A General Class of Throughput Optimal Routing Policies in Multi-hop Wireless Networks

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Abstract—This paper considers the problem of routing packets across a multi-hop wireless network while ensuring throughput optimality. One of the main challenges in the design of throughput optimal routing policies is identifying appropriate and universal Lyapunov functions with negative expected drift. The few well-known throughput optimal routing policies in the literature are constructed using simple quadratic or exponential Lyapunov functions of the queue backlogs and as such they do not use any metric of closeness to the destination. Consequently, these routing policies exhibit poor delay performance under many network topologies and traffic conditions.

By considering a class of continuous, differentiable, and piece-wise quadratic Lyapunov functions, this paper provides a large class of throughput optimal routing policies. The proposed class of Lyapunov functions allow for the routing policies to control the traffic along short paths for a large portion of state-space while ensuring a negative expected drift, hence, enabling the design of routing policies with much improved delay performance. In particular, an opportunistic routing policy with congestion diversity is proved to be throughput optimal.

I. INTRODUCTION

Opportunistic routing for multi-hop wireless ad-hoc networks has seen recent research interest to overcome deficiencies of conventional routing [1]–[5]. Opportunistic routing mitigates the impact of poor wireless links by exploiting the broadcast nature of wireless transmissions and the path diversity. More precisely, the routing decisions are made in an online manner by choosing the next relay based on the actual transmission outcomes as well as a rank ordering of neighboring nodes. The authors in [5] provided a Markov decision theoretic formulation for opportunistic routing. In particular, it is shown that for any given packet and at any relaying epoch, the optimal routing decision, in the sense of minimum cost or hop-count, is to select the next relay node based on an index. This index is equal to the expected cost or hop-count of relaying the packet along the least costly or the shortest feasible path to the destination. Furthermore, this index is computable in a distributed manner and with low complexity using a time-invariant probabilistic description of wireless links and the time-invariant transmission costs or transmission times. As such, [5] provides a unifying framework for almost all versions of opportunistic routing [1]–[3], where the variations are due to the authors’ choices of costs; e.g. for ExOR [3], the cost to be minimized is the expected hop-counts (ETX).

When multiple streams of packets are to traverse the network, however, it might be necessary to route some packets along longer paths, if these paths eventually lead to links that are less congested. More precisely, and as noted in [6], [7], the above opportunistic routing schemes can potentially cause severe congestion and unbounded delays (see examples given in [7]). In other words, these routing schemes are said to fail to stabilize otherwise stabilizable traffic. In contrast, it is known that a simple routing policy, known as backpressure [8], ensures bounded expected total backlog for all stabilizable arrival rates. Most interestingly, this routing policy provides throughput optimality without knowledge of the network topology or the traffic rates. In the opportunistic context, diversity backpressure routing (DIVBAR) [6] provides an opportunistic generalization of backpressure which incorporates the wireless diversity.

Note that to ensure throughput optimality, backpressure-based policies [6], [8] do something very different from [1]–[5]; rather than any metric of closeness to the destination (or cost), they choose the receiver with the largest positive differential queue backlog (routing responsibility is retained by the transmitter if no such receiver exists). This very property of ignoring the cost to the destination, however, becomes the bane of this approach, leading to poor delay performance (see [6], [7]). In [7], the authors proposed a routing policy, known as Opportunistic Routing with Congestion Diversity (ORCD) with an improved delay performance. ORCD combines the congestion information with the shortest path calculations inherent in opportunistic routing [7]. The throughput optimality of ORCD was conjectured in [7] but was left unproven, due to the difficulty of identifying appropriate (and universal) Lyapunov functions with negative expected drift. In fact backpressure [8] and its variants [6], [9]–[11], with quadratic Lyapunov function, and randomized strategies [12] with an exponential Lyapunov function remain to be the only known throughput optimal routing policies. The strict schur-convex structure of these Lyapunov functions, however, are such that their negative drift is ensured only at the cost of potentially large delays. In this paper, we provide a large class of throughput optimal policies by considering a class of piece-wise quadratic Lyapunov functions. The proposed class of Lyapunov functions allow for the routing policies to control
the traffic along short paths for a large portion of state-space while ensuring a negative expected drift, hence, enabling the design of routing policies without many of the deficiencies of backpressure-based policies. We also specialize our result to prove the throughput optimality of ORCD (whose throughput optimality only was conjectured in [7]).

