proposition 3 states that the optimal performance $d_{i}^{*}$ cannot exceed $\min_{j \in \mathcal{M}_{i}} \max_{q \in \Omega} \{d_{i,j}^{*}, d_{j,i}^{*}\}$. Since all the “strong” divergences in matrix (17) are of the same order of magnitude, $\min_{j \in \mathcal{M}_{i}} \max_{q \in \Omega} \{d_{i,j}^{*}, d_{j,i}^{*}\}$ is close to $\min_{j \in \mathcal{M}_{i}} d_{i,j}$, and therefore $\min_{i,j \in \mathcal{M}} d_{i,j} \approx d_{i}^{*}$.

It is worth mentioning that the results of the above propositions can be obtained from a game-theoretic perspective, by formulating the optimization problem as a two-person zero-sum game. This viewpoint is developed in Appendix B.

Taked together, the results in Propositions 2 and 3 suggest the following rule-of-thumb signal selection strategy:
\[
q_{0}^{*}(u) = \frac{1}{m}, \quad \forall u,
\]
\[
q_{i}^{*}(u) = \begin{cases} 1 & \text{if } u = i, \\ 0 & \text{else}, \end{cases} \quad i \in \mathcal{M}_{0}.
\]

(30)
(31)

Note that this strategy is also the optimal solution in the symmetric Gaussian example with $m = 2$. The strategy has a straightforward physical interpretation: when there is no signal, choosing one or another control makes little difference; in contrast, when a target is on the scene it makes sense to select the waveform specifically designed for it.

In Section III we have already examined the gains achievable with respect to the static and the blind strategies, for the particular case of a Gaussian model with $m = 2$. The availability of the closed-form PMF solution in (30) and (31) allows to extend the comparison to broader settings. For instance, in a Gaussian case with arbitrary $m$, where $\text{snr}_{i} = \text{snr}$ for all $i \in \mathcal{M}_{0}$, and $\alpha_{i}^{u} \approx \epsilon \ll 1$ for all $u \neq i$, we get
\[
d_{i}^{0} = \text{snr} \frac{1 + (m - 1)\epsilon^{2}}{m},
\]
\[
d_{i}^{*} = \text{snr} (1 - \epsilon)^{2}, \quad i \in \mathcal{M}_{0}.
\]

(32)
(33)

Consider now the gain over the static and over the blind signal selection. The derivation of the expected number under the different hypotheses for the static and the blind systems is not detailed for the sake of brevity. We give the final result:
\[
G_{0}^{\text{sta}} \approx \frac{1 + (m - 1)\epsilon^{2}}{m \epsilon^{2}}, \quad G_{1}^{\text{sta}} \approx 1, \quad G_{i}^{\text{sta}} \gg 1, \quad i > 1,
\]
\[
G_{i}^{\text{blind}} \approx 1, \quad G_{i}^{\text{blind}} \approx \frac{m(1-\epsilon)^{2}}{\epsilon^{2}(m-1)\epsilon}, \quad m, \quad i \in \mathcal{M}_{0}.
\]

V. ANALYTICAL APPROXIMATIONS OF SYSTEM PERFORMANCE

For the Chernoff detector, from (3), we have
\[
E_{i}[N] \approx \min_{j \in \mathcal{M}_{i}} \frac{-\log R_{i}}{\max_{u \in \mathcal{M}_{0}} q_{i}^{*}(u) D(p_{i}^{u}||p_{j}^{u})}
\]
and the approximation is valid in the limit of the risks $R_{i}$’s tending to zero. In some case, the accuracy of this first-order result is limited and improved analytical expressions are needed. We now derive useful analytical approximations by assuming that the true hypothesis has been correctly estimated, i.e., in (5) the estimate $\hat{n}_{i}$ can be replaced by the index $i$ of the actual hypothesis. Then, from (5) we have
\[
\log \frac{p_{i}(y^{n}, u^{n})}{\max_{j \in \mathcal{M}_{i}} p_{j}(y^{n}, u^{n})} = \min_{j \in \mathcal{M}_{i}} L_{i,j}(y^{n}),
\]
where
\[
L_{i,j}(y^{n}) = \sum_{k=1}^{n} \log \frac{p_{i}^{u_{k}}(y_{k})}{p_{j}^{u_{k}}(y_{k})}.
\]

Using the above assumptions, we derive approximations for the system performance, considering separately the case in which the true hypothesis is $\mathcal{H}_{i}$, $i > 0$, and the case of $\mathcal{H}_{0}$.

