On Factorization of M-Channel Paraunitary Filterbanks

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***Abstract*—We systematically investigate the factorization of causal finite impulse response (FIR) paraunitary filterbanks with given filter length. Based on the singular value decomposition of the coefficient matrices of the polyphase representation, a fundamental order-one factorization form is first proposed for general paraunitary systems. Then, we develop a new structure for the design and implementation of paraunitary system based on the decomposition of Hermitian unitary matrices. Within this framework, the linear-phase filterbank and pairwise mirror-image symmetry filterbank are revisited. Their structures are special cases of the proposed general structures. Compared with the existing structures, more efficient ones that only use approximately half the number of free parameters are derived. The proposed structures are complete and minimal. Although the factorization theory with or without constraints is discussed in the framework of M-channel filterbanks, the results can be applied to wavelets and multiwavelet systems and could serve as a general theory for paraunitary systems.**

I. Introduction

In the research area of multirate filterbanks, wavelets, and multiwavelets, the paraunitary concept plays a central role [1]–[7]. The design and implementation of orthogonal systems are based on multi-input and multi-output systems, of which the polyphase transfer matrices are paraunitary. A polyphase transfer function matrix $\mathbf{E}(z)$ is paraunitary if $\mathbf{E}^{\mathrm{H}}(z^{-1})\mathbf{E}(z) = \mathbf{I}$, where the superscript H stands for the conjugated transpose, and $\mathbf{I}$ is the identity matrix [1]. A complete factorization of the paraunitary matrix with or without constraints often provides an efficient structure for optimal design and fast implementation.

In this paper, we systematically investigate the factorization of paraunitary matrices in the framework of M-channel maximally decimated FIR filterbanks. The results can be applied to M-band wavelets as well as multiwavelet systems. Fig. 1(a) shows a typical M-channel maximally decimated filterbank, where $H_k(z)$ and $F_k(z)$ are the kth analysis and synthesis subband filters. The polyphase implementation of the filterbank is illustrated in Fig. 1(b), where $\mathbf{E}(z)$ and $\mathbf{R}(z)$ are, respectively, type-I analysis polyphase matrix and type-II synthesis polyphase matrix [1], i.e.,

$$[H_0(z)\ H_1(z) \cdots H_{M-1}(z)]^{\mathrm{T}} = \mathbf{E}(z^M)\left[1\ z^{-1} \cdots z^{-(M-1)}\right]^{\mathrm{T}}$$

and

$$[F_0(z)\ F_1(z) \cdots F_{M-1}(z)] = \left[z^{-(M-1)}\ z^{-(M-2)} \cdots 1\right]\mathbf{R}(z^M).$$

In a causal FIR orthogonal filterbank with filters of length $L = MK$, $\mathbf{E}(z)$ and $\mathbf{R}(z)$ are paraunitary matrices of order $K-1$, and $\mathbf{R} = z^{-(K-1)}\mathbf{E}^{\mathrm{H}}(z^{-1})$. We use "order" to denote the highest power of z^{-1} in $\mathbf{E}(z)$ and $\mathbf{R}(z)$. The order comes directly from the filter lengths. Our objective is to develop complete and minimal factorization structures for paraunitary matrices *of given order* with or without further constraints.

Many excellent works on this topic have been reported by other researchers [8]–[19]. For general paraunitary filterbanks, Vaidyanathan *et al.* propose a complete and minimal structure in [9] and [10], which shows that any paraunitary matrix of McMillan degree N can be factorized into a product of N degree-one paraunitary building blocks and an additional unitary matrix. The McMillan degree of a multi-input and multi-output causal rational system is the minimum number of delay units (i.e., z^{-1} elements) required in implementation. With filters of length MK, the McMillan degree of the polyphase matrix may range from $(K-1)$ to $M(K-1)$. Conversely, paraunitary matrices of different orders can have the same degree. In practice, we care more about the filter length than the degree of the system. In this case, it is expected to have a factorization form for paraunitary matrices of *a given order*. This is the initial motivation of this work. With constraints such as linear phase and/or pairwise mirror-image symmetry on subband filters, complete and minimal lattice structures of the paraunitary polyphase matrices have been developed [12]–[18]. The constrained paraunitary matrix of order $K-1$ can be expressed as the product of $K-1$ order-one paraunitary building blocks and an additional unitary matrix. These factorizations provide efficient structures for implementing linear-phase and mirror-image paraunitary filterbanks with length constraint. More recently, Rault and Guillemot prove that an $M \times M$ paraunitary matrix of order $K-1$ can be factorized into a $(K-1)$-stage order-one form if the ranks of the first and last coefficient matrices are not less than $M/2$ for even M or $(M-1)/2$ for odd

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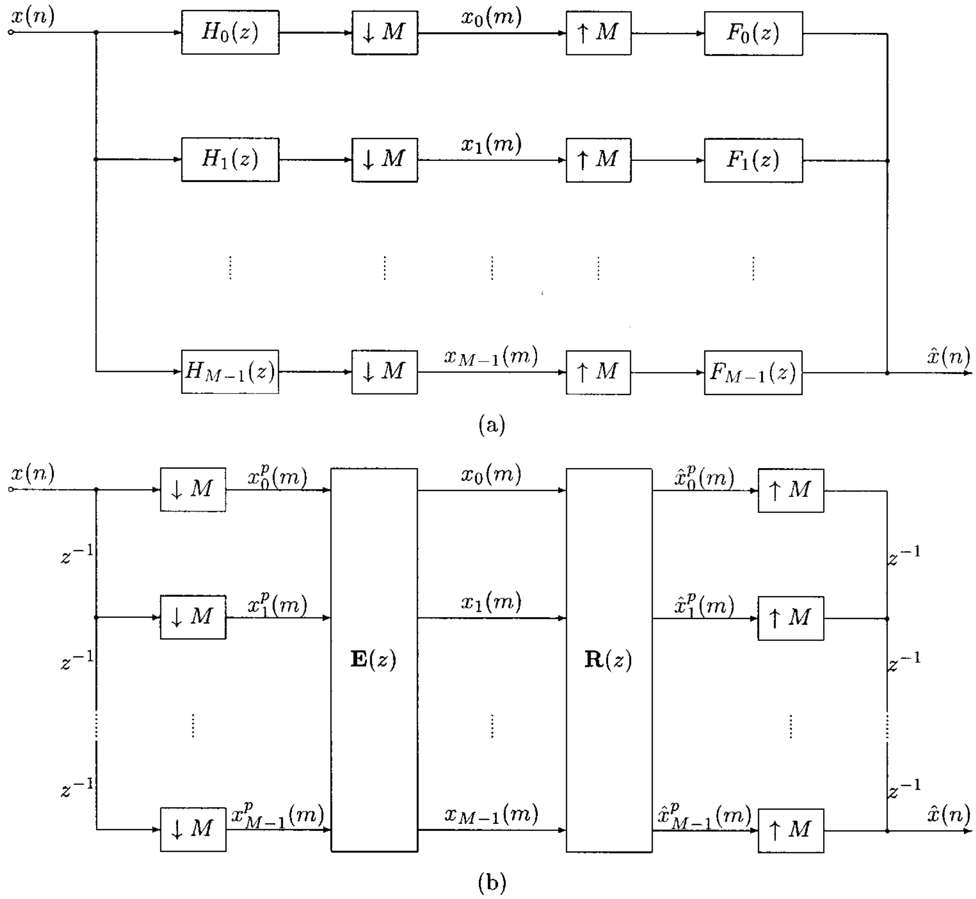

Fig. 1. M-channel maximally decimated filterbank. (a) Basic structure. (b) Polyphase implementation.

M[19]. The resulting structure has the symmetric delay property that has been found to be valuable in object-based audiovisual signal compression.

It is interesting that all the reported results on factorizing a special class of a paraunitary system take the $(K-1)$-stage order-one form. These special systems also include the two-channel filterbanks [8] and cosine-modulated paraunitary filterbanks [20]–[25]. Does a generalized factorization for paraunitary system exist? It is reasonable to handle the factorization problem of causal FIR paraunitary systems with or without constraints in one framework.

Section II shows that if the degree-one building blocks are separated into $K-1$ groups in which the parameter vectors within each group are orthogonal, then the product is paraunitary of order $K-1$, and the filter length is MK. This leads to the fundamental order-one factorization form. In this section, we also prove that the order-one factorization is complete; any causal FIR paraunitary matrix of order $K-1$ is allowed. Our approach is based on the singular value decomposition (SVD) of coefficient matrices. Notice that the SVD has also been used in the design of M-channel linear-phase biorthogonal filterbanks in [26]. The difference is that it was used to represent invertible matrices that appear in the proposed lattice structure, whereas we use it to derive the structure of a given paraunitary filterbank. We develop a more efficient structure for the design and implementation in Section III. The CS decomposition and spectral decomposition of Hermitian unitary matrices are valuable for paraunitary systems. In this framework, we revisit the linear-phase and mirror-image filterbanks in Section IV. More efficient structures than those reported are obtained, which implies more efficient design and implementation of these classes of filterbanks. Section V discusses several design examples.

Notations: Bold-faced letters indicate vectors and matrices. The tilde operation on a matrix function $\mathbf{E}(z)$ is defined by $\tilde{\mathbf{E}}(z) = \mathbf{E}^{\mathrm{H}}(z^{-1})$, where the superscript H denotes the conjugated transpose. For real coefficient systems, H is replaced by T (the transpose). The rank of the matrix $\mathbf{X}$ is $\rho(\mathbf{X})$. $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the integer floor and ceiling of x. $\mathbf{I}_N$ and $\mathbf{J}_N$ are the identity and reverse identity matrices, respectively.

II. Degree-One Factorization and Order-One Factorization

A. Brief Review of Degree-One Factorization

Consider an M-channel causal FIR filterbank with filter length $L = MK$. The polyphase matrix of the analysis bank can be expressed as

$$\mathbf{E}(z) = \sum_{k=0}^{K-1} \mathbf{E}_k z^{-k} \tag{1}$$

where $\mathbf{E}_{K-1} \neq 0$. The filterbank is paraunitary if and only if $\mathbf{E}(z)$ is paraunitary, i.e.,

$$\tilde{\mathbf{E}}(z)\mathbf{E}(z) = \mathbf{I}_M. \tag{2}$$

For $K > 1$, this implies particularly that the following equation holds:

$$\mathbf{E}_0^{\mathrm{H}}\mathbf{E}_{K-1} = \mathbf{0} \tag{3}$$

which implies that $\mathbf{E}_0$ must be singular since $\mathbf{E}_{K-1}$ is nonzero. This property is the starting point for a degree-one factorization [1], [10]. In fact, (3) contains more information on $\mathbf{E}_0$ and $\mathbf{E}_{K-1}$, which will be used to complete the factorization.

Suppose that the McMillan degree of $\mathbf{E}(z)$ is N. Since $\mathbf{E}(z)$ is paraunitary, the order of its determinant is N. Since $\mathbf{E}_0$ is singular, there exists an M-dimensional unit vector $\mathbf{v}_N$ such that $\mathbf{v}_N^{\mathrm{H}}\mathbf{E}_0 = \mathbf{0}$. Using $\mathbf{v}_N$, we can define an $M \times M$ degree-one paraunitary matrix $\mathbf{V}_N(z) = \mathbf{I}_M - \mathbf{v}_N\mathbf{v}_N^{\mathrm{H}} + z^{-1}\mathbf{v}_N\mathbf{v}_N^{\mathrm{H}}$. Then, $\tilde{\mathbf{V}}_N(z)\mathbf{E}(z)$ is a causal FIR paraunitary system of degree $N-1$. As mentioned above, the first coefficient matrix of $\tilde{\mathbf{V}}_N(z)\mathbf{E}(z)$ must be singular if $N-1>0$. Therefore, the degree reduction process can be repeated until a constant unitary matrix is obtained. Therefore, for any $M \times M$ paraunitary $\mathbf{E}(z)$ of degree N, there exist N vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N$ and a unitary matrix $\mathbf{X}_0$ so that $\mathbf{E}(z)$ can be factorized into [1]

$$\mathbf{E}(z) = \mathbf{V}_N(z)\mathbf{V}_{N-1}(z)\cdots\mathbf{V}_1(z)\mathbf{X}_0 \tag{4}$$

where $\mathbf{V}_n(z) = \mathbf{I}_M - \mathbf{v}_n\mathbf{v}_n^{\mathrm{H}} + z^{-1}\mathbf{v}_n\mathbf{v}_n^{\mathrm{H}}$. Conversely, with arbitrary unit-norm vectors $\mathbf{v}_n$, $1 \le n \le N$, and arbitrary unitary matrix $\mathbf{X}_0$, the product $\mathbf{E}(z)$ in (4) is always a causal FIR paraunitary system of degree N.

It is easy to see that $\mathbf{E}(z)$ produced by (4) is not necessarily of order $K-1$. If we want to design a paraunitary filterbank with given filter lengths, the parameters in (4) can not be arbitrary, and it is not reasonable to fix the McMillan degree of the system in advance. For order $K-1$ paraunitary $\mathbf{E}(z)$, its McMillan degree can range from $(K-1)$ to $M(K-1)$. Any order $K-1$ paraunitary $\mathbf{E}(z)$ of degree $K-1$ can be produced by (4) with $N = K-1$. This fact was used to design paraunitary filterbanks with filters of given length in [10].