In this paper we assume each network node transmits over an orthogonal channel, so that there is no inter-channel interference. Furthermore, we assume that the network topology as well as probability of successful transmissions are fixed. These assumptions allow for a clear presentation of the routing problem and illuminate the main concepts in their simplest forms. However, we emphasize that the generalization to the networks with inter-channel interference follows directly from [6]. In [6], the price of this generalization is shown to be the centralization of the routing/scheduling globally across the network or a constant factor performance loss of the distributed variants. The generalization to the case of 1) multi-destination scenario and 2) ergodic time-varying network topology and transmission probabilities are believed to be also straightforward but remain as future areas of work.

We close this section with a note on the notations used. Let \( [x]^+ = \max\{x, 0\} \). The indicator function \( 1_{\{x\}} \) takes the value 1 whenever \( x \) occurs, and 0 otherwise. For any set \( S \), \( |S| \) denotes the cardinality of \( S \), while for any vector \( v \), \( \|v\| \) denotes the euclidean norm of \( v \). For any set \( S \), \( \text{int}(S) \) is the set of all interior points of \( S \). When dealing with a sequence of sets \( C_1, C_2, \ldots \), we define \( C^t = \cup_{j=1}^t C_j \).

II. PROBLEM FORMULATION AND OVERVIEW OF THE RESULTS

A. Problem Setup

We consider a time slotted system with slots indexed by \( t \in \{0, 1, 2, \ldots \} \) where slot \( t \) refers to the time interval \([t, t+1)\). There are \( N+1 \) nodes in the network labeled by \( \Omega := \{0, 1, \ldots, N\} \), where node 0 is assumed to be the destination.

Let random variable \( A_i(t) \) represent the amount of data that exogenously arrives to node \( i \) during time slot \( t \). Arrivals are assumed to be i.i.d. over time and bounded by a constant \( A_{\text{max}} \). Let \( \lambda_i = \mathbb{E}[A_i(t)] \) denote the exogenous arrival rate to node \( i \). We define \( \lambda = [\lambda_1, \lambda_2, \ldots, \lambda_N] \) to be the arrival rate vector. We assume packets that arrive exogenously at node \( i \) as well as packets routed to node \( i \) from other nodes are queued at node \( i \) in a buffer with infinite queuing space. Let \( Q_i(t) \) denote the queue backlog of node \( i \) at time slot \( t \). We assume any data that is successfully delivered to the destination will exit the network and hence, \( Q_0(t) = 0 \) for all time slots \( t \). We define \( Q(t) = [Q_1(t), Q_2(t), \ldots, Q_N(t)] \) to be the vector of queue backlogs of nodes \( 1, 2, \ldots, N \).

We assume each node transmits at most one packet during a single time slot. Let \( S_i(t) \) represent the (random) set of nodes that have received the packet transmitted by node \( i \) at time slot \( t \). We refer to \( S_i(t) \) as the set of potential forwarders for node \( i \). Furthermore, we assume that node \( i \) has perfect knowledge of \( S_i(t) \) and \( i \in S_i(t) \) for all time \( t \). We characterize the behavior of the wireless channel using a probabilistic local broadcast model [5]. The local broadcast model is defined using conditional probabilities \( P(S|i) := \text{Prob}(\{S_i(t) = S \text{ when } i \text{ transmits a packet at time } t\}) \), \( S \subseteq \Omega \), \( i \in \Omega \). Note that, by definition, for all \( S \neq S' \), successful reception at \( S \) and \( S' \) are mutually exclusive and \( \sum_{S \subseteq \Omega} P(S|i) = 1 \). We say node \( i \) reaches node \( j \) (we write \( i \rightarrow j \)), if there exists a set of nodes \( S \subseteq \Omega \) such that \( j \in S \) and \( P(S|i) > 0 \).

We define a routing decision \( \mu_{ij}(t) \) to be the number of packets whose relaying responsibility is shifted from node \( i \) to node \( j \) during time slot \( t \). Note that \( \mu_{ij}(t) \) forms the departure process from node \( i \), while it is an element of the endogenous arrival to node \( j \), and hence,

\[
\mu_{ij}(t) \in \{0, 1\}, \quad \mu_{ij}(t) \leq 1_{(j \in S_i(t))}, \quad \sum_{j=0}^{N} \mu_{ij}(t) \leq 1. \tag{1}
\]

If \( \mu_{ii}(t) = 1 \), then node \( i \) retains the packet for future retransmissions. Without loss of generality, we assume that after a packet is successfully received at the destination, the packet would not be (re)transmitted by any other node, i.e. \( \mu_{i0}(t) = 1 \) if 0 \( \in S_i(t) \).