A. Approximations in the Presence of Targets

Let us consider first the hypothesis $\mathcal{H}_{i}$, $i > 0$, and consider the Chernoff detector using the rule-of-thumb signal selection, as described in the previous section. Then, with the assumption that the hypothesis has been correctly estimated, the average stopping time for the Chernoff detector reduces to the average stopping time of the multifihypothesis sequential probability ratio test studied in [7], [8]. In particular, using tools from nonlinear renewal theory [6], [28]–[30], in [8] accurate asymptotic expressions—up to a vanishing term— for the average stopping time $E_{i}[N]$ have been obtained, by taking into account the overshoot $\Omega$, which is a random variable defined as the excess over the boundary of the decision statistics:
\[
\Omega := \log \frac{p_{i}(y^{n}, u^{n})}{\max_{j \in \mathcal{M}_{i}} p_{j}(y^{n}, u^{n})} - \gamma.
\]

Assuming that the minimum divergence $\min_{j \in \mathcal{M}_{i}} D(p_{i}^{u}||p_{j}^{u})$ is unique for all $i > 0$, in [8, eq. (3.15)] it is proved that
\[
E_{i}[N] = \frac{\gamma + E_{i}[\Omega]}{\min_{j \in \mathcal{M}_{i}} D(p_{i}^{u}||p_{j}^{u})} + o(1),
\]

2The exact equality $\alpha_{i}^{u} = \epsilon$ would be hardly encountered in practice and, indeed, when $m \geq 2$, it would cause some pairs of hypotheses to be indistinguishable, i.e., $D(p_{i}^{u}||p_{j}^{u}) = 0$. Thus, replacing all the $\alpha_{i}^{u}$ with $\epsilon$ in the calculations is only used to provide approximate closed-form performance figures.

3Lemma 1 in [1] states that, in the limit of vanishing risks, the estimated hypothesis tends to be equal to the true one for all but the first “few” observations, provided that the stopping time is disregarded and the process continues ad infinitum.
where \( \lim_{y \to \infty} o(1) = 0 \). Precisely, let \( j_s = \arg \min_{j \in \mathcal{M}} D(p_j' || p_j') \), we have

\[
\Omega = \sum_{k=1}^{N_{js}} \log \frac{p_j'(y_k)}{p_j(y_k)} - \gamma, \tag{39}
\]

where \( N_{js} = \min\{n \geq 1 : \sum_{k=1}^{n} \log \frac{p_j'(y_k)}{p_j(y_k)} \geq \gamma \} \). It is seen form (39) that the overshoot \( \Omega \) depends on the hypothesis \( \mathcal{H}_i \), but for easier notation we omit to make explicit this dependence.

According to (38), a simple approximation to compute the average sample number of the Chernoff detector under \( \mathcal{H}_i \), \( i > 0 \), is therefore

\[
\mathbb{E}_s[N] \approx \frac{\gamma + \mathbb{E}_s[\Omega]}{\min_{j \in \mathcal{M}} D(p_j'||p_j'),} \tag{40}
\]

whose computation boils down to that of \( \mathbb{E}_s[\Omega] \). This is the average stopping time of a single random walk with positive drift for which simple exact asymptotic results can be used, see Appendix A. Equation (40), with \( \mathbb{E}_s[\Omega] \) computed by means of (61) in Appendix A, provides the sought approximation.

We now give some informal arguments that support the result in (40). Let us consider the \( m \) random walk processes (log-likelihoods) \( L_i, j(y^n), j \in \mathcal{M} \), appearing in (36), and note that \( \mathbb{E}_s[L_i, j(y)] \) is a divergence and is therefore strictly positive by assumption. Now, since \( N \) is the minimum index \( n \) at which all the \( m \) log-likelihoods lie above \( \gamma \), a reasonable assumption is that when the “slowest” process first crosses \( \gamma \), all the other random walks also lie above \( \gamma \). Under this assumption, the stopping time \( N \) can be then approximated as the minimum index at which the slowest random walk \( L_i, j(y^n) \), with \( j_s = \arg \min E_s[L_i, j(y)] \), crosses \( \gamma \). In other words, let \( N_j \) be the stopping time of the random walk \( L_{i_s,j}(y^n) \) with respect to the threshold \( \gamma \). We have the two approximations

\[
N \approx \max\{N_j, j \in \mathcal{M}\} \approx N_{js}. \tag{41}
\]

This reduces the computation of \( \mathbb{E}_s[N] \) to that of the average stopping time \( \mathbb{E}_s[N_{js}] \) of a single random walk with positive drift, and (40) follows.

B. Approximations in the Absence of Targets

Let us now consider the case in which the true hypothesis is \( \mathcal{H}_0 \). In this case, the second approximation in (41) cannot be used. Indeed, consider the symmetric scenario in which, with reference to matrix \( D_0 \) in (17), all the entries over the main diagonal have one and the same value, and all the off-diagonal entries have one and the same value. Then, under \( \mathcal{H}_0 \), the steps of the \( m \) individual random walks have all the same distribution, and therefore they evolve similarly: all the stopping times \( N_j, j \in \mathcal{M}_0 \), have the same distribution and a process substantially slower than the others does not exist. In other words, with reference to (41), we can still approximate \( N \approx \max\{N_j, j \in \mathcal{M}_0\} \), where \( N_j \) is the stopping time of the random walk \( L_{0,j}(y^n) \), but the further approximation \( N \approx N_{js} \) for some \( j_s \) is not valid anymore, since the maximum now matters. It is worth noting that the general results presented in [8] would allow to address also this more complex scenario. However, we now show that the peculiar features of the strong-or-weak model allow to derive simple and insightful approximations.