B. Order-One Factorization

We now derive the order-one factorization from the degree-one structure by constraining the unit norm vectors and then prove completeness for fixed order $K-1$. Separate the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N\}$ of unit vectors into $K-1$ disjoint subsets. The kth group contains $r_k = N_k - N_{k-1}$ vectors ending with $\mathbf{v}_{N_k}$. Suppose that the vectors within each group are orthonormal. In this case, $r_k \le M$. Let $\mathbf{w}_k$ be the $M \times r_k$ matrix $\left[\mathbf{v}_{N_{k-1}+1}\ \mathbf{v}_{N_{k-1}+2}\ \cdots\ \mathbf{v}_{N_k}\right]$ with columns from group k; then, (4) can be rewritten as

$$\mathbf{E}(z) = \mathbf{W}_{K-1}(z)\mathbf{W}_{K-2}(z)\cdots\mathbf{W}_1(z)\mathbf{X}_0 \tag{5}$$

where

$$\mathbf{W}_k(z) = \mathbf{I}_M - \mathbf{w}_k\mathbf{w}_k^{\mathrm{H}} + z^{-1}\mathbf{w}_k\mathbf{w}_k^{\mathrm{H}}. \tag{6}$$

Since the r_k columns of $\mathbf{w}_k$ are orthonormal, we can find $(M - r_k)$ vectors to complete an $M \times M$ unitary matrix $\mathbf{X}_k$ (of which the right $M \times r_k$ submatrix is $\mathbf{w}_k$). Then

$$\mathbf{W}_k(z) = \mathbf{X}_k\mathbf{\Lambda}_k(z)\mathbf{X}_k^{\mathrm{H}} \tag{7}$$

where

$$\mathbf{\Lambda}_k(z) = \mathrm{diag}\left(\mathbf{I}_{M-r_k}, z^{-1}\mathbf{I}_{r_k}\right). \tag{8}$$

Substituting (7) into (5), we obtain

$$\mathbf{E}(z) = \mathbf{U}_{K-1}\mathbf{\Lambda}_{K-1}(z)\mathbf{U}_{K-2}\mathbf{\Lambda}_{K-2}(z)\cdots\mathbf{U}_1\mathbf{\Lambda}_1(z)\mathbf{U}_0 \tag{9}$$

where $\mathbf{U}_k = \begin{cases}\mathbf{X}_{k+1}^{\mathrm{H}}\mathbf{X}_k, & 0 \le k \le K-2 \\ \mathbf{X}_{K-1}, & k = K-1\end{cases}$. Since $\mathbf{X}_k$ are unitary matrices, so are $\mathbf{U}_k$. The structure is demonstrated in Fig. 2(a), where $\bar{r}_k = M - r_k$.

With the orthogonality assumption on the parameter vectors, we have shown that $\mathbf{E}(z)$ of the N-stage degree-one structure can be expressed as the product of $K-1$ order-one paraunitary matrices. From (9), with arbitrary unitary matrices $\mathbf{U}_k$, the order of $\mathbf{E}(z)$ cannot be higher than $K-1$. Therefore, we can use these structures to design filterbanks with constrained filter length. In fact, the structure of (9) was used in the design of paraunitary system [27]. This order-one form is more suitable than the degree-one form when the filter length is constrained. In the rest of this subsection, we will show that any causal FIR paraunitary matrix $\mathbf{E}(z)$ of order $K-1$ can be factorized into the form of (9).

Given a causal FIR paraunitary matrix $\mathbf{E}(z)$, it can be expressed as in (1), where $\mathbf{E}_0^{\mathrm{H}}\mathbf{E}_{K-1} = \mathbf{0}$. Thus, the ranks of $\mathbf{E}_0$ and $\mathbf{E}_{K-1}$ satisfy

$$\rho(\mathbf{E}_0) + \rho(\mathbf{E}_{K-1}) \le M. \tag{10}$$

The singular value decompositions of $\mathbf{E}_0$ and $\mathbf{E}_{K-1}$ are

$$\mathbf{E}_0 = \mathbf{Q}_{00}\mathbf{\Sigma}_0\mathbf{Q}_{01}^{\mathrm{H}}, \quad \mathbf{E}_{K-1} = \mathbf{Q}_{10}\mathbf{\Sigma}_1\mathbf{Q}_{11}^{\mathrm{H}} \tag{11}$$

where

$\mathbf{\Sigma}_0$ and $\mathbf{\Sigma}_1$	square diagonal matrices containing the $\rho(\mathbf{E}_0)$ and $\rho(\mathbf{E}_{K-1})$ singular values of $\mathbf{E}_0$ and $\mathbf{E}_{K-1}$;
$\mathbf{Q}_{00}$ and $\mathbf{Q}_{01}$	$M \times \rho(\mathbf{E}_0)$ matrices;
$\mathbf{Q}_{10}$ and $\mathbf{Q}_{11}$	$M \times \rho(\mathbf{E}_{K-1})$ matrices.

The column vectors of each $\mathbf{Q}_{ij}$ are orthonormal. Substituting (11) into $\mathbf{E}_0^{\mathrm{H}}\mathbf{E}_{K-1} = \mathbf{0}$, we have

$$\mathbf{Q}_{10}^{\mathrm{H}}\mathbf{Q}_{00} = \mathbf{0}. \tag{12}$$

This means that the column vectors of $\mathbf{Q}_{00}$ are orthogonal to those of $\mathbf{Q}_{10}$. Similarly, the column vectors of $\mathbf{Q}_{01}$ are orthogonal to those of $\mathbf{Q}_{11}$. The paraunitary property also implies $\mathbf{E}(z)\tilde{\mathbf{E}}(z) = \mathbf{I}_M$, which leads to $\mathbf{E}_0\mathbf{E}_{K-1}^{\mathrm{H}} = \mathbf{0}$ and $\mathbf{Q}_{01}^{\mathrm{H}}\mathbf{Q}_{11} = \mathbf{0}$. Since the $\rho(\mathbf{E}_0) + \rho(\mathbf{E}_{K-1})$ columns of $\mathbf{Q}_{00}$ and $\mathbf{Q}_{10}$ are orthogonal, there are $M - \rho(\mathbf{E}_0) - \rho(\mathbf{E}_{K-1})$ orthonormal vectors orthogonal to these columns. We place these vectors into the columns of a matrix $\mathbf{Q}_\perp$ and create the square unitary matrix

$$\mathbf{U}_{K-1} = [\,\mathbf{Q}_{00} \quad \mathbf{Q}_\perp \quad \mathbf{Q}_{10}\,]. \tag{13}$$

Then, $\mathbf{E}_0$ and $\mathbf{E}_{K-1}$ can be expressed as

$$\mathbf{E}_0 = \mathbf{U}_{K-1}\begin{bmatrix}\mathbf{\Sigma}_0\mathbf{Q}_{01}^{\mathrm{H}} \\ \mathbf{0}\end{bmatrix}, \quad \mathbf{E}_{K-1} = \mathbf{U}_{K-1}\begin{bmatrix}\mathbf{0} \\ \mathbf{\Sigma}_1\mathbf{Q}_{11}^{\mathrm{H}}\end{bmatrix}. \tag{14}$$

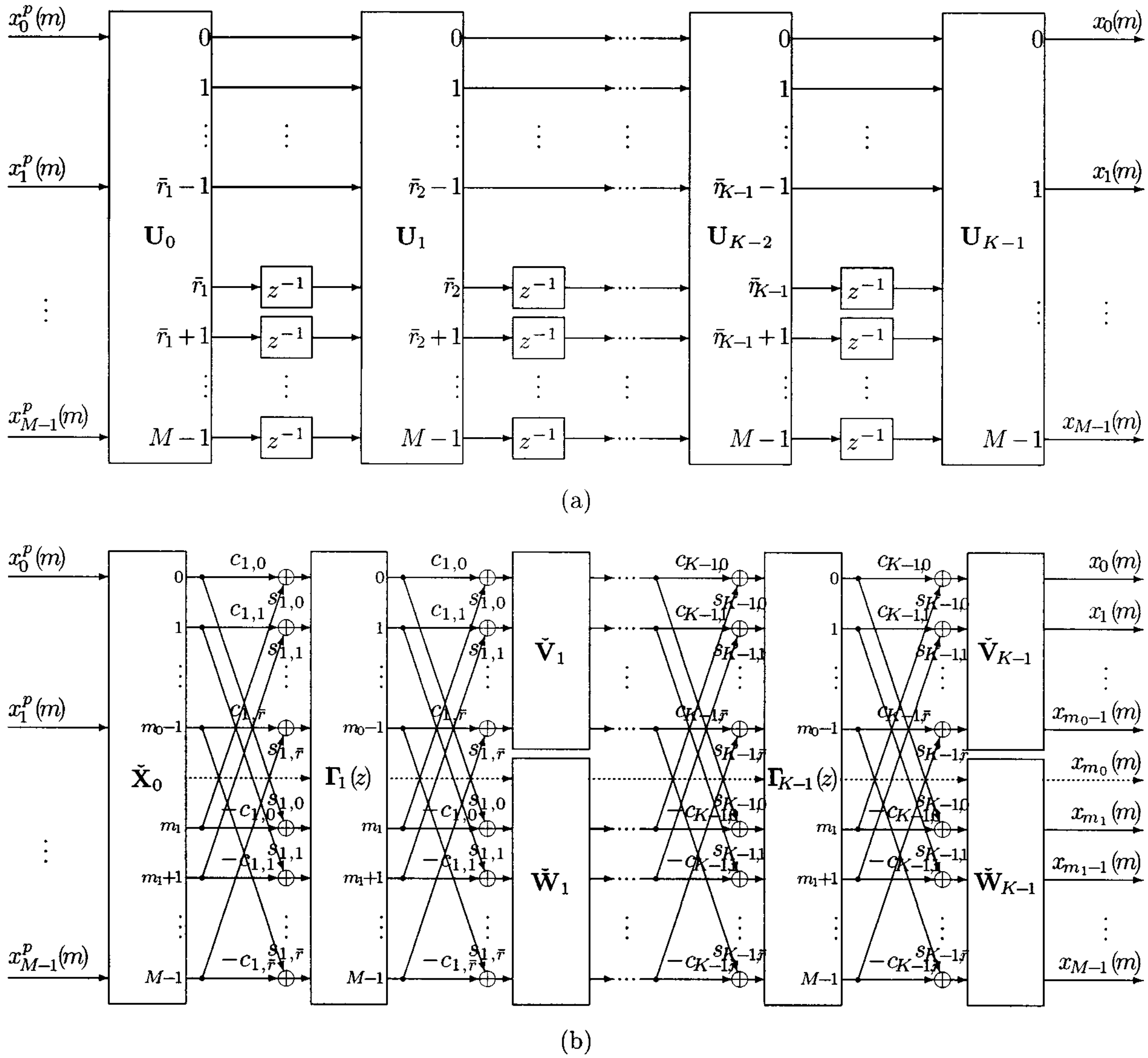

Fig. 2. Order-one structure for paraunitary systems. (a) Basic form. (b) More efficient form.

Substituting (14) into (1), we have

$$\begin{aligned}\mathbf{E}(z) &= \mathbf{U}_{K-1}\begin{bmatrix}\boldsymbol{\Sigma}_0\mathbf{Q}_{01}^{\mathrm{H}}\\ \mathbf{0}\end{bmatrix}\\ &\quad + \sum_{k=1}^{K-2}\mathbf{E}_k z^{-k} + \mathbf{U}_{K-1}\begin{bmatrix}\mathbf{0}\\ \boldsymbol{\Sigma}_1\mathbf{Q}_{11}^{\mathrm{H}}\end{bmatrix} z^{-(K-1)}\\ &= \mathbf{U}_{K-1}\boldsymbol{\Lambda}_{K-1}(z)\hat{\mathbf{E}}(z) \qquad (15)\end{aligned}$$

where $\boldsymbol{\Lambda}_{K-1}(z) = \mathrm{diag}(\mathbf{I}_{M-r_{K-1}}, z^{-1}\mathbf{I}_{r_{K-1}})$ with $\rho(\mathbf{E}_{K-1}) \le r_{K-1} \le M - \rho(\mathbf{E}_0)$; $\hat{\mathbf{E}}(z)$ is a causal FIR transfer function matrix of order $K-2$. Equation (15) shows that the order $K-1$ of $\mathbf{E}(z)$ can be reduced to $K-2$ for $\hat{\mathbf{E}}(z)$. Since $\mathbf{E}(z)$, $\mathbf{U}_{K-1}$ and $\boldsymbol{\Lambda}_{K-1}(z)$ are paraunitary, $\hat{\mathbf{E}}(z)$ is also paraunitary. Therefore, the above order-one reduction process can be repeated until the remainder is a constant unitary matrix. Then, $\mathbf{E}(z)$ is factorized into (9). The above form also holds for system with real coefficients. In summary, we have the following result.

Theorem 1: Let $\mathbf{E}(z)$ be an $M \times M$ causal FIR transfer function matrix with order not exceeding $K-1$. It is paraunitary if and only if there exist unitary matrices $\mathbf{U}_0, \mathbf{U}_1, \ldots, \mathbf{U}_{K-1}$ and integers $r_1, r_2, \ldots, r_{K-1}$ so that $\mathbf{E}(z)$ can be factorized as (9) with $\boldsymbol{\Lambda}_k(z)$ defined by (8). In the real-coefficient case, $\mathbf{U}_k$ are real and orthogonal.