For a set \( C \) of nodes, we define \( A_C(t) = \sum_{i \in C} A_i(t) \), \( Q_C(t) = \sum_{i \in C} Q_i(t) \), \( \mu_{C,in}(t) = \sum_{j \notin C} \sum_{k \in C} \mu_{jk}(t) \), and \( \mu_{C,out}(t) = \sum_{j \in C} \sum_{k \notin C} \mu_{jk}(t) \).

The selection of routing decisions together with the exogenous arrivals impact the queue backlog of node \( i \), \( i \in \Omega \), in the following manner:

\[
Q_i(t+1) = [Q_i(t) - \sum_{j \in \Omega} \mu_{ij}(t)]^+ + \sum_{j \in \Omega} \mu_{jk}(t)1_{(Q_j(t) \geq \mu_{jk}(t))} + A_i(t). \tag{2}
\]

Definition. A routing policy is a collection of routing decisions \( \bigcup_{i,j \in \Omega} \bigcup_{t=0}^{\infty} \{\mu_{ij}(t)\} \) where for all \( i,j \in \Omega \) and \( \theta \in \{0, 1\} \), the decision \( \{\mu_{ij}(t) = \theta\} \) belongs to the space generated by \( \bigcup_{i,j \in \Omega} \{Q_i(0), S_0(0), \mu_{ij}(0), \ldots, Q_i(t-1), S_i(t-1), \mu_{ij}(t-1), Q_i(t), S_i(t)\} \).

Definition. A routing policy \( \Pi \) is said to stabilize the network if the time average queue backlog of all nodes remain finite when packets are routed according to \( \Pi \). The stability region of the network (denoted by \( \mathcal{S} \)) is the set of all arrival rate vectors \( \lambda \) for which there exists a routing policy that stabilizes the network.

Definition. A routing policy is said to be throughput optimal if it stabilizes the network for all arrival rate vectors that belong to the interior of the stability region.

**Fact 1** (Corollary 1 in [6]). An arrival rate vector \( \lambda \) is in the stability region \( \mathcal{S} \) if and only if there exists a stationary randomized routing policy that makes routing decisions \( \{\mu_{ij}(t)\}_{i,j \in \Omega} \) solely based on the collection of potential forwarders at time \( t \), \( \{S_i(t)\}_{i \in \Omega} \), and for which

\[
\mathbb{E}\left[ \sum_{j \in \Omega} \hat{\mu}_{ij}(t) - \sum_{i \in \Omega} \hat{\mu}_{ik}(t) \right] \geq \lambda_k.
\]

In this paper we are interested in a class of routing policies which are throughput optimal but do not require knowledge of arrival rates.
B. Priority-Based Routing

In this section, we introduce the class of priority-based routing policies. To define the priority-based routing policy we need the following definitions.

**Definition.** A rank ordering \( R = (C_1, C_2, \ldots, C_M) \) is an ordered list of non-empty sets \( C_1, C_2, \ldots, C_M \) (1 \( \leq \) \( M \) \( \leq \) \( N \)), referred to as ranking classes, that make up a partition of \( \{1, 2, \ldots, N\} \) (all nodes except the destination node), i.e.

\[
\bigcup_{i=1}^{M} C_i = \{1, 2, \ldots, N\} \quad \text{and} \quad C_i \cap C_j = \emptyset, \quad i \neq j.
\]

We denote the set of all possible rank orderings of \( \{1, 2, \ldots, N\} \) by \( \mathcal{R} \). Note that when \( C_i \)'s are singleton, \( R \) reduces to a simple permutation of the nodes \( \{1, 2, \ldots, N\} \). Given a rank ordering \( R = (C_1, C_2, \ldots, C_M) \), we write \( a \prec_R b \) to indicate that node \( a \in C_i \) has a lower rank than \( b \in C_j \), \( i < j \). And we write \( a \preceq_R b \), if \( a \prec_R b \) or \( a, b \in C_i \) for some \( i \).