First, using the rule-of-thumb signal selection under \( i = 0 \), \( u \) takes on the values \( \{1, \ldots, m\} \) with equal probability \( 1/m \). Thus, the step of the generic random walk \( L_{0,j}(y^n), j \in \mathcal{M}_0 \), takes on the values

\[
\log \frac{p_0^0(y)}{p_j(y)}, \log \frac{p_0^1(y)}{p_j(y)}, \ldots, \log \frac{p_0^m(y)}{p_j(y)}, \tag{42}
\]

with probability \( 1/m \). Complying with the small-risk assumption, let us assume that the number of samples collected by the detector is large enough. The fraction of samples for which the control is \( u = l \), for each \( l = 1, \ldots, m \), is close to \( 1/m \); let us assume that this fraction is exactly \( 1/m \), and that the sequence of controls is \( u_1 = 1, u_2 = 2, \ldots, u_m = m, u_{m+1} = 1, u_{m+2} = 2, \ldots \), and so forth in a deterministic round-robin fashion. This basically amounts to neglect the randomness of the controls and to confuse the long-term arithmetic mean with the average value.

Taking to one extreme the strong-or-weak assumption, we disregard the contribution to the random walk \( L_{0,j}(y^n) \) of all the mixture components \( \log \frac{p_0^0(y)}{p_j(y)}, u \neq j \), shown in (42), having small average value, and retain only the dominant contribution \( \log \frac{p_0^0(y)}{p_j(y)} \). This assumption corresponds to subsampling the \( j \)-th random walk \( L_{0,j}(y^n) \) by a factor \( m \), retaining only the samples corresponding to the time instants where the control \( u = j \) is applied. This approximation can be applied to all processes \( L_{0,j}(y^n), j \in \mathcal{M}_0 \), and leads to \( m \) subsampled random walks. The key point, exploited below, is that each such random walk involves disjoint subsets of the data collected by the detector, so that the \( m \) subsampled random walks are mutually independent.

Denoting by \( N_j' \) the stopping time of the \( j \)-th subsampled walk, Wald’s identity [30] gives:

\[
\mathbb{E}_s[N_j'] = \frac{\gamma + \mathbb{E}_s[\Omega_j']}{d_{j,j}}, \tag{43}
\]

where the divergence \( d_{j,j} \) pertains to the random walk with step \( \log \frac{p_0^0(y)}{p_j(y)} \), and \( \Omega_j' \) is the corresponding overshoot.

Let \( N' = \max\{N_j', j \in \mathcal{M}_0\} \). Given the independence and the identical distribution of the \( N_j', j \in \mathcal{M}_0 \), and denoting by \( G(x) \) their common CDF, it turns out that the CDF of \( N' \) is \( G^m(x) \). Since the expectation of a nonnegative random variable can be computed by integrating one minus its CDF function, we also have \( \mathbb{E}_s[N'] = \int_0^\infty [1 - G^m(x)] \, dx \).

For any \( j \in \mathcal{M}_0 \), consider now the normalized random variable \( N_j'/\mathbb{E}_s[N_j'] \) and denote by \( \mathcal{W}(x) \) its CDF. The relationship between \( \mathcal{W}(x) \) and \( G(x) \) is simply found: \( \mathcal{W}(x) = G(x \mathbb{E}_s[N_j']) \). Thus, by exploiting the relationship \( 1 - x^m = 1 - x + (1 - x) \sum_{k=1}^{m-1} x^k \), straightforward algebra yields

\[
\mathbb{E}_s[N'] = \int_0^\infty [1 - G^m(x)] \, dx
\]

\[= \mathbb{E}_s[N_j'] \left( 1 + \int_0^\infty [1 - \mathcal{W}(x)] \sum_{k=1}^{m-1} \mathcal{W}(x) \, dx \right). \tag{44}\]
It remains to compute the CDF of the random variable $N'_i/E_0[N_i]$. We argue that the distribution of the normalized random variable $N'_i/E_0[N_i]$ is the Wald’s distribution (also known as inverse Gaussian) [31], [32]. This follows from taking the following exact asymptotic result as an approximation, see also [33] for similar arguments.

Wald’s CDF [31, Chap. 15]. Let $S_0 = 0$ and, for $n \geq 1$, $S_n = \sum_{k=1}^{n} X_k$, where the $X_k$’s are iid. Let $M = \min\{n \geq 1: S_n \geq \gamma\}$. Suppose that $0 < E[X_1] < \infty$ and $\text{VAR}[X_1] < \infty$. In the limit $E[X_1] \to 0$, $\text{VAR}[X_1] \to 0$ with the ratio $E[X_1]/\text{VAR}[X_1]$ held constant, it turns out that

$$\lim_{E[M] \to \infty} \mathbb{P}\left( \frac{M}{E[M]} \leq x \right) = W(x),$$

(45)

where

$$W(x) = 1 - Q\left(\frac{x-1}{\sqrt{x}}\right) + e^{x\varphi} Q\left(\frac{x+1}{\sqrt{x}}\right)$$

(46)

is called Wald’s distribution, and the parameter involved is

$$\varphi = \frac{E[X_1]}{\text{VAR}[X_1]} \gamma.$$  

(47)

By identifying $N'_i$ with $M$, so that the parameter $\varphi$ involves the mean and the variance of $\log[p_i(y)/p_i'(y)]$, we end up with an approximate formula to compute $E_0[N_i]$ by using expressions (46) and (43) in (44).