Comments:

1) There are K unitary matrices $\mathbf{U}_k$ in this order-one factorization. Any unitary matrix can be completely factorized by M^2 free parameters [1], [28]. For the real-coefficient case, $M(M-1)/2$ rotation parameters are required. Thus, the total numbers of parameters are KM^2 and $KM(M-1)/2$ for the complex-coefficient and real-coefficient cases, respectively.
2) When $\mathbf{E}_{K-1} \neq 0$, the set of free integers r_k can be constrained as

$$1 \le r_{K-1} \le r_{K-2} \le \cdots \le r_1 \le M. \qquad (16)$$

To see this, we can check $\hat{\mathbf{E}}(z)$ in (15). From $\hat{\mathbf{E}}(z) = \sum_{k=0}^{K-2}\hat{\mathbf{E}}_k z^{-k}$, the first $\rho(\mathbf{E}_0)$ rows of $\hat{\mathbf{E}}_0$ and the last $\rho(\mathbf{E}_{K-1})$ rows of $\hat{\mathbf{E}}_{K-2}$ are $\boldsymbol{\Lambda}_0\mathbf{Q}_{01}^{\mathrm{H}}$ and $\boldsymbol{\Lambda}_1\mathbf{Q}_{11}^{\mathrm{H}}$, respectively. Consequently, $\rho(\hat{\mathbf{E}}_0) \geq \rho(\mathbf{E}_0)$, and $\rho(\hat{\mathbf{E}}_{K-2}) \geq \rho(\mathbf{E}_{K-1})$. If we choose $r_{K-1} = \rho(\mathbf{E}_{K-1})$ in the first reduction step and repeat in the subsequent step, then $r_{K-1} \leq r_{K-2}$. In this way, the paraunitary matrix $\mathbf{E}(z)$ can be factorized as in (9) with a set of integers r_k satisfying (16). If we choose $r_{K-1} = M - \rho(\mathbf{E}_0)$ in the first step and repeat in the subsequent steps, the integers are in the following order:

$$M \geq r_{K-1} \geq r_{K-2} \geq \cdots \geq r_1 \geq 1. \tag{17}$$

3) For a two-channel filterbank with real coefficients and with $\mathbf{E}_0 \neq \mathbf{0}$ and $\mathbf{E}_{K-1} \neq \mathbf{0}$, the set of integers must be $r_{K-1} = r_{K-2} = \cdots = r_1 = 1$, and $\mathbf{U}_k$ are 2×2 real orthogonal matrices. Any 2×2 orthogonal matrix can be completely factorized as

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u_1 & 0 \\ 0 & u_2 \end{bmatrix}$$

where θ is the rotation angle $u_1, u_2 \in \{1, -1\}$. Let $\mathbf{U}_{K-1} = \mathbf{R}_{K-1}\mathbf{u}_{K-1}$ and $\mathbf{u}_{k+1}\mathbf{U}_k = \mathbf{R}_k\mathbf{u}_k$, $0 \leq k \leq K-2$, where

$$\mathbf{R}_k = \begin{bmatrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{bmatrix}$$

and $\mathbf{u}_k$, $0 \leq k \leq K-1$ are diagonal matrices with diagonal entries ± 1; then, $\mathbf{E}(z)$ reduces to the following two-channel lossless lattice structure [8]:

$$\mathbf{E}(z) = \mathbf{R}_{K-1}\mathbf{\Lambda}(z)\mathbf{R}_{K-2}\mathbf{\Lambda}(z)\cdots\mathbf{R}_1\mathbf{\Lambda}(z)\mathbf{R}_0 \cdot \mathrm{diag}(\pm 1, \pm 1) \tag{18}$$

where $\mathbf{\Lambda}(z) = \mathrm{diag}(1, z^{-1})$.

4) Although any $\mathbf{E}(z)$ with order $K-1$ and with $\mathbf{E}_{K-1} \neq 0$ can be factorized as in (9), the order of $\mathbf{E}(z)$ produced by this structure with arbitrary unitary matrices $\mathbf{U}_k$ is not necessarily equal to $K-1$, i.e., $\mathbf{E}_{K-1}$ can be the zero matrix. However, the order cannot be higher than $K-1$. This situation appears in other special factorization forms for special classes of filterbanks [8], [13]–[18]. To ensure that $\mathbf{E}_{K-1} \neq 0$, one needs to impose additional constraint on $\mathbf{U}_k$ and r_k such that

$$\prod_{k=K-1}^{1} \{\mathbf{U}_k \cdot \mathrm{diag}(\mathbf{0}, \mathbf{I}_{r_k})\}\mathbf{U}_0 \neq \mathbf{0}. \tag{19}$$

In practice, it seems that there is no reason to do so. If r_k can take arbitrary integer values from 0 to M, the whole space of $\mathbf{E}(z)$ produced by (9) with arbitrary unitary matrices $\mathbf{U}_k$ is composed of causal paraunitary matrices with order not exceeding $K-1$.

III. More Efficient Structure

Since the order-one factorization form in (9) is complete for order $K-1$, it can be used to design paraunitary filterbanks with the constrained filter lengths. However, as the number of channels and the filter length increase, the number of free parameters is very large, and the optimal solution is difficult to obtain. In this section, we present a more efficient structure. To achieve this, we first investigate the Hermitian unitary matrix, which plays an important role in the development.

A. Hermitian Unitary Matrix

Let $\mathbf{A}$ be Hermitian and unitary

$$\mathbf{A} = \mathbf{A}^{\mathrm{H}}, \quad \mathbf{A}^{\mathrm{H}}\mathbf{A} = \mathbf{I}_M. \tag{20}$$

If $\mathbf{A}$ is real, it is symmetric and orthogonal. Any complex unitary matrix can be expressed by M^2 parameters, whereas any orthogonal matrix can be represented by $M(M-1)/2$ angle parameters. The Hermitian constraint $\mathbf{A} = \mathbf{A}^{\mathrm{H}}$ reduces these degrees of freedom. The problem is how to express this special class of unitary matrix with fewer parameters. Similar to the cosine-sine (CS) decomposition for a general unitary matrix [29], we have the following result on Hermitian unitary matrix.

Lemma 1: Let $\mathbf{A}$ be an $M \times M$ matrix. Partition it as $\mathbf{A} = \begin{bmatrix} \mathbf{A}_{00} & \mathbf{A}_{01} \\ \mathbf{A}_{10} & \mathbf{A}_{11} \end{bmatrix}$, where $\mathbf{A}_{ij}$ is $m_i \times m_j$ with $m_0 = \lfloor M/2 \rfloor$ and $m_1 = \lceil M/2 \rceil$. $\mathbf{A}$ is Hermitian and unitary if and only if it can be expressed as

$$\mathbf{A} = \mathrm{diag}(\mathbf{V}, \mathbf{W}) \cdot \mathbf{D}_{cs} \cdot \mathrm{diag}(\mathbf{V}^{\mathrm{H}}, \mathbf{W}^{\mathrm{H}}) \tag{21}$$

where $\mathbf{V}$ and $\mathbf{W}$ are $m_0 \times m_0$ and $m_1 \times m_1$ unitary matrices, respectively; $\mathbf{D}_{cs} = \begin{bmatrix} \mathbf{C} & \mathbf{S} \\ \mathbf{S} & \hat{\mathbf{C}} \end{bmatrix}$, for even M and $\begin{bmatrix} \mathbf{C} & \mathbf{0} & \mathbf{S} \\ \mathbf{0} & \pm 1 & \mathbf{0} \\ \mathbf{S} & \mathbf{0} & \hat{\mathbf{C}} \end{bmatrix}$ for odd M, where $\mathbf{C}$, $\mathbf{S}$, and $\hat{\mathbf{C}}$ are $m_0 \times m_0$ real diagonal matrices with entries $[\mathbf{C}]_{kk} = \cos\theta_k$, $[\mathbf{S}]_{kk} = \sin\theta_k$, and

$$[\hat{\mathbf{C}}]_{kk} = \begin{cases} \pm 1, & \sin\theta_k = 0 \\ \cos\theta_k, & \sin\theta_k \neq 0 \end{cases} \quad \theta_k \in \mathbf{R}.$$

If $\mathbf{A}$ is real, $\mathbf{V}$ and $\mathbf{W}$ can be real orthogonal.

Appendix A provides the proof for Lemma 1. Compared with the CS decomposition for a general unitary matrix [29], the proposed form for Hermitian unitary matrix contains unconstrained angles θ_k. More importantly, only two reduced-size unitary matrices of order $\lfloor M/2 \rfloor$ and $\lceil M/2 \rceil$ are required, plus $\lfloor M/2 \rfloor$ additional angle parameters. The total number of parameters is $(2\lfloor M/2 \rfloor + 1)\lceil M/2 \rceil$ for a complex Hermitian unitary matrix and $\lfloor M/2 \rfloor \lceil M/2 \rceil$ for a real symmetric orthogonal matrix. The Hermitian constraint reduces the number of parameters nearly by half. In fact, to factorize an $M \times M$ orthogonal matrix by Givens rotations, M sign parameters ± 1 are required besides planar rotation parameters [1]. The rotation angles take arbitrary values. The factorization of an orthogonal matrix is $\mathbf{R} = \mathbf{TD}$, where the matrix $\mathbf{T}$ represents the planar rotation matrix, and the sign parameters are in the diagonal matrix $\mathbf{D}$. From (21), the sign parameters of $\mathbf{V}$ and $\mathbf{W}$ can be moved into $\mathbf{D}_{cs}$, which only affects the angles θ_k. Therefore, the sign parameters of $\mathbf{V}$ and $\mathbf{W}$ are not necessary for a symmetric orthogonal matrix. This is an advantage of using the unconstrained angles in the proposed CS decomposition.

The special matrix $\mathbf{D}_{cs}$ is characterized by $\lfloor M/2 \rfloor$ 2×2 symmetric orthogonal matrices $\mathbf{B}_k = \begin{bmatrix} [\mathbf{C}]_{kk} & [\mathbf{S}]_{kk} \\ [\mathbf{S}]_{kk} & [\hat{\mathbf{C}}]_{kk} \end{bmatrix}$. The eigenvalues of any symmetric orthogonal matrix are 1 or -1. It is easy to show that

$$\mathbf{B}_k = \begin{bmatrix} \cos\alpha_k & \sin\alpha_k \\ \sin\alpha_k & -\cos\alpha_k \end{bmatrix} \mathbf{\Omega}_k \begin{bmatrix} \cos\alpha_k & \sin\alpha_k \\ \sin\alpha_k & -\cos\alpha_k \end{bmatrix} \quad (22)$$

where

$$\alpha_k = \begin{cases} 0, & \sin\theta_k = 0 \\ \theta_k/2, & \sin\theta_k \neq 0 \end{cases}$$

and

$$\Omega_k = \begin{cases} \text{diag}([\mathbf{C}]_{kk}, [\mathbf{S}]_{kk}), & \sin\theta_k = 0 \\ \text{diag}(1, -1), & \sin\theta_k \neq 0. \end{cases}$$

Therefore, $\mathbf{D}_{cs}$ can be expressed as

$$\mathbf{D}_{cs} = \mathbf{Q}\mathbf{\Gamma}\mathbf{Q} \quad (23)$$

where $\mathbf{\Gamma}$ is a real diagonal matrix with diagonal entries of absolute value 1

$$\mathbf{Q} = \begin{bmatrix} \check{\mathbf{C}} & \check{\mathbf{S}} \\ \check{\mathbf{S}} & -\check{\mathbf{C}} \end{bmatrix}, \text{ for even } M \text{ and}$$
$$\mathbf{Q} = \begin{bmatrix} \check{\mathbf{C}} & \mathbf{0} & \check{\mathbf{S}} \\ \mathbf{0} & \pm 1 & \mathbf{0} \\ \check{\mathbf{S}} & \mathbf{0} & -\check{\mathbf{C}} \end{bmatrix}, \text{ for odd } M \quad (24)$$

where $\check{\mathbf{C}}$ and $\check{\mathbf{S}}$ are real diagonal matrices with entries $[\check{\mathbf{C}}]_{kk} = \cos\alpha_k$ and $[\check{\mathbf{S}}]_{kk} = \sin\alpha_k$. Conversely, with arbitrary α_k and arbitrary real diagonal matrix $\mathbf{\Gamma}$ with diagonal entries of absolute value 1, $\mathbf{D}_{cs}$ produced by (23) is symmetric orthogonal, and therefore, $\mathbf{A}$ produced by (21) is Hermitian unitary. In summary, we have the following result.

Lemma 2: Let $\mathbf{A}$ be an $M \times M$ matrix, and let it be partitioned as in Lemma 1. $\mathbf{A}$ is Hermitian and unitary if and only if it can be expressed as in (21) with $\mathbf{D}_{cs}$ defined by (23). If $\mathbf{A}$ is real, the two unitary matrices are real orthogonal.

This lemma gives the spectral decomposition for a Hermitian unitary matrix. The transform matrix takes the special form $\text{diag}(\mathbf{V}, \mathbf{W}) \cdot \mathbf{Q}$, where $\mathbf{V}$ and $\mathbf{W}$ are unitary, and $\mathbf{Q}$ is defined by (24).

B. More Efficient Structure for Paraunitary System

For $\mathbf{U}_k$ in (9), we can find K unitary matrices $\mathbf{X}_k$ so that

$$\mathbf{U}_k = \begin{cases} \mathbf{X}_{k+1}^{\mathrm{H}}\mathbf{X}_k, & 0 \le k \le K-2 \\ \mathbf{X}_{K-1}, & k = K-1. \end{cases}$$

Therefore, any paraunitary causal FIR $\mathbf{E}(z)$ of order $K-1$ can also be expressed as the form in (5) with the order-one matrix $\mathbf{W}_k(z)$ defined by (7).

$$\mathbf{E}(z) = \mathbf{W}_{K-1}(z)\mathbf{W}_{K-2}(z)\cdots\mathbf{W}_1(z)\mathbf{X}_0 \quad (25)$$

where

$$\mathbf{W}_k(z) = \mathbf{X}_k\mathbf{\Lambda}_k(z)\mathbf{X}_k^{\mathrm{H}} \quad (26)$$
$$\mathbf{X}_k = \begin{cases} \mathbf{X}_{k+1}\mathbf{U}_k, & 0 \le k \le K-2 \\ \mathbf{U}_{K-1}, & k = K-1. \end{cases} \quad (27)$$

It is easy to see that $\mathbf{\Lambda}_k(z)$ can be expressed as

$$\mathbf{\Lambda}_k(z) = \frac{1}{2}(\mathbf{I}_M + \mathbf{\Lambda}_k(-1)) + \frac{1}{2}(\mathbf{I}_M - \mathbf{\Lambda}_k(-1))z^{-1} \quad (28)$$

where $\mathbf{\Lambda}_k(-1) = \text{diag}(\mathbf{I}_{M-r_k}, -\mathbf{I}_{r_k})$. Substituting (28) into (26), we have

$$\mathbf{W}_k(z) = \frac{1}{2}(\mathbf{I}_M + \mathbf{A}_k) + \frac{1}{2}(\mathbf{I}_M - \mathbf{A}_k)z^{-1} \quad (29)$$

where

$$\mathbf{A}_k = \mathbf{X}_k\mathbf{\Lambda}_k(-1)\mathbf{X}_k^{\mathrm{H}}. \quad (30)$$

The matrix $\mathbf{A}_k$ is Hermitian unitary matrix. From Lemma 2, we can express $\mathbf{A}_k$ as

$$\mathbf{A}_k = \text{diag}(\mathbf{V}_k, \mathbf{W}_k) \cdot \mathbf{Q}_k\mathbf{\Gamma}_k\mathbf{Q}_k \cdot \text{diag}\left(\mathbf{V}_k^{\mathrm{H}}, \mathbf{W}_k^{\mathrm{H}}\right) \quad (31)$$

where

- $\mathbf{V}_k$ and $\mathbf{W}_k$ — $\lfloor M/2 \rfloor \times \lfloor M/2 \rfloor$ and $\lceil M/2 \rceil \times \lceil M/2 \rceil$ unitary matrices, respectively;
- $\mathbf{\Gamma}_k$ — real diagonal matrix with diagonal entries 1 or -1;
- $\mathbf{Q}_k$ — real orthogonal matrix of the form

$$\mathbf{Q}_k = \begin{bmatrix} \check{\mathbf{C}}_k & \check{\mathbf{S}}_k \\ \check{\mathbf{S}}_k & -\check{\mathbf{C}}_k \end{bmatrix}, \text{ for even } M \text{ and}$$
$$\mathbf{Q}_k = \begin{bmatrix} \check{\mathbf{C}}_k & \mathbf{0} & \check{\mathbf{S}}_k \\ \mathbf{0} & \pm 1 & \mathbf{0} \\ \check{\mathbf{S}}_k & \mathbf{0} & -\check{\mathbf{C}}_k \end{bmatrix}, \text{ for odd } M \quad (32)$$

where $\check{\mathbf{C}}_k$ and $\check{\mathbf{S}}_k$ are real diagonal matrices with entries $[\check{\mathbf{C}}_k]_{ll} = \cos\alpha_{k,l}$ and $[\check{\mathbf{S}}_k]_{ll} = \sin\alpha_{k,l}$.

Substituting (31) into (29)

$$\mathbf{W}_k(z) = \text{diag}(\mathbf{V}_k, \mathbf{W}_k) \cdot \mathbf{Q}_k\mathbf{\Gamma}_k(z)\mathbf{Q}_k \cdot \text{diag}\left(\mathbf{V}_k^{\mathrm{H}}, \mathbf{W}_k^{\mathrm{H}}\right) \quad (33)$$

where $\mathbf{\Gamma}_k(z)$ is a diagonal matrix with

$$[\mathbf{\Gamma}_k(z)]_{ll} = \begin{cases} 1, & [\mathbf{\Gamma}_k]_{ll} = 1 \\ z^{-1}, & [\mathbf{\Gamma}_k]_{ll} = -1. \end{cases}$$

Substituting (33) into (25), we obtain

$$\mathbf{E}(z) = \prod_{k=K-1}^{1} \{\text{diag}(\check{\mathbf{V}}_k, \check{\mathbf{W}}_k) \cdot \mathbf{Q}_k\mathbf{\Gamma}_k(z)\mathbf{Q}_k\}\check{\mathbf{X}}_0 \quad (34)$$

where

- $\check{\mathbf{V}}_k$ — $\begin{cases} \mathbf{V}_{K-1}, & k = K-1 \\ \mathbf{V}_{k+1}^{\mathrm{H}}\mathbf{V}_k, & 1 \le k \le K-2; \end{cases}$
- $\check{\mathbf{W}}_k$ — $\begin{cases} \mathbf{W}_{K-1}, & k = K-1 \\ \mathbf{W}_{k+1}^{\mathrm{H}}\mathbf{W}_k, & 1 \le k \le K-2; \end{cases}$
- $\check{\mathbf{X}}_0$ — $\text{diag}(\mathbf{V}_1^{\mathrm{H}}, \mathbf{W}_1^{\mathrm{H}})\mathbf{X}_0$.

In (34), $\check{\mathbf{V}}_k$, $\check{\mathbf{W}}_k$, and $\check{\mathbf{X}}_0$ are unitary matrices. Conversely, with arbitrary unitary matrices $\check{\mathbf{V}}_k$, $\check{\mathbf{W}}_k$, and $\check{\mathbf{X}}_0$ and arbitrary angles $\alpha_{k,l}$ and $\mathbf{E}(z)$ produced by (34), they are causal FIR paraunitary with order equal to or less than $K-1$. For the real-coefficient case, $\check{\mathbf{V}}_k$, $\check{\mathbf{W}}_k$, and $\check{\mathbf{X}}_0$ are real orthogonal. In summary, we have proved the following theorem.

Theorem 2: Let $\mathbf{E}(z)$ be an $M \times M$ causal FIR transfer function matrix with order not exceeding $K-1$. It is paraunitary if and only if there exist $\lfloor M/2 \rfloor \times \lfloor M/2 \rfloor$ unitary matrices $\check{\mathbf{V}}_k$, $\lceil M/2 \rceil \times \lceil M/2 \rceil$ unitary matrices $\check{\mathbf{W}}_k$, an $M \times M$ unitary matrix $\check{\mathbf{X}}_0$, and a set of angles $\alpha_{k,l}$ so that it can be factorized as in (34) with $\mathbf{Q}_k$ defined by (32) and the diagonal matrices $\mathbf{\Gamma}_k(z)$ of diagonal entries 1 or z^{-1}. For real coefficients, all the unitary matrices are real orthogonal.

Comments:

1) The above theorem shows that the causal FIR paraunitary system can be characterized by $(K-1)$ unitary matrices of size $\lfloor M/2 \rfloor \times \lfloor M/2 \rfloor$, $(K-1)$ unitary matrices of size $\lceil M/2 \rceil \times \lceil M/2 \rceil$, an $M \times M$ unitary matrix, and $(K-1)\lfloor M/2 \rfloor$ additional angles. The total numbers of free parameters is $(K-1)(2\lfloor M/2 \rfloor+1)\lceil M/2 \rceil + M^2$ for the complex case and $(K-1)\lfloor M/2 \rfloor \lceil M/2 \rceil + M(M-1)/2$ for the real case. Compared with the structure of (9), a significant reduction of parameters is achieved. This gives more efficient design for the paraunitary system. Fig. 2(b) shows the structure in (34). At each stage of the lattice structure, the $\lfloor M/2 \rfloor \times \lfloor M/2 \rfloor$ unitary matrix $\check{\mathbf{V}}_k$, $\lceil M/2 \rceil \times \lceil M/2 \rceil$ unitary matrix $\check{\mathbf{W}}_k$, and $\lfloor M/2 \rfloor 2 \times 2$ matrices are to be implemented instead of the $M \times M$ matrix $\mathbf{U}_k$ in the structure of (9). This implies a significant saving of the implementation cost.

2) $\mathbf{\Gamma}_k(z)$ in (34) is generally different from $\mathbf{\Lambda}_k(z)$ in (9), although (34) is derived from (9). However, they contain the same number of delay terms. To verify, one only needs to check their determinants

$$|\mathbf{\Gamma}_k(z)| = |\mathbf{W}_k(z)| = |\mathbf{X}_k \mathbf{\Lambda}_k(z) \mathbf{X}_k^{\mathrm{H}}| = |\mathbf{\Lambda}_k(z)|. \tag{35}$$

The power of z^{-1} indicates the number of z^{-1} in $\mathbf{\Gamma}_k(z)$ and $\mathbf{\Lambda}_k(z)$ since they are diagonal matrices with diagonal entries 1 or z^{-1}.

3) For real coefficients, the unitary matrices are real orthogonal. The sign parameters of $\check{\mathbf{V}}_k$ and $\check{\mathbf{W}}_k$ are not necessary. The rotation parameters are sufficient. For a two-channel system, $\check{\mathbf{V}}_k$ and $\check{\mathbf{W}}_k$ are scalars and can be set to 1, which simplifies the polyphase matrix as

$$\begin{aligned} E(z) =& \prod_{k=K-1}^{1} \left\{ \begin{bmatrix} \cos\alpha_{k,0} & \sin\alpha_{k,0} \\ \sin\alpha_{k,0} & -\cos\alpha_{k,0} \end{bmatrix} \right. \\ & \left. \times \mathbf{\Gamma}_k(z) \begin{bmatrix} \cos\alpha_{k,0} & \sin\alpha_{k,0} \\ \sin\alpha_{k,0} & -\cos\alpha_{k,0} \end{bmatrix} \right\} \\ & \times \begin{bmatrix} \cos\alpha_{0,0} & \sin\alpha_{0,0} \\ \sin\alpha_{0,0} & -\cos\alpha_{0,0} \end{bmatrix} \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix} \\ =& \prod_{k=K-1}^{1} \left\{ \begin{bmatrix} \cos\alpha_{k} & \sin\alpha_{k} \\ \sin\alpha_{k} & -\cos\alpha_{k} \end{bmatrix} \mathbf{\Gamma}_k(z) \right\} \\ & \times \begin{bmatrix} \cos\alpha_{0} & \sin\alpha_{0} \\ \sin\alpha_{0} & -\cos\alpha_{0} \end{bmatrix} \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix} \end{aligned} \tag{36}$$

where

$$\alpha_k = \begin{cases} \alpha_{K-1,0}, & k = K-1 \\ \alpha_{k+1,0} - \alpha_{k,0}, & 0 \le k \le K-2. \end{cases}$$

It has the same implementation complexity as in (18).

IV. Factorization of Paraunitary System with Constraints

In this section, we consider two special classes of paraunitary filterbanks using the above factorization for general paraunitary systems. In applications such as image coding, linear phase is important. Several researchers have reported results on lattice structure for linear-phase filterbanks [13], [15], [17]. We will revisit it in the following subsection and derive an alternative structure with fewer parameters. The other class of filterbank considered in this section is the pairwise mirror-image symmetry filterbank, which was first investigated by Nguyen and Vaidyanathan in [12]. With the mirror-image constraint on the subband filters, the number of free parameters can be reduced significantly, whereas a good stopband attenuation can be preserved. A factorization was proposed in [12], and its completeness was proved in [14]. For these special filterbanks, we will first obtain the same or equivalent results as the reported works, which correspond to (9), and then achieve more efficient structures related to the form in (34). The filterbanks considered here have real coefficients, and M is assumed to be even for simplicity.

A. Linear Phase Filterbank

In a linear phase paraunitary filterbank, all filters are either symmetric or antisymmetric, i.e.,

$$h_k(L-1-n) = j_k h_k(n) \tag{37}$$

where j_k is 1, and -1 for symmetry and antisymmetry, respectively. Since $f_k(n) = h_k(L-1-n)$, (37) also implies that synthesis filters are symmetric or antisymmetric. Equation (37) holds if and only if the polyphase matrix satisfies

$$\mathbf{E}(z) = z^{-(K-1)} \mathbf{D}_{lp} \mathbf{E}(z^{-1}) \mathbf{J}_M \tag{38}$$

where $\mathbf{D}_{lp} = \mathrm{diag}(j_0, j_1, \ldots, j_{M-1})$. For perfect reconstruction, the symmetry polarities of the filters can not be independent. As shown in [13], there are $M/2$ symmetric and $M/2$ antisymmetric filters for even M. For simplicity, it is often assumed that the first $M/2$ filters are symmetric, i.e., $\mathbf{D}_{lp} = \mathrm{diag}(\mathbf{I}_{M/2}, -\mathbf{I}_{M/2})$.