Computing $E_0[N_i] = E_0[\max\{N_j, \ j \in \mathcal{M}_0\}]$ is now a simple task. Since $N'_i$ is the stopping time of the original random walk undersampled by a factor $m$ according to the described round-robin procedure, in the asymptotic regime of vanishingly small risks, we can safely assume $N_j \approx mN'_j$, whence $E_0[N_i] \approx m E_0[N'_i]$, yielding:

$$E_0[N_i] \approx \gamma + E_0[\Omega_i] + \frac{1}{m} \sum_{u=1}^{m} \mathbb{W}_k(x)dx.$$  

(48)

In Section VI we discuss how to exploit the above equation to assess the performance of the radar system under the strong-or-weak echo model.

C. Approximations for the Blind and the Static Rules

So far, we have considered the Chernoff detector with rule-of-thumb signal selection. Approximations for the blind and the static signal selection can be similarly found.

For the blind detector, under $\mathcal{H}_0$, the situation is identical to that of the Chernoff case with rule-of-thumb selection. Under $\mathcal{H}_i$, $i > 0$, we must focus on the quantity $(1/m) \sum_{u=1}^{m} d_{u,i}$. Inspection of matrix (17) reveals that, under the assumption that all the strong divergences are approximately equal:

$$j_s = \arg \min_{j \in \mathcal{M}} \frac{1}{m} \sum_{u=1}^{m} d_{u,j} = 0,$$

(49)

since the sum over the columns is minimum for the first column. Accordingly, there exists a slower process that determines the stopping time, i.e., $N \approx N_0$, see (41), now with

$$E_i[N] \approx \frac{\gamma + E_i[\Omega]}{m} \approx \frac{\gamma + E_i[\Omega]}{d_{i,0}/m},$$

(50)

where in the last step we have used the fact that $d_{i,0}$ is the dominant divergence. In the previous formula the overshoot distribution can be approximated as the one corresponding to the dominant component of the random walk $L_{i,0}(y')$, namely, $E_i[\Omega]$ can be computed by applying (61) in Appendix A to the random walk with step $\log[p_i'(y)/p_i(y')]$.

Consider now the static signal selection in which $u = 1$. Here, for all $\mathcal{H}_i$, $i > 0$, we have the approximation $N \approx N_i$, see (41), where, obviously: $j_s = \arg \min_j D(p_i(y)/p_i'(y))$. This implies

$$E_i[N] \approx \frac{\gamma + E_i[\Omega]}{\min_{j \in \mathcal{M}} D(p_i(y)/p_i'(y))},$$

(51)

where the computation of the expected overshoot can be made using eq. (61), for a random walk with step $\log[p_i'(y)/p_i(y')]$.

VI. THEORETICAL APPROXIMATIONS VERSUS NUMERICAL SIMULATIONS

We now present a comparison between the theoretical approximations and computer experiments based on a standard Monte Carlo counting approach. We consider both the Gaussian model, as well as another popular model of radar returns: the exponential shift-in-scale.

A. The Exponential Strong-or-Weak Model

The details of the exponential model, under a strong-or-weak assumption, are as follows: (i) Under $\mathcal{H}_0$, radar returns are exponential random variables with unit rate, regardless of the control; (ii) the radar return under $\mathcal{H}_i$, $i \in \mathcal{M}_0$, using the probing signal matched to that hypothesis, is an exponential with rate $1/(1 + \text{SNR})$, and (iii) the radar return under $\mathcal{H}_i, i \in \mathcal{M}_0$, probed by any signal other than that matched to $\mathcal{H}_i$, follows an exponential distribution with rate $1/(1 + \text{SNR})$. This means that for $i \in \mathcal{M}, u \in \mathcal{M}_0$, and $y \geq 0$,

$$p_i^u(y) = \frac{e^{-y/(1+\text{SNR})}}{1 + \alpha^u \text{SNR}},$$

(52)

with

$$\begin{cases} 
\alpha^u_0 = 0, \\
\alpha^u_1 = 1, \ u = i, \\
\alpha^u_i \ll 1, \ u \neq i.
\end{cases}$$

In the numerical experiments that follow we set $\text{SNR} = \text{SNR}$, for all $i \in \mathcal{M}_0$.  

B. Overshoot Evaluation and Computation of $\varphi$

To exploit the analytical approximations developed in Section V, the computation of the series in (61) of Appendix A for the cases of interest is needed. Also, under $\mathcal{H}_0$, the parameter $\varphi$ appearing in (46) must be determined.