The symmetry condition (38) implies particularly that $\mathbf{E}_{K-1} = \mathbf{D}_{lp} \mathbf{E}_0 \mathbf{J}_M$. Therefore, $\rho(\mathbf{E}_{K-1}) = \rho(\mathbf{E}_0) \le M/2$, and the SVD of $\mathbf{E}_{K-1}$ can be derived from that of $\mathbf{E}_0$. $\mathbf{Q}_{10}$ and $\mathbf{Q}_{11}$ in (11) can be chosen as $\mathbf{Q}_{10} = \mathbf{D}_{lp} \mathbf{Q}_{00}$ and $\mathbf{Q}_{11} = \mathbf{J}_M \mathbf{Q}_{01}$. Let $\mathbf{Q}_{00} = \begin{bmatrix} (\mathbf{Q}_{00}^0)^{\mathrm{T}} & (\mathbf{Q}_{00}^1)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$, where both $\mathbf{Q}_{00}^0$ and $\mathbf{Q}_{00}^1$ are $(M/2) \times \rho(\mathbf{E}_0)$ matrices. Then, $(\mathbf{Q}_{00}^0)^{\mathrm{T}} \mathbf{Q}_{00}^0 + (\mathbf{Q}_{00}^1)^{\mathrm{T}} \mathbf{Q}_{00}^1 = \mathbf{I}_{\rho(\mathbf{E}_0)}$. From (12), we have $(\mathbf{Q}_{00}^0)^{\mathrm{T}} \mathbf{Q}_{00}^0 - (\mathbf{Q}_{00}^1)^{\mathrm{T}} \mathbf{Q}_{00}^1 = \mathbf{0}$. Therefore, $(\mathbf{Q}_{00}^0)^{\mathrm{T}} \mathbf{Q}_{00}^0 = (\mathbf{Q}_{00}^1)^{\mathrm{T}} \mathbf{Q}_{00}^1 = (1/2)\mathbf{I}_{\rho(\mathbf{E}_0)}$. This means that the column vectors of each $\mathbf{Q}_{00}^l$ are orthogonal. There exist $(M/2) \times (M/2 - \rho(\mathbf{E}_0))$ matrices $\mathbf{Q}_\perp^l$, $l = 0, 1$ so that $(\mathbf{Q}_\perp^l)^{\mathrm{T}} \mathbf{Q}_\perp^l = (1/2)\mathbf{I}_{M/2-\rho(\mathbf{E}_0)}$ and $(\mathbf{Q}_\perp^l)^{\mathrm{T}} \mathbf{Q}_{00}^l = \mathbf{0}$. Therefore, we can choose the orthogonal matrix $\mathbf{U}_{K-1}$ as

$$\mathbf{U}_{K-1} = \frac{\sqrt{2}}{2} \begin{bmatrix} \mathbf{U}_{K-1,0} & \mathbf{U}_{K-1,0} \\ \mathbf{U}_{K-1,1} & -\mathbf{U}_{K-1,1} \end{bmatrix} \tag{39}$$

where $\mathbf{U}_{K-1,l} = \sqrt{2}\left[\mathbf{Q}_{00}^{l}\ \mathbf{Q}_{\perp}^{l}\right]$. With this matrix, (14) and (15) hold, and we can set $r_{K-1} = M/2$. Thus, $\mathbf{E}(z)$ can be expressed as

$$\begin{aligned}\mathbf{E}(z) &= \mathbf{U}_{K-1}\mathbf{\Lambda}_{K-1}(z)\hat{\mathbf{E}}(z)\\ &= \frac{1}{2}\cdot \text{diag}(\mathbf{U}_{K-1,0}, \mathbf{U}_{K-1,1})\cdot \mathbf{Q}_{lp}\\ &\quad \cdot \text{diag}\left(\mathbf{I}_{M/2}, z^{-1}\mathbf{I}_{M/2}\right)\cdot \mathbf{Q}_{lp}\breve{\mathbf{E}}(z) \end{aligned} \tag{40}$$

where

$$\mathbf{Q}_{lp} = \begin{bmatrix}\mathbf{I}_{M/2} & \mathbf{I}_{M/2}\\ \mathbf{I}_{M/2} & -\mathbf{I}_{M/2}\end{bmatrix}$$

and $\breve{\mathbf{E}}(z) = (\sqrt{2}/2)\mathbf{Q}_{lp}\hat{\mathbf{E}}(z)$. Substituting (40) into (38), we have

$$\breve{\mathbf{E}}(z) = z^{-(K-2)}\mathbf{D}_{lp}\breve{\mathbf{E}}(z^{-1})\mathbf{J}_M. \tag{41}$$

This means that the causal FIR paraunitary matrix $\breve{\mathbf{E}}(z)$ of order $K-2$ also satisfies the linear phase condition. The above process is repeated on $\breve{\mathbf{E}}(z)$ until the order is reduced to zero. The zero-order paraunitary matrix satisfying the linear phase condition can be expressed as $(\sqrt{2}/2)\text{diag}(\mathbf{U}_{0,0}, \mathbf{U}_{0,1})\cdot \mathbf{P}_{lp}$, where $\mathbf{U}_{0,0}$ and $\mathbf{U}_{0,1}$ are $(M/2)\times(M/2)$ orthogonal matrices, and $\mathbf{P}_{lp} = \begin{bmatrix}\mathbf{I}_{M/2} & \mathbf{J}_{M/2}\\ \mathbf{I}_{M/2} & -\mathbf{J}_{M/2}\end{bmatrix}$. Therefore, for the linear phase paraunitary filterbank, the polyphase matrix can be factorized as in (9) with the unitary matrices of the following form:

$$\mathbf{U}_k = \begin{cases}\frac{\sqrt{2}}{2}\cdot \text{diag}(\mathbf{U}_{K-1,0}, \mathbf{U}_{K-1,1})\cdot \mathbf{Q}_{lp}, & k = K-1\\ \frac{1}{2}\mathbf{Q}_{lp}\cdot \text{diag}(\mathbf{U}_{k,0}, \mathbf{U}_{k,1})\cdot \mathbf{Q}_{lp}, & 1\le k\le K-2\\ \frac{1}{2}\mathbf{Q}_{lp}\cdot \text{diag}(\mathbf{U}_{0,0}, \mathbf{U}_{0,1})\cdot \mathbf{P}_{lp}, & k=0\end{cases} \tag{42}$$

and

$$\mathbf{\Lambda}_k(z) = \mathbf{\Lambda}(z) = \text{diag}\left(\mathbf{I}_{M/2}, z^{-1}\mathbf{I}_{M/2}\right). \tag{43}$$

Conversely, with arbitrary orthogonal matrices $\mathbf{U}_{k,l}$, $\mathbf{E}(z)$ produced by (9), (42), and (43) is a causal FIR paraunitary matrix and satisfies the linear phase condition (38). This result is the same as that presented in [15] and equivalent to that in [13]. The structure is shown in Fig. 3(a), where the scale factors are ignored. It is interesting that this structure is similar to that of (34) with $\breve{\mathbf{C}}_k$ and $\breve{\mathbf{S}}_k$ replaced by $(\sqrt{2}/2)\mathbf{I}_{M/2}$ and the initial stage of the special form. The structure of (34) is for general paraunitary systems. With the linear phase constraint, further reduction of parameters is possible.

Substituting (42) into (27), and then (30), we get

$$\mathbf{X}_k = \begin{cases}\frac{\sqrt{2}}{2}\cdot \text{diag}(\mathbf{X}_{k,0}, \mathbf{X}_{k,1})\cdot \mathbf{Q}_{lp}, & 1\le k\le K-1\\ \frac{\sqrt{2}}{2}\cdot \text{diag}(\mathbf{X}_{0,0}, \mathbf{X}_{0,1})\cdot \mathbf{P}_{lp}, & k=0\end{cases} \tag{44}$$

$$\mathbf{A}_k = \begin{bmatrix}\mathbf{0} & \mathbf{A}_{k,0}\\ \mathbf{A}_{k,0}^{\mathrm{T}} & \mathbf{0}\end{bmatrix} \tag{45}$$

where

$\mathbf{X}_{k,l}$ $\prod_{m=K-1}^{k}\mathbf{U}_{m,l}$;
$\mathbf{A}_{k,0}$ $\mathbf{X}_{k,0}\mathbf{X}_{k,1}^{\mathrm{T}}$;
$\mathbf{A}_k$ special symmetric orthogonal matrix.

The CS decomposition and spectral decomposition of $\mathbf{A}_k$ are the following.

$$\begin{aligned}\mathbf{A}_k &= \text{diag}\left(\mathbf{I}_{M/2}, \mathbf{A}_{k,0}^{\mathrm{T}}\right)\cdot \begin{bmatrix}\mathbf{0} & \mathbf{I}_{M/2}\\ \mathbf{I}_{M/2} & \mathbf{0}\end{bmatrix}\cdot \text{diag}(\mathbf{I}_{M/2}, \mathbf{A}_{k,0})\\ &= \frac{1}{2}\cdot \text{diag}\left(\mathbf{I}_{M/2}, \mathbf{A}_{k,0}^{\mathrm{T}}\right)\cdot \mathbf{Q}_{lp}\cdot \text{diag}(\mathbf{I}_{M/2}, -\mathbf{I}_{M/2})\\ &\quad \cdot \mathbf{Q}_{lp}\cdot \text{diag}\left(\mathbf{I}_{M/2}, \mathbf{A}_{k,0}\right). \end{aligned} \tag{46}$$

Therefore $\mathbf{E}(z)$ can be rewritten as

$$\begin{aligned}\mathbf{E}(z) = \frac{\sqrt{2}}{2^K}\prod_{k=K-1}^{1}&\left\{\text{diag}(\mathbf{I}_{M/2}, \breve{\mathbf{W}}_k)\cdot \mathbf{Q}_{lp}\right.\\ &\left.\cdot \text{diag}\left(\mathbf{I}_{M/2}, z^{-1}\mathbf{I}_{M/2}\right)\cdot \mathbf{Q}_{lp}\right\}\\ &\cdot \text{diag}(\breve{\mathbf{V}}_0, \breve{\mathbf{W}}_0)\cdot \mathbf{P}_{lp}\end{aligned} \tag{47}$$

where $\breve{\mathbf{V}}_0 = \mathbf{X}_{0,0}$ and

$$\breve{\mathbf{W}}_k = \begin{cases}\mathbf{A}_{K-1,0}^{\mathrm{T}}, & k=K-1\\ \mathbf{W}_{k+1}^{\mathrm{T}}\mathbf{A}_{k,0}^{\mathrm{T}}, & 1\le k\le K-2\\ \mathbf{W}_1^{\mathrm{T}}\mathbf{X}_{0,1}, & k=0.\end{cases}$$

Conversely, with arbitrary orthogonal matrices $\breve{\mathbf{W}}_k$ and $\breve{\mathbf{V}}_0$, $\mathbf{E}(z)$, which is produced by (47), is a causal FIR paraunitary matrix and satisfies the linear phase condition (38). In summary, we have the following result.

Theorem 3: Let $\mathbf{E}(z)$ be an $M\times M$ real causal FIR transfer function matrix with order not exceeding $K-1$. Then, it is paraunitary and satisfies the linear phase condition (38) if and only if there exist $(M/2)\times(M/2)$ orthogonal matrices $\breve{\mathbf{V}}_0$ and $\breve{\mathbf{W}}_k$ so that it can be factorized as in (47).

Comments:

1) In this new structure, only one $(M/2)\times(M/2)$ orthogonal matrix is required at each stage, except for the first one. Therefore, a significant reduction of free parameters is achieved, and the implementation cost is reduced. Fig. 3(b) illustrates this new structure, where the scale factor $\sqrt{2}/(2^K)$ is not included.
2) The spectral decomposition of $\mathbf{A}_k$ in (46) implies that we restrict $\breve{\mathbf{C}}_k = \breve{\mathbf{S}}_k = (\sqrt{2}/2)\mathbf{I}_{M/2}$. The sign parameters of $\breve{\mathbf{W}}_k$ are generally necessary. In other words, if we just consider the planar rotation part of $\breve{\mathbf{W}}_k$ as in (34), $\breve{\mathbf{S}}_k$ should be allowed to be any diagonal matrix with diagonal entries 1 or -1 to keep the completeness of this factorization.

B. Pairwise Mirror-Image Symmetry Filterbank

For the filterbank with even M, the mirror-image condition can be imposed by the following relation [12]:

$$H_{M-1-k}(z) = z^{-(L-1)}H_k(-z^{-1}) \quad 0\le k\le M-1. \tag{48}$$

For simplicity, we can reorder the filters so that (48) becomes

$$H_{M/2+k}(z) = z^{-(L-1)}H_k(-z^{-1}) \quad 0\le k\le \frac{M}{2}-1. \tag{49}$$

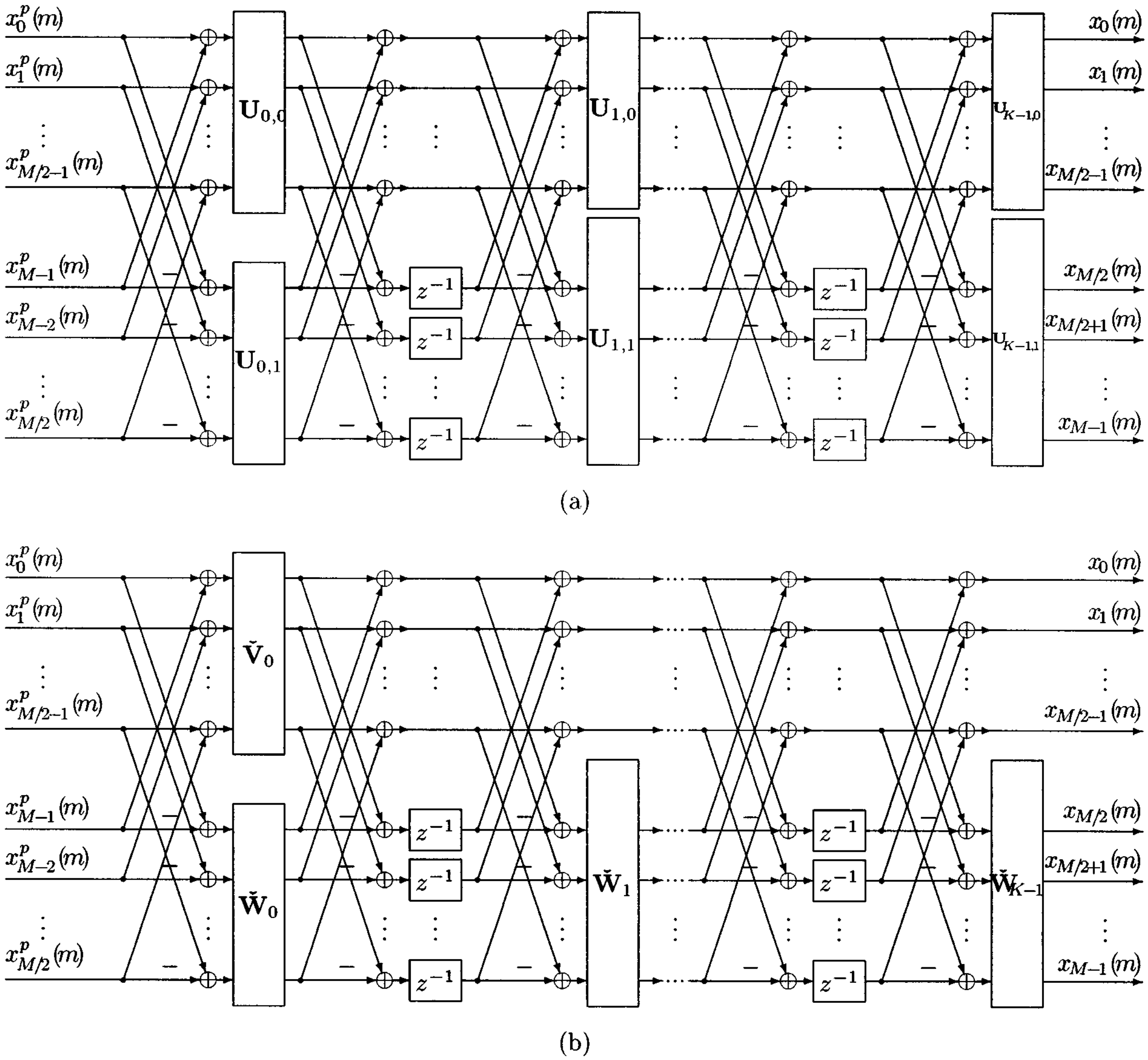

Fig. 3. Structure for paraunitary system with linear phase property. (a) Existing form. (b) New form.

This relation is equivalent to the following condition on the polyphase matrix:

$$\mathbf{E}(z) = z^{-(K-1)}\mathbf{D}_{pm}\mathbf{E}(z^{-1})\mathbf{P}_{pm} \tag{50}$$

where

$$\mathbf{D}_{pm} = \begin{bmatrix} \mathbf{0} & -\mathbf{I}_{M/2} \\ \mathbf{I}_{M/2} & \mathbf{0} \end{bmatrix}$$

and

$$\mathbf{P}_{pm} = \begin{bmatrix} \mathbf{0} & \mathbf{V}_{pm}\mathbf{J}_{M/2} \\ \mathbf{J}_{M/2}\mathbf{V}_{pm} & \mathbf{0} \end{bmatrix}$$

with $\mathbf{V}_{pm} = \text{diag}(1, -1, \ldots, (-1)^{M/2-1})$. Equation (50) implies particularly that $\mathbf{E}_{K-1} = \mathbf{D}_{pm}\mathbf{E}_0\mathbf{P}_{pm}$. Therefore, as in the linear phase filterbank, $\rho(\mathbf{E}_{K-1}) = \rho(\mathbf{E}_0) \leq M/2$, and the SVD of $\mathbf{E}_{K-1}$ can be derived from that of $\mathbf{E}_0$. $\mathbf{Q}_{10}$ and $\mathbf{Q}_{11}$ in (11) can be chosen as $\mathbf{Q}_{10} = \mathbf{D}_{pm}\mathbf{Q}_{00}$ and $\mathbf{Q}_{11} = \mathbf{P}_{pm}\mathbf{Q}_{01}$. Let an M-dimensional vector $\mathbf{q}$ be orthogonal to the column vectors of both $\mathbf{Q}_{10}$ and $\mathbf{Q}_{00}$, i.e., $\mathbf{q}^{\mathrm{T}}\mathbf{Q}_{00} = \mathbf{0}$ and $\mathbf{q}^{\mathrm{T}}\mathbf{Q}_{10} = \mathbf{0}$. Using the fact $\mathbf{D}_{pm}^{\mathrm{T}} = -\mathbf{D}_{pm}$, we have $(\mathbf{D}_{pm}\mathbf{q})^{\mathrm{T}}\mathbf{Q}_{10} = \mathbf{0}$ and $(\mathbf{D}_{pm}\mathbf{q})^{\mathrm{T}}\mathbf{Q}_{00} = \mathbf{0}$. Thus, the vector $\mathbf{D}_{pm}\mathbf{q}$ is also orthogonal to the column vectors of both $\mathbf{Q}_{00}$ and $\mathbf{Q}_{10}$. It is easy to verify that $\mathbf{q}$ is also orthogonal to $\mathbf{D}_{pm}\mathbf{q}$. Therefore, $\mathbf{q}$ and $\mathbf{D}_{pm}\mathbf{q}$ could be two column vectors of $\mathbf{Q}_{\perp}$ in $\mathbf{U}_{K-1}$. In this way, we can choose the orthogonal matrix $\mathbf{U}_{K-1}$ of the following form:

$$\mathbf{U}_{K-1} = \begin{bmatrix} \mathbf{U}_{K-1,0} & -\mathbf{U}_{K-1,1} \\ \mathbf{U}_{K-1,1} & \mathbf{U}_{K-1,0} \end{bmatrix} \tag{51}$$

where $\mathbf{U}_{K-1,l}$, $l = 0, 1$ are $(M/2) \times (M/2)$ matrices. Notice that the left and right $M \times \rho(\mathbf{E}_0)$ submatrices of $\mathbf{U}_{K-1}$ are $\mathbf{Q}_{00}$ and $\mathbf{Q}_{10}$, respectively. Using this matrix, (14) and (15) hold, and we can set $r_{K-1} = M/2$. Thus, $\mathbf{E}(z)$ can be expressed as

$$\mathbf{E}(z) = \begin{bmatrix} \mathbf{U}_{K-1,0} & -\mathbf{U}_{K-1,1} \\ \mathbf{U}_{K-1,1} & \mathbf{U}_{K-1,0} \end{bmatrix} \cdot \text{diag}\left(\mathbf{I}_{M/2}, z^{-1}\mathbf{I}_{M/2}\right) \cdot \hat{\mathbf{E}}(z). \tag{52}$$

It is easy to verify that the causal FIR paraunitary matrix $\hat{\mathbf{E}}(z)$ of order $K-2$ also satisfies the pairwise mirror-image symmetry condition (50) with K replaced by $K-1$. We can repeat the above process on $\hat{\mathbf{E}}(z)$ until the order is reduced to zero, whereas the mirror-image property is

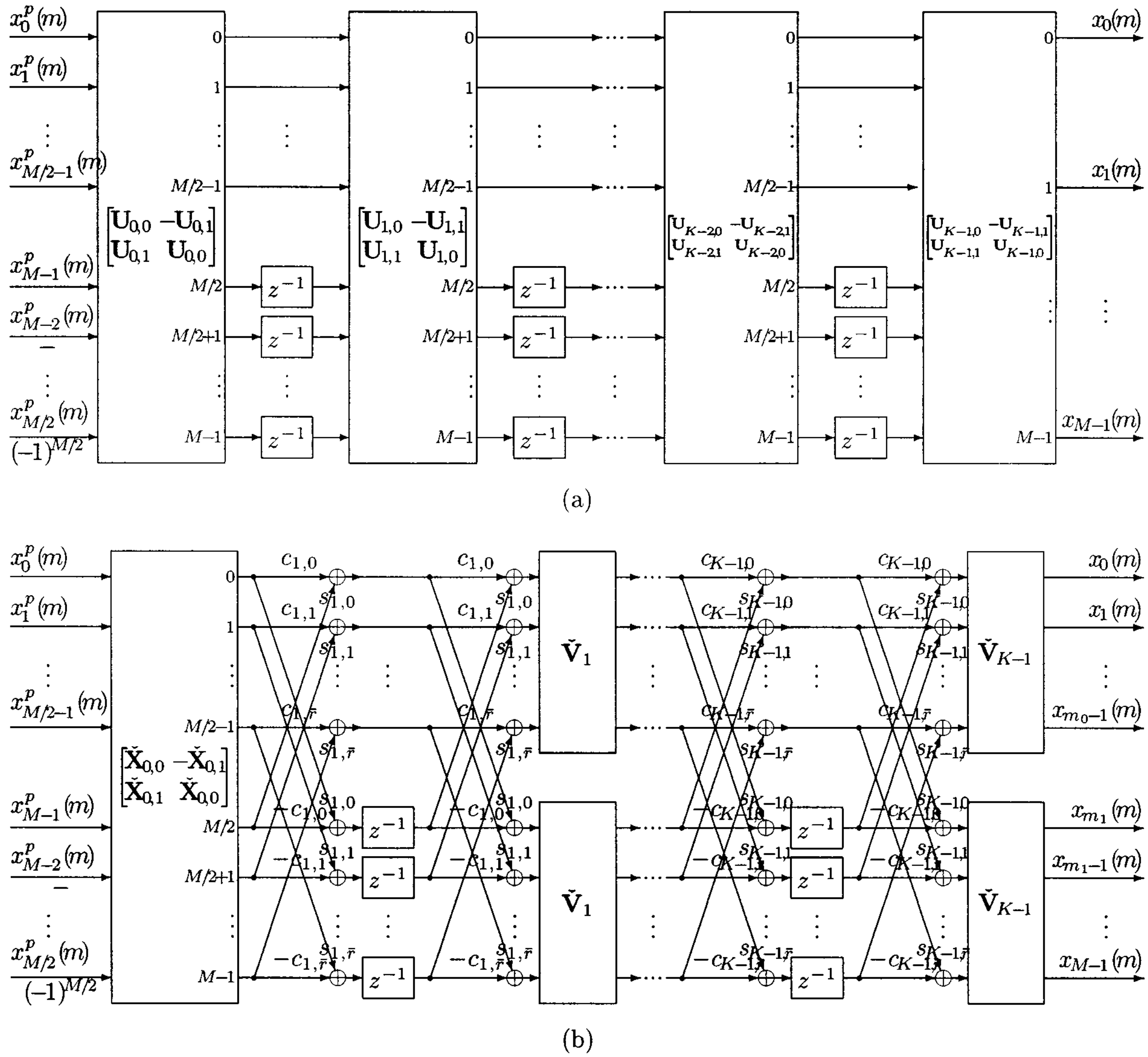

Fig. 4. Structure for paraunitary system with pairwise mirror image symmetry property. (a) Form equivalent to the existing structures. (b) New form.

preserved at each step. The zero order paraunitary matrix satisfying the mirror-image condition can be expressed as $\begin{bmatrix} \mathbf{U}_{0,0} & -\mathbf{U}_{0,1} \\ \mathbf{U}_{0,1} & \mathbf{U}_{0,0} \end{bmatrix} \cdot \text{diag}(\mathbf{I}_{M/2}, \mathbf{V}_{pm}\mathbf{J}_{M/2})$, where $\mathbf{U}_{0,0}$ and $\mathbf{U}_{0,1}$ are $(M/2) \times (M/2)$ matrices. Therefore, for the pairwise mirror-image symmetry paraunitary filterbank, the polyphase matrix can be factorized as in (9) with the orthogonal matrices as in (53), shown at the bottom of the page, and $\mathbf{\Lambda}_k(z)$ defined by (43). Conversely, with arbitrary orthogonal matrices $\mathbf{U}_k$ of the form (53) and $\mathbf{\Lambda}_k(z)$ defined by (43), $\mathbf{E}(z)$, which is produced by (9), is a causal FIR paraunitary matrix and satisfies the pairwise mirror-image symmetry condition (50). This result is equivalent to those in [12] and [14]. The structure is shown in Fig. 4(a). The $M \times M$ real matrix $\mathbf{U}_k$ is orthogonal if and only if the $(M/2) \times (M/2)$ complex matrix $\mathbf{U}_{k,0} + j\mathbf{U}_{k,1}$ is unitary. Thus, $\mathbf{U}_k$ can be expressed by $M^2/4$ parameters, whereas general orthogonal matrix needs $M(M-1)/2$ parameters besides M sign parameters. The reduction of the parameters is significant. This structure corresponds to our basic order-one factorization form. Further reduction of the parameters is still possible, which is related to the structure of (34).