1) Presence of Targets: With reference to the random walk $S_n = \sum_{k=1}^{n} X_k$ in Appendix A, assuming $i > 0$ and the rule-of-thumb signal selection $u = i$, we have $X_k = \log[p_i'(y_k)/p_i(y_k)]$. Let $j_s = \arg \min_j d_{j,i}$ be the index of the minimum of the $i$-th row of matrix (17), and $E_i[X_k] = d_{j,i}$. Let $d = d_{i,j}$, by disregarding the subscripts for simplicity of notation, and consider the Gaussian example first. Under the
Gaussian model the process \( S_n \) is also Gaussian, and simple computation of (61) yields (see also [6, Examples 3.1 and 6.4]):

\[
E_n[\Omega] \to \frac{d}{2} - \sqrt{\frac{2d}{k}} \times \sum_{k=1}^{\infty} \left[ f \left( \sqrt{\frac{d}{k}} \right) - \sqrt{\frac{d}{k}} F \left( \sqrt{\frac{d}{k}} \right) \right],
\]

where \( f(\cdot) \) and \( F(\cdot) \) represent the PDF and the cumulative distribution function (CDF) of a standard Gaussian random variable.

Under the exponential model, we have \( d = d_{i,j} = D(p_i || p_j) \) where the two distributions are exponentials with different rates, say \( \gamma \) and \( \gamma_0 \), respectively, and \( \gamma < \gamma_0 \). The step of the random walk is \( X_k = -\log \frac{\alpha}{1 + \alpha} + (\gamma_0 - \gamma) y_k \), where \( y_k \) is exponentially distributed with rate \( \gamma_0 \), implying that

\[
P_i(X_k > z) = C \exp\{-z\beta\}, \quad \text{for } z \geq 0,
\]

with \( C = \exp\{-r_1 \log(r_0/r_1)\} \) and \( \beta = \frac{r_1 - r_0}{r_1} \). The exponential right tail property (54) of \( X_k \) can be used to show that the overshoot \( \Omega \) is exponentially distributed with rate \( \beta \), see [6, Example 2.2, p. 19], yielding

\[
E_n[\Omega] = \frac{r_0 - r_1}{r_1} = \frac{\text{snr}}{1 + \alpha_i} + \frac{1}{1 + \alpha_i},
\]

Thus, under hypothesis \( H_i, i > 0 \), we have the theoretical approximation (40), where \( E_n[\Omega] \) is computed by the RHS of (53) for the Gaussian model, and by (55) in the exponential case.

2) Absence of Targets: Under hypothesis \( H_0 \), the approximate expected stopping time is given by (48) where, provided that the \( d_{i,j} \)'s are almost equal, the index \( j \in M(0 \) can be chosen arbitrarily. To compute the expected overshoot \( E_0[\Omega] \), we must consider the component with largest expected value. In the Gaussian case, it suffices to set \( d = d_{i,j} \) and apply eq. (53).

The same approach can be pursued for the exponential model: the generic step \( \log[p_i(y)/p_j(y)] \) of the random walk is now given by the log-likelihood between two exponentials of different rates, say \( r_1 = 1 \) and \( r_0 = 1/(1 + \text{snr}) \). However, since now \( r_1 > r_0 \), the exponential right tail property (54) does not hold anymore: we cannot conclude that the overshoot is exponentially distributed. Instead, we resort to the general expression (61) and, after some algebra, we find:

\[
E_0[\Omega] \to \frac{1}{2} \left[ \frac{(r_1 - r_0)^2}{r_1^2 \log \frac{r_0}{r_1} - r_1 (r_1 - r_0)} + \log \frac{r_1}{r_0} - r_1 \rho \left( 1 - F(r_1 k \rho) \right) \right] \times \frac{r_1 - r_0}{r_1} \sum_{k=1}^{\infty} \frac{\Gamma(k + 1, r_1 k \rho)}{k!} - r_1 \rho (1 - F(r_1 k \rho)) \right],
\]

where \( \rho = \log \frac{r_0}{r_1} \), \( F(\cdot) \) is the CDF of a Gamma random variable with shape parameter \( k \) and unit rate, whose PDF, for \( z \geq 0 \), is \( f(z) = z^{k-1} e^{-z} (k-1)! \), and \( \Gamma(n, r) = \int_0^\infty t^{n-1} e^{-t} dt \) is the incomplete gamma function.

Summarizing, under hypothesis \( H_0 \), we use the theoretical approximation (48), where \( E_n[\Omega] \) is computed by the RHS of (53) with \( d = d_{i,j} \) (any \( j \in M(0 \) for the Gaussian model, and by the RHS of (56) with \( r_1 = 1 \) and \( r_0 = 1/(1 + \text{snr}) \) in the exponential case.

It only remains to compute the parameter \( \varphi \) appearing in (46). This amounts to computing the average and the variance of the step \( \log[p_i(y)/p_j(y)] \) of the \( j \)-th subsampled random walk, see Section V-B. This is a very simple task: for the two examples of interest we find

\[
E_0 \left[ \log \frac{p_i(y)}{p_j(y)} \right] = \frac{1}{2} \text{VAR}_0 \left[ \log \frac{p_i(y)}{p_j(y)} \right] = \text{snr} (\text{Gauss}),
\]

\[
E_0 \left[ \log \frac{p_i(y)}{p_j(y)} \right] = \text{VAR}_0 \left[ \log \frac{p_i(y)}{p_j(y)} \right] = \log(1 + \text{snr}) - \frac{\text{snr}}{1 + \text{snr}} (\text{exponential}),
\]

and therefore \( \varphi = \gamma/2 \) for the Gaussian example, and \( \varphi = \gamma \) in the exponential case.