Substituting (53) into (27) and then (30), we can see that $\mathbf{X}_k$ takes the same form as $\mathbf{U}_k$, i.e., we have (54), shown at the bottom of the page, and $\mathbf{A}_k$ is of the following form:

$$\mathbf{A}_k = \begin{bmatrix} \mathbf{A}_{k,0} & \mathbf{A}_{k,1} \\ \mathbf{A}_{k,1} & -\mathbf{A}_{k,0} \end{bmatrix}, 1 \le k \le K-1 \tag{55}$$

$$\mathbf{U}_k = \begin{cases} \begin{bmatrix} \mathbf{U}_{k,0} & -\mathbf{U}_{k,1} \\ \mathbf{U}_{k,1} & \mathbf{U}_{k,0} \end{bmatrix}, & 1 \le k \le K-1 \\ \begin{bmatrix} \mathbf{U}_{0,0} & -\mathbf{U}_{0,1} \\ \mathbf{U}_{0,1} & \mathbf{U}_{0,0} \end{bmatrix} \cdot \text{diag}(\mathbf{I}_{M/2}, \mathbf{V}_{pm}\mathbf{J}_{M/2}), & k = 0 \end{cases} \tag{53}$$

where $\mathbf{A}_{k,0}$ and $\mathbf{A}_{k,1}$ are symmetric. In addition, $\mathbf{A}_k$ is a special symmetric orthogonal matrix. From Appendix B, we can express $\mathbf{A}_k$ as

$$\mathbf{A}_k = \text{diag}(\mathbf{V}_k, \mathbf{V}_k) \cdot \mathbf{Q}_k \cdot \text{diag}(\mathbf{I}_{M/2}, -\mathbf{I}_{M/2}) \cdot \mathbf{Q}_k \cdot \text{diag}\left(\mathbf{V}_k^{\mathrm{T}}, \mathbf{V}_k^{\mathrm{T}}\right) \tag{56}$$

where $\mathbf{V}_k$ is orthogonal, and $\mathbf{Q}_k$ is defined as in (32) with even M. This implies that we can express $\mathbf{E}(z)$ as in (34) with the matrices as

$$\check{\mathbf{V}}_k = \check{\mathbf{W}}_k = \begin{cases} \mathbf{V}_{K-1}, & k = K-1 \\ \mathbf{V}_{k+1}^{\mathrm{T}}\mathbf{V}_k, & 1 \le k \le K-2 \end{cases}$$

$$\mathbf{Q}_k = \begin{bmatrix} \check{\mathbf{C}}_k & \check{\mathbf{S}}_k \\ \check{\mathbf{S}}_k & -\check{\mathbf{C}}_k \end{bmatrix}$$

$$\check{\mathbf{X}}_0 = \text{diag}\left(\mathbf{V}_1^{\mathrm{T}}, \mathbf{V}_1^{\mathrm{T}}\right)\mathbf{X}_0$$

$$\mathbf{\Gamma}_k(z) = \text{diag}\left(\mathbf{I}_{M/2}, z^{-1}\mathbf{I}_{M/2}\right).$$

For easy reference, we give the expression of $\mathbf{E}(z)$ in the following:

$$\mathbf{E}(z) = \prod_{k=K-1}^{1} \left\{\text{diag}(\check{\mathbf{V}}_k, \check{\mathbf{V}}_k) \cdot \mathbf{Q}_k \cdot \text{diag}\left(\mathbf{I}_{M/2}, z^{-1}\mathbf{I}_{M/2}\right) \cdot \mathbf{Q}_k\right\}\check{\mathbf{X}}_0 \cdot \text{diag}(\mathbf{I}_{M/2}, \mathbf{V}_{pm}\mathbf{J}_{M/2}). \tag{57}$$

Conversely, with arbitrary orthogonal matrices $\check{\mathbf{V}}_k$ and $\check{\mathbf{X}}_0$ of the form $\begin{bmatrix} \check{\mathbf{X}}_{0,0} & -\check{\mathbf{X}}_{0,1} \\ \check{\mathbf{X}}_{0,1} & \check{\mathbf{X}}_{0,0} \end{bmatrix}$ and a set of angles $\alpha_{k,l}$, $\mathbf{E}(z)$, which is produced by (57), is a causal FIR paraunitary matrix and satisfies the pairwise mirror-image symmetry condition (50). In summary, we have the following result.

Theorem 4: Let $\mathbf{E}(z)$ be an $M \times M$ real causal FIR transfer function matrix with order not exceeding $K-1$. It is paraunitary and satisfies the pairwise mirror-image symmetry condition (50) if and only if there exist orthogonal matrices $\check{\mathbf{V}}_k$, an orthogonal matrix $\check{\mathbf{X}}_0$ of the form $\begin{bmatrix} \check{\mathbf{X}}_{0,0} & -\check{\mathbf{X}}_{0,1} \\ \check{\mathbf{X}}_{0,1} & \check{\mathbf{X}}_{0,0} \end{bmatrix}$, and a set of angles $\alpha_{k,l}$ so that it can be factorized as (57).

Comments:

1) Fig. 4(b) illustrates the structure of (57). At each stage of this new structure except the initial one, only one free $(M/2) \times (M/2)$ orthogonal matrix is required besides $M/2$ angles in $\check{\mathbf{C}}_k$ and $\check{\mathbf{S}}_k$. Compared with the structure of (34) for general paraunitary systems and the one for this special class of filterbank as shown in Fig. 4(a), this structure needs far fewer parameters.
2) In the implementation aspect, the structure in Fig. 4(b) has the same complexity as that in Fig. 2(b), although it is more efficient than that in Fig. 4(a). This means that the implementation cost does not decrease for the pairwise mirror-image symmetry. In the linear phase filterbank, the linear phase constraint leads to improvements in both design and implementation cost.

V. Design Example

In this section, we present several design examples of real-coefficient filterbanks with even M based on the proposed structures in (34), (47), and (57). In different applications, various objective functions can be constructed to optimize the filter coefficients. The most common functions are the stopband attenuation and coding gain. Our examples presented in this paper are based on minimizing the stopband energy and maximizing the coding gain. The stopband energy criterion measures the sum of the filters' energy outside the designated passbands:

$$\Phi_{\text{stopband energy}} = \frac{M}{M_s}\sum_{k=0}^{M_s-1}\int_{\omega\in\Omega_k}|H_k(e^{j\omega})|^2 d\omega \tag{58}$$

where Ω_k denotes the stopband of the filter $H_k(z)$, and M_s is equal to M for the general paraunitary filterbank and linear phase filterbank and $M/2$ for the even-channel mirror-image filterbank. The coding gain measures the energy compaction ability of a filterbank for the input signal. Let σ_x^2 be the variance of the input signal $x(n)$, and let $\sigma_{x_k}^2$ be the variance of the kth subband signal $x_k(n)$. The objective function is defined as [32]

$$\Phi_{\text{coding gain}} = 10\log 10\frac{\sigma_x^2}{\left(\prod_{k=0}^{M-1}\sigma_{x_k}^2\right)^{1/M}}. \tag{59}$$

The commonly used AR(1) process with autocorrelation coefficient $\rho = 0.95$ was used in the optimization.

In the design, we factorize the orthogonal matrices with the Givens rotation. The complete parameter sets for the general paraunitary filterbank and linear-phase filterbank contain sign parameters, whereas the mirror-image filterbank only involves angle parameters. For the general paraunitary filterbank, M sign parameters are required to express each $\mathbf{\Gamma}_k(z)$ in (34). For the linear phase filterbank, the sign parameters of the $\check{\mathbf{W}}_k$ in (47) are needed to ensure the structure's completeness. For simplicity, we fixed the sign parameters in the design. All the $\Gamma_k(z)$ in (34) were set to be diag$(\mathbf{I}_{M/2}, z^{-1}\mathbf{I}_{M/2})$, and the sign parameters of the $\check{\mathbf{W}}_k$ in (47) were set to be 1. For the general paraunitary filterbank, this setting of the sign parameters assures that the designed filterbank has the symmetric delay property as discussed in [19]. To speed up the design procedure, the gradients of the objective functions can be calculated with explicit expressions. In the design, we can use a constrained filterbank to initialize a general paraunitary system since the factorization form of the former is a special one of the later. In addition, a high-order system can be initialized by a lower one. In

$$\mathbf{X}_k = \begin{cases} \begin{bmatrix} \mathbf{X}_{k,0} & -\mathbf{X}_{k,1} \\ \mathbf{X}_{k,1} & \mathbf{X}_{k,0} \end{bmatrix}, & 1 \le k \le K-1 \\ \begin{bmatrix} \mathbf{X}_{0,0} & -\mathbf{X}_{0,1} \\ \mathbf{X}_{0,1} & \mathbf{X}_{0,0} \end{bmatrix} \cdot \text{diag}(\mathbf{I}_{M/2}, \mathbf{V}_{pm}\mathbf{J}_{M/2}), & \text{k=0} \end{cases} \tag{54}$$

TABLE I
STOPBAND ENERGY OF FOUR-CHANNEL FILTER BANKS

	Stopband energy (in dB)		
Order K	General	Linear phase	Mirror-image
0	1.8626	1.8626	1.8626
1	11.0154	7.1914	11.0154
2	19.6973	9.7537	19.6973
3	27.8650	9.8371	27.8650
4	35.8598	12.6722	35.8598
5	44.6084	12.9423	44.6084
6	52.6200	13.1049	52.6200
7	60.6938	13.1471	60.6938

TABLE II
CODING GAIN OF FOUR-CHANNEL FILTER BANKS

	Coding Gain (in dB)		
Order K	General	Linear phase	Mirror-image
0	7.5825	7.5825	7.2098
1	8.1882	7.9605	8.1752
2	8.4003	8.2157	8.3950
3	8.4838	8.3593	8.4824
4	8.5213	8.4115	8.5196
5	8.5421	8.4532	8.5416
6	8.5550	8.4532	8.5545
7	8.5635	8.4551	8.5632

our design, these two methods were used. Tables I and II list the results of four-channel paraunitary systems with different orders. The stopbands were set to be $\Omega_k = [0, \max(0, (2k - 1.2)\pi/8)] \cup [\min(\pi, (2k+3.2)\pi/8), \pi]$. From the tables, it can be seen that both the stopband energy and coding gain of the pairwise mirror-image symmetry system are very close to the general one, except in the zero-order case. The coding gain of the linear phase system is comparable with the others, whereas the performance of the stopband attenuation is worse. The magnitude responses of four-channel filterbanks with $K = 7$ are shown in Fig. 5.

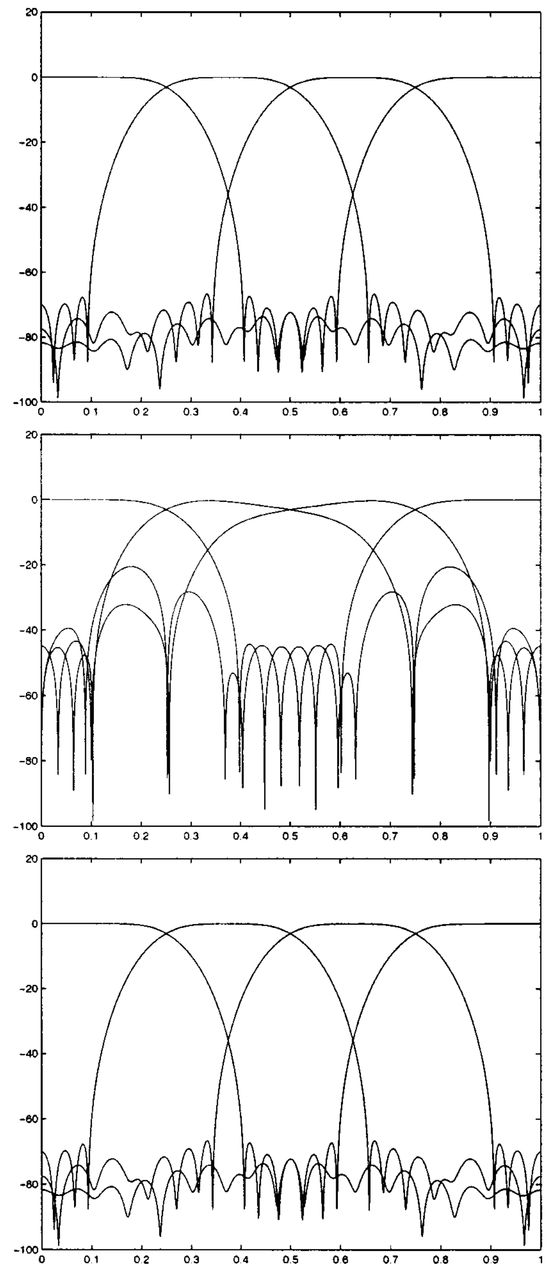

Fig. 5. Magnitude responses of four-channel paraunitary filterbanks with filter length $L = 32$. The horizontal and vertical axes represent frequency and magnitude response, respectively. Top: General paraunitary filterbank. Middle: Linear phase filterbank. Bottom: Mirror-image filterbank.