C. Computer Experiments

In the computer experiments that follow we assume that all the risks are equal \( R_i = R, \forall i \in M \), and take uniform priors: \( \pi(j) = 1/(m+1), j \in M \), see (2). Also, when not stated otherwise, it is assumed that the Chernoff detector employs the rule-of-thumb signal selection (30) and (31).

We fix the threshold value \( \gamma \) appearing in eq. (5), as described by [7, eq. (2.4)], see also [12, eq. (21)]: \( \gamma = -\log \frac{(m+1) R}{m} \), where \( R \) is a nominal value of the risk. Then, Monte Carlo trials are designed in order to simulate the various detectors –Chernoff, blind, static– under all the \( m+1 \) hypotheses, and in particular to compute their final decisions and the number of steps needed to cross the threshold. From these data, the empirical average stopping times and the empirical risks\(^4\) are derived.

The figures that follow report the empirical value of \( E_n[N] \) versus the empirical value of \( \Omega \). We also show the approximated theoretical values of \( E_n[N] \), obtained as discussed earlier, again as function of the empirical value of \( \Omega \).

Let us start by considering the case \( m = 2 \). Both for the Gaussian and for the exponential examples, we set \( \text{snr} = 1 \) and we take the symmetric case in which \( \alpha_1^2 = \alpha_2^2 = \epsilon = 0.1 \). We note that the assumption leading to (38) that a unique minimum divergence \( \min_{i \in M} D(p_i || p_j) \) exists, is certainly met in the case \( m = 2 \). Also, the assumption that \( \text{snr} = \text{snr} \) implies that the index \( j = 1, 2 \), in (48) is indeed arbitrary, when dealing with the \( H_0 \) hypothesis.

Figs. 1–3 refer to \( 10^6 \) Monte Carlo runs for each simulated point. As shown in the legends, C, B, and S, refer to the Chernoff, the blind, and the static detectors, respectively. The simulated points are shown as symbols (circle, + and ×), while the theoretical approximations are shown as lines.

Consider first Fig. 1. Clearly, the Chernoff and the blind detector have the same performance. In the blind strategy the signal selection is made uniformly at random from the beginning of the probing actions, while the active system learns from data and eventually selects the optimal probing strategy, which in this case is just the uniformly random one. This means that

\(^4\)Note that the formula for setting the threshold as function of the nominal risk \( R \) does not ensure that the actual risk is less than \( R \), which is a further reason why it is appropriate to evaluate it numerically.
the Chernoff detector quickly learns the true state of the nature and selects with the right probability (uniformly) the probing signals. The static detector performs worse, as expected. It is also remarkable that the proposed theoretical approximations are in excellent agreement with the simulations for all the three signal selection strategies – they are almost undistinguishable from the simulation points, except for very large values of $\bar{R}$.

Fig. 2 addresses the case in which the true hypothesis is $H_1$. Now we see that the static signal selection achieves the lowest value of $E_1[N]$, but this is due to the fact that, by assumption, the static detector always use $u = 1$ and, under $H_1$, this is the correct waveform to be used; obviously this is paid under the other hypotheses with a very large stopping time.

The Chernoff detector is only slightly slower in making the decision, i.e., the average stopping time is slightly larger. Presumably, this can be ascribed to the initial steps of the detection algorithm, as discussed earlier, when the detector has to learn about the state of the nature. In fact, it should be noted that at the first step of the detection algorithm the signal is uniformly chosen among the $m$ waveforms, which was the optimal signal selection under $H_0$, but not under the other hypotheses. Note also that the difference in performance tends to disappear at low values of risk. We also see that the theoretical approximation for the Chernoff detector is a bit less accurate with respect to Fig. 1. However, the analytical approximation is very satisfying and improves at low values of the risk. Indeed, it is evident that the simulation points and the theoretical curve approach each other as the risk decreases.

The blind signal selection causes a loss factor approximately equal to $1/2$, namely, the stopping time is twice larger with respect to the Chernoff signal selection. The reason is that, half of time, it chooses the wrong waveform $u = 2$. We stress the perfect superposition between the theoretical approximations and the simulation points for the blind and for the static cases.

In Fig. 3 the true hypothesis is $H_2$. The static signal selection performs similarly to what we have seen under hypothesis $H_0$, and is the worst. The Chernoff and the blind signal selections behave like under hypothesis $H_1$. The accuracy of the theoretical approximations remains satisfying, and becomes excellent as the risk decreases. Another important remark is that, for $m = 2$, the optimal signal selection for the Chernoff detector coincides with the rule-of-thumb strategy in (30) and (31), discussed in Section IV-A and used in the simulations.

We have considered so far a three-hypotheses radar problem. Let us now consider a case where there are $m = 5$ possible targets and therefore 6 hypotheses. Figs. 4 and 5, obtained by $10^5$ Monte Carlo runs for each point of simulation, refer to that scenario. Note that the static strategy is not shown in Figs. 4 and 5 because the average number is so large that it is difficult to obtain reliable simulations for the empirical average stopping time and for the empirical risk.

In Figs. 4 and 5, both for the Gaussian and for the exponential case we assume, as before, $\text{SNR} = 1$, and we now set $\alpha_0^*$, see (8) and (52), as shown in the following matrix

$$
\begin{pmatrix}
1 & 0.0093 & 0.0169 & 0.0311 & 0.0934 \\
0.0650 & 1 & 0.0757 & 0.0158 & 0.0447 \\
0.0972 & 0.0572 & 1 & 0.0420 & 0.0888 \\
0.0179 & 0.0879 & 0.0696 & 1 & 0.0891 \\
0.0408 & 0.0691 & 0.0415 & 0.0290 & 1
\end{pmatrix},
$$

(57)
where $\alpha_i^u$ is given by the $(i, u)$-th entry.

In Fig. 4 the hypothesis $\mathcal{H}_0$ is addressed. The blind and the Chernoff signal selection coincide and, indeed, the simulation points are very close to each other. The theoretical approximations predict in an excellent way the relationship between the risk and the average stopping time. Fig. 4 also shows the theoretical performance of the Chernoff detector obtained by using the first-order approximation (34), see the curves labeled as “C (34)”. It is particularly evident in this case that the simple approximation provided by eq. (34) is not accurate.

Note also that in the case of $m = 5$, under hypothesis $\mathcal{H}_0$, $\text{SNR} = 1$, and using the values $\alpha_i^u$ in (57), the optimal signal selection is obtained by exploiting the results in Proposition 2, and amounts to

$$q^*_0 = (1.999, 1.998, 2.005, 2.022, 1.977)^T \text{ (Gauss.),}$$

$$q^*_0 = (1.997, 1.995, 2.010, 2.051, 1.946)^T \text{ (exp.),}$$

(58)

with the corresponding optimal values given by

$$d^*_0 = .2030 \text{ (Gaussian),}$$

$$d^*_0 = .0400 \text{ (exponential).}$$

(59)

The rule-of-thumb selection strategy in this case is the uniform form $q^*_0 = (2.2, 2.2, 2.2)^T$, for which the corresponding divergences are: $d^*_0 = .2008$ for the Gaussian case and $d^*_0 = .0390$ for the exponential case. The simulations shown in Fig. 4 are obtained by using such uniform PMF. Note that the uniform PMF is only slightly different from the one in (58), so that one does not expect that using the optimal PMFs in (58) the performance would change significantly. Indeed, computer experiments (not shown here) confirm that the performance of the Chernoff detector using the optimal signal selection is practically undistinguishable from that obtained with the rule-of-thumb strategy.

Consider now Fig. 5, which shows the performance under $\mathcal{H}_1$, but it can be considered representative of all $\mathcal{H}_i$, $i > 0$, since the behavior of the curves for $i = 1, \ldots, m$, is almost identical. The theoretical approximation is excellent for the blind detector, while some discrepancy occurs for the Chernoff detector at large values of the risk. Again, it is evident that the matching improves as long as the risk decreases.

A possible explanation of this effect is as follows. Using the values of $\alpha_i^u$ in (57) the assumption of the uniqueness of the minimum divergence leading to (38) is indeed met; however, differently from the case $m = 2$, other divergences may have values very close to this minimum, so that the emergence of a substantially dominant process might appear at lower values of the risk. This is exactly what we observe in Fig. 5.

Note finally that, under $\mathcal{H}_i$, $i > 0$, in the exponential example the optimal signal selection coincides with the rule-of-thumb strategy. Not so for the Gaussian case: under $\mathcal{H}_1$, the rule-of-thumb signal selection amounts to the deterministic choice $u = 1$, yielding $d^*_1 = .8219$, while the optimal PMF with the parameters used in the example results in $q^*_1 = (.8873 \ 0 \ 0 \ 0 \ .1127)^T$, with the corresponding divergence $d^*_1 = .8330$.

Numerically, the differences are still minimal, but slightly larger than those seen under $\mathcal{H}_0$, and they might deserve further analysis. Accordingly, in Fig. 5 (Gaussian case, upper panel) we also show by square symbols (“C-opt sim.” in the legend) the results of simulations in which the Chernoff detector employs the optimal signal selection strategy in place of the rule-of-thumb one. As expected, no appreciable difference arises, which further corroborates the choice of the rule-of-thumb signal selection under the strong-or-weak observation model.

VII. SUMMARY

We studied the Chernoff test under a strong-or-weak model for the radar returns, with focus on the design of the optimal signal selection strategy and on the performance evaluation in terms of average number of samples needed to make a decision, and in terms of the related risk (probability of wrong deciding for a certain hypothesis). We first explored a Gaussian shift-in-mean model with $m = 2$ possible targets (three-hypothesis test), and derived simple closed-form formulas for the signal selection and for the detection performance. Then, we addressed the optimal signal selection problem under arbitrary radar echo distributions and arbitrary $m$. In general, the signal selection procedure, and consequently the system performance, can only be expressed in algorithmic form as the output of a linear program. Under the strong-or-weak model, however, we derived simple closed-form solutions with straightforward physical interpretation. In particular, elaborating on recent generalizations of Chernoff’s results [12], the performance analysis allows us to derive analytical expressions relating the risks $R_i$’s and the average sample number of the test, which are valid in the limit $\max_i R_i \rightarrow 0$.