VI. CONCLUSION

In this paper, we presented a general factorization theory for M-channel causal FIR paraunitary filterbanks with constrained filter length. Based on the singular value decomposition of the coefficient matrices, any paraunitary matrix of order $K-1$, which corresponds to general paraunitary filterbanks with filters of length KM, can be factorized into $K-1$ order-one paraunitary building blocks in addition to the initial unitary matrix. Furthermore, the CS decomposition and spectral decomposition of Hermitian unitary matrices are investigated and applied to develop more efficient structures for paraunitary systems. Under the theoretical framework for the general paraunitary systems, the linear phase filterbank and pairwise mirror-image symmetry filterbank are revisited. It has been shown that the existing factorization structures of the two special paraunitary systems fall into the proposed general structures. Moreover, we obtain more efficient structures for these special paraunitary systems.

Although we have not mentioned the minimality of the structures, it is easy to verify that all the structures as shown in Fig. 2(a) through Fig. 4(b) are minimal. Moreover, the factorization theory for the paraunitary matrix with or without constraints is not restricted to the filterbank. It can be applied to wavelets and multiwavelet systems and serve as a general theory for paraunitary systems.

APPENDIX A
PROOF OF LEMMA 1

Proof: It is easy to check that with any unitary $\mathbf{V}$, $\mathbf{W}$ and angles, the matrix produced by (21) satisfies $\mathbf{A} = \mathbf{A}^{\mathrm{H}}$ and $\mathbf{A}^{\mathrm{H}}\mathbf{A} = \mathbf{I}_M$. We only need to prove the converse.

The Hermitian property of $\mathbf{A}$ implies $\mathbf{A}_{10} = \mathbf{A}_{01}^{\mathrm{H}}$, $\mathbf{A}_{00} = \mathbf{A}_{00}^{\mathrm{H}}$, and $\mathbf{A}_{11} = \mathbf{A}_{11}^{\mathrm{H}}$. The unitary property of $\mathbf{A}$ implies $\mathbf{A}_{0i}^{\mathrm{H}}\mathbf{A}_{0i} + \mathbf{A}_{1i}^{\mathrm{H}}\mathbf{A}_{1i} = \mathbf{I}_{m_i}$, $i = 0, 1$, and $\mathbf{A}_{0i}^{\mathrm{H}}\mathbf{A}_{0j} + \mathbf{A}_{1i}^{\mathrm{H}}\mathbf{A}_{1j} = \mathbf{0}$, $i \neq j$. Since $\mathbf{A}_{00}$ is Hermitian, there exists an $m_0 \times m_0$ unitary matrix $\mathbf{V}$ and a real diagonal matrix $\mathbf{C}$ so that

$$\mathbf{A}_{00} = \mathbf{V}\mathbf{C}\mathbf{V}^{\mathrm{H}}. \tag{60}$$

Substituting into the unitary condition on $\mathbf{A}_{ij}$, we have $\mathbf{A}_{10}^{\mathrm{H}}\mathbf{A}_{10} = \mathbf{V}(\mathbf{I}_{m_0} - \mathbf{C}^2)\mathbf{V}^{\mathrm{H}}$. Since $\mathbf{A}_{10}^{\mathrm{H}}\mathbf{A}_{10}$ is a non-negative definite matrix, the diagonal entries of the diagonal matrix $\mathbf{I}_{m_0} - \mathbf{C}^2$ are non-negative. Therefore, we can express $\mathbf{C}$ as $\mathbf{C} = \mathrm{diag}(\cos\theta_0, \cos\theta_1, \ldots, \cos\theta_{m_0-1})$. Let $\mathbf{S} = \mathrm{diag}(\sin\theta_0, \sin\theta_1, \ldots, \sin\theta_{m_0-1})$; then, $\mathbf{S}^2 = \mathbf{I}_{m_0} - \mathbf{C}^2$, and $\mathbf{A}_{10}^{\mathrm{H}}\mathbf{A}_{10} = (\mathbf{S}\mathbf{V}^{\mathrm{H}})^{\mathrm{H}}(\mathbf{S}\mathbf{V}^{\mathrm{H}})$. From [30], we know that there exists an $m_1 \times m_0$ matrix $\mathbf{X}$ so that $\mathbf{X}^{\mathrm{H}}\mathbf{X} = \mathbf{I}_{m_0}$ and $\mathbf{A}_{10} = \mathbf{X}\mathbf{S}\mathbf{V}^{\mathrm{H}}$. When M is even, $\mathbf{X}$ is a square unitary matrix. When M is odd, there exists a vector $\mathbf{x}$ so that $[\mathbf{x}\;\mathbf{X}]$ is a unitary matrix. Let $\mathbf{W} = \begin{cases} \mathbf{X}, & M \text{ even} \\ [\mathbf{x}\mathbf{X}], & M \text{ odd} \end{cases}$; then

$$\mathbf{A}_{10} = \mathbf{W}\begin{bmatrix} \mathbf{0} \\ \mathbf{S} \end{bmatrix}\mathbf{V}^{\mathrm{H}} \tag{61}$$

where the zero vector does not appear above $\mathbf{S}$ for even M. Without loss of generality, we assume that $\mathbf{S} = \mathrm{diag}(\mathbf{0}, \mathbf{S}_{11})$, where the diagonal entries of $\mathbf{S}_{11}$ are nonzero. In this case, $\mathbf{C} = \mathrm{diag}(\mathbf{C}_{00}, \mathbf{C}_{11})$, where $\mathbf{C}_{00}$ contains all the diagonal entries of absolute value 1 of $\mathbf{C}$. Using (60), (61), and the relation $\mathbf{A}_{01} = \mathbf{A}_{10}^{\mathrm{H}}$, we have

$$\begin{aligned}\mathbf{D}_{cs} &\triangleq \mathrm{diag}(\mathbf{V}^{\mathrm{H}}, \mathbf{W}^{\mathrm{H}}) \cdot \mathbf{A} \cdot \mathrm{diag}(\mathbf{V}, \mathbf{W}) \\ &= \begin{bmatrix} \mathbf{C}_{00} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{11} & \mathbf{0} & \mathbf{S}_{11} \\ \mathbf{0} & \mathbf{0} & \mathbf{D}_{00} & \mathbf{D}_{01} \\ \mathbf{0} & \mathbf{S}_{11} & \mathbf{D}_{10} & \mathbf{D}_{11} \end{bmatrix}\end{aligned} \tag{62}$$

where $\begin{bmatrix} \mathbf{D}_{00} & \mathbf{D}_{01} \\ \mathbf{D}_{10} & \mathbf{D}_{11} \end{bmatrix} = \mathbf{W}^{\mathrm{H}}\mathbf{A}_{11}\mathbf{W} \triangleq \mathbf{D}$. The left side of (62) is unitary, and so is the right side. The unitary property implies $\mathbf{S}_{11}\mathbf{D}_{01} = \mathbf{0}$, $\mathbf{S}_{11}\mathbf{D}_{10} = \mathbf{0}$, $\mathbf{C}_{11}\mathbf{S}_{11} + \mathbf{S}_{11}\mathbf{D}_{11} = \mathbf{0}$, and $\mathbf{D}_{00}^{\mathrm{H}}\mathbf{D}_{00} + \mathbf{D}_{10}^{\mathrm{H}}\mathbf{D}_{10} = \mathbf{I}$. Therefore, we have $\mathbf{D}_{01} = \mathbf{D}_{10} = \mathbf{0}$, $\mathbf{D}_{11} = -\mathbf{C}_{11}$, and $\mathbf{D}_{00}^{\mathrm{H}}\mathbf{D}_{00} = \mathbf{I}$. Now that $\mathbf{D}_{00}$ is Hermitian unitary, it can be expressed as $\mathbf{D}_{00} = \mathbf{U}_{00}\hat{\mathbf{C}}_{00}\mathbf{U}_{00}^{\mathrm{H}}$, where $\mathbf{U}_{00}$ is unitary, and $\hat{\mathbf{C}}_{00}$ is a real diagonal matrix with diagonal entries of absolute value 1. By replacing $\mathbf{W}$ with $\mathrm{diag}(\mathbf{U}_{00}, \mathbf{I}) \cdot \mathbf{W}$, (62) becomes

$$\mathbf{D}_{cs} = \begin{bmatrix} \mathbf{C}_{00} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{11} & \mathbf{0} & \mathbf{S}_{11} \\ \mathbf{0} & \mathbf{0} & \hat{\mathbf{C}}_{00} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_{11} & \mathbf{0} & -\mathbf{C}_{11} \end{bmatrix}. \tag{63}$$

Therefore, $\mathbf{A}$ can be expressed as (21). For real $\mathbf{A}$, the Hermitian matrices $\mathbf{A}_{00}$, $\mathbf{A}_{11}$, and $\mathbf{D}_{00}$ are real symmetric, and all the unitary matrices in the above derivation can be real orthogonal. This completes the proof. □

APPENDIX B
DECOMPOSITION OF SPECIAL SYMMETRIC ORTHOGONAL MATRICES IN SECTION IV-B

Lemma 3: Let $\mathbf{A}$ be a real even-order matrix of the form $\mathbf{A} = \begin{bmatrix} \mathbf{A}_0 & \mathbf{A}_1 \\ \mathbf{A}_1 & -\mathbf{A}_0 \end{bmatrix}$, where $\mathbf{A}_0$ and $\mathbf{A}_1$ are symmetric. $\mathbf{A}$ is orthogonal if and only if it can be expressed as

$$\mathbf{A} = \mathrm{diag}(\mathbf{V}, \mathbf{V}) \cdot \begin{bmatrix} \mathbf{C} & \mathbf{S} \\ \mathbf{S} & -\mathbf{C} \end{bmatrix} \cdot \mathrm{diag}(\mathbf{V}^{\mathrm{H}}, \mathbf{V}^{\mathrm{H}}) \tag{64}$$

where $\mathbf{V}$ is orthogonal, and $\mathbf{C}$ and $\mathbf{S}$ are diagonal matrices with entries $[\mathbf{C}]_{kk} = \cos\theta_k$ and $[\mathbf{S}]_{kk} = \sin\theta_k$.

Proof: It is easy to verify that with arbitrary orthogonal $\mathbf{V}$ and angles θ_k, the matrix $\mathbf{A}$ produced by (64) is an orthogonal matrix of the form $\mathbf{A} = \begin{bmatrix} \mathbf{A}_0 & \mathbf{A}_1 \\ \mathbf{A}_1 & -\mathbf{A}_0 \end{bmatrix}$, where $\mathbf{A}_0$ and $\mathbf{A}_1$ are symmetric. In the following, we prove the converse.

The orthogonality of $\mathbf{A}$ implies that $\mathbf{A}_1^{\mathrm{T}}\mathbf{A}_0 = \mathbf{A}_0^{\mathrm{T}}\mathbf{A}_1$. Since $\mathbf{A}_0$ and $\mathbf{A}_1$ are symmetric, $\mathbf{A}_1\mathbf{A}_0 = \mathbf{A}_0\mathbf{A}_1$. For these commuting matrices, there exists an orthogonal matrix $\mathbf{V}$ so that [31]

$$\mathbf{A}_0 = \mathbf{V}\mathbf{C}\mathbf{V}^{\mathrm{T}}, \quad \mathbf{A}_1 = \mathbf{V}\mathbf{S}\mathbf{V}^{\mathrm{T}} \tag{65}$$

where $\mathbf{C}$ and $\mathbf{S}$ are diagonal matrices containing the eigenvalues of $\mathbf{A}_0$ and $\mathbf{A}_1$. The above equation means that we can express $\mathbf{A}$ as in (64). The orthogonality of $\mathbf{A}$ also implies that $\mathbf{A}_0^{\mathrm{T}}\mathbf{A}_0 + \mathbf{A}_1^{\mathrm{T}}\mathbf{A}_1 = \mathbf{I}$. Substituting (65) into it, we have $\mathbf{C}^2 + \mathbf{S}^2 = \mathbf{I}$. Therefore, we can express $\mathbf{C}$ and $\mathbf{S}$ by angles as stated in the lemma. This completes the proof, and we have an easy corollary. □

Corollary 1: Let $\mathbf{A}$ be a real even-order matrix of the form $\mathbf{A} = \begin{bmatrix} \mathbf{A}_0 & \mathbf{A}_1 \\ \mathbf{A}_1 & -\mathbf{A}_0 \end{bmatrix}$, where $\mathbf{A}_0$ and $\mathbf{A}_1$ are symmetric. $\mathbf{A}$ is orthogonal if and only if it can be expressed as

$$\begin{aligned}\mathbf{A} = \mathrm{diag}(\mathbf{V}, \mathbf{V}) \cdot \begin{bmatrix} \check{\mathbf{C}} & \check{\mathbf{S}} \\ \check{\mathbf{S}} & -\check{\mathbf{C}} \end{bmatrix} \cdot \mathrm{diag}(\mathbf{I}_{M/2}, -\mathbf{I}_{M/2}) \\ \cdot \begin{bmatrix} \check{\mathbf{C}} & \check{\mathbf{S}} \\ \check{\mathbf{S}} & -\check{\mathbf{C}} \end{bmatrix} \cdot \mathrm{diag}(\mathbf{V}^{\mathrm{H}}, \mathbf{V}^{\mathrm{H}})\end{aligned} \tag{66}$$

where $\mathbf{V}$ is orthogonal, and $\check{\mathbf{C}}$ and $\check{\mathbf{S}}$ are diagonal with entries $[\check{\mathbf{C}}]_{kk} = \cos\alpha_k$ and $[\check{\mathbf{S}}]_{kk} = \sin\alpha_k$.

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