The applicability of these results to real-world radar problems may be limited by the weak accuracy of the formulas for values of the risk close to those of practical interest. Accordingly, we further studied approximations exploiting results on stopped random walks and on nonlinear renewal theory. This yields simple and accurate analytical expressions, in closed form or in a form that is very easy to solve numerically. Finally, extensive computer experiments have been conducted to validate our analytical approximations, confirming their good accuracy for a wide range of parameters.

Our strong-or-weak model inherently assumes that some signal design has been performed beforehand, but this design is not specifically tailored to the Chernoff test to be implemented. One possible line of research for future studies is to incorporate the waveform design in the optimization procedure.
Appendix A
Expected Overshoot of a Random Walk With Positive Drift

We refer to [34, Th. 4.4]. Let \( S_0 = 0 \) and, for \( n \geq 1 \), \( S_n = \sum_{k=1}^{n} X_k \), where the \( X_k \)'s are iid. Let \( M = \min \{ n \geq 1 : S_n \geq \gamma \} \). Suppose that \( 0 < \mu = \mathbb{E}[X_1] < \infty \). Then, Wald’s identity (see, e.g., [30, Prop. 2.18]) yields

\[
\mathbb{E}[M] = (\gamma + \mathbb{E}[\Omega]) / \mathbb{E}[X_1],
\]

(60)

where the random variable \( \Omega := S_M - \gamma \) is called overshoot and represents the excess over the boundary. Suppose now that \( 0 < \sigma^2 = \text{VAR}[X_1] < \infty \), that the third moment \( \mathbb{E}[|X_1|^3] \) is finite, that \( X_1 \) has a density continuous a.e. with respect to the Lebesgue measure, and that some of the characteristic function of \( X_1 \) is integrable. Then, we have [34, eq. (4.5)]:

\[
\mathbb{E}[\Omega] \xrightarrow{\gamma \rightarrow \infty} \frac{\sigma^2 + \mu^2}{2\mu} - \sum_{k=1}^{\infty} \frac{1}{k} \mathbb{E}[S_k^\gamma],
\]

(61)

where \( S_k^\gamma = -\min(0, S_k^\gamma) \).

Appendix B
Two-Person Zero-Sum Game

The results of Section IV are amenable to a game-theoretic formulation, and can be formulated in terms of a two-person zero-sum game—a viewpoint also suggested in [1], [11]. It is therefore instructive to highlight how the problem of finding the optimal pair \( q^* \) and \( d^* \) can be formulated in this setting.

Let us imagine that the system designer is playing a game against an adversary: the designer selects the \( u \)-th probing signal \( u \in M_0 \), and simultaneously the adversary selects one hypothesis, say \( H_j, j \in M_1 \). The payoff for the designer is \( d_{u,j} \), and that for the adversary is \( -d_{u,j} \). The goal of each of the two players is to maximize his payoff. This game, in strategic form, can be represented by a matrix of payoffs in which the strategy of the designer is to select a row and that of the adversary is to select a column. Matrices \( D_i = \{ d_{u,j} \} \) in (16) and (17) are exactly of this kind. Now, the celebrated von Neumann minimax result on two-person zero-sum finite games states that [35]

\[
\max_u \min_{g \in \mathcal{G}} \sum_{j \in M_1} q(u,g(j)) d_{u,j} = \min_v \max_{g \in \mathcal{G}} \sum_{j \in M_1} q(u,g(j)) d_{u,j}
\]

(62)
i.e., there always exist two optimal PMFs, say \( q^*_g \) and \( g^*_q \), such that (62) holds. The quantity in (62) is referred to as the value \( d^* \) of the game. Namely, von Neumann minimax theorem, states that the game has always a solution, provided that both players are allowed to choose their strategy according to some distribution (mixed strategies), and are not constrained to make their choices deterministically (pure strategies).

Since

\[
\max_u \min_{q \in \mathcal{Q}_q} \sum_{j \in M_1} q(u,g(j)) d_{u,j} = \max_u \min_{q \in \mathcal{Q}_q} \sum_{j \in M_1} q(u) d_{u,j}
\]

we immediately see that the optimal \( q^*_g \) and the optimal \( d^* \) for the Chernoff test, can be found as the optimal solutions to the two-person zero-sum game with matrix \( D_i = \{ d_{u,j} \} \) given in (16) and (17).

Many results derived in Section IV can be revisited from this perspective. Indeed, the linear programs (19) and (21) represent one standard way to solve the game. Also, Proposition 2 can be found in the literature as a game-theoretic result, see, e.g., [26, Th. 3.2], as well as eq. (25), related to Proposition 3, is nothing else that the condition ensuring the existence of a pure saddlepoint solution of the game.

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