**Definition.** A rank ordering \( R \) is referred to as path-connected if for each node \( i \) there exist distinct nodes \( j_1, j_2, \ldots, j_l \) such that \( i \rightarrow j_1 \rightarrow j_2 \rightarrow \ldots \rightarrow j_l \rightarrow 0 \) and \( j_n \preceq_R i \) for all \( 1 \leq n \leq l \). The set of all path-connected rank orderings is denoted by \( \mathcal{R}_c \), \( \mathcal{R}_c \subseteq \mathcal{R} \).

**Definition.** A priority-based routing policy \( \Pi_R(t) \) is a routing policy under which node \( i \), at time \( t \) and among its set of potential forwarders \( S_i(t) \), selects a node with the lowest rank according to \( R(t) \) in \( \mathcal{R}_c \). In other words, under \( \Pi_R(t) \), \( \mu_{ij}(t) = 1 \), only when \( j \in S_i(t) \) and \( j \preceq_R k \) for all \( k \in S_i(t) \).

Next we give a few definitions which allow us to compare rank orderings \( R \) and \( R' \):

**Definition.** Let \( R = (C_1, C_2, \ldots, C_M) \) and \( R' = (C'_1, C'_2, \ldots, C'_M) \). Define a mismatch \( m : \mathcal{R} \times \mathcal{R} \rightarrow \mathbb{N} \) as

\[
m(R, R') = \min \{ i \in \mathbb{N} : C_i \neq C_i' \}.
\]

For two rank orderings \( R \) and \( R' \), \( m(R, R') \) compares ranking classes of \( R \) and \( R' \) from low to high and determines the index of the first ranking class in which they differ.

**Definition.** Given two rank orderings \( R \) and \( R' \), we say \( R' \) is a refinement of \( R \) (and \( R \) is a confinement of \( R' \)) if \( i \prec_R j \) implies that \( i \prec_R' j \) for any \( i, j \in \Omega \).

**Definition.** Given two rank orderings \( R = (C_1, C_2, \ldots, C_M) \) and \( R' = (C'_1, C'_2, \ldots, C'_{M+1}) \), we say \( R' \) is a one-step refinement of \( R \) (and \( R \) is a one-step confinement of \( R' \)) with regard to ranking class \( C_i \) if

\[
\begin{align*}
C_k &= C'_k & \text{if } 1 \leq k \leq i - 1, \\
C_i &= C'_i \cup C'_{i+1}, \\
C_k &= C'_{k+1} & \text{if } i + 1 \leq k \leq M.
\end{align*}
\]

The set of all one-step refinements and one-step confinements of \( R \) is referred to as adjacency of \( R \) and is denoted by \( \mathcal{A}(R) \). Similarly, we denote the set of all path-connected one-step refinements and confinements of \( R \) by \( \mathcal{A}_c(R) \).

Next we introduce a class of priority-based routing policies under which \( R(t) \) is chosen as a time-invariant function of \( Q(t) \) and the network topology, i.e. for any given network topology, there exists a function \( \pi : \mathbb{R}_+^N \rightarrow \mathcal{R}_c \) such that \( R(t) = \pi(Q(t)) \). In Section II-D we proceed to establish the throughput optimality of this class of routing policies.

C. Path-Connected f-policy

In this section, we introduce a class of priority-based routing policies each of which is associated with a bivariate function \( f \), hence referred to as an \( f \)-policy. Each such policy partitions the space of queue backlogs, \( \mathbb{R}_+^N \), into \( \mathcal{R}_c \) routing decision cones. The specific shape of each cone is dictated by the corresponding function \( f \) and to each cone a unique path-connected rank ordering of nodes \( R \in \mathcal{R}_c \) is assigned. Now define the mapping \( \pi_f : \mathbb{R}_+^N \rightarrow \mathcal{R}_c \) such that at any time \( t \) and for all \( Q(t) \) in the cone associated with \( R \), \( \pi_f(Q(t)) = R \). In order to give the precise description of path-connected \( f \)-policy, we need the following definitions.

**Definition.** Given a bivariate function \( f \), a penalty function \( \Lambda_f \) is defined on backlog vector \( Q \in \mathbb{R}_+^N \), rank ordering \( R = (C_1, C_2, \ldots, C_M) \in \mathcal{R} \), and natural number \( n, n \leq M \):

\[
\Lambda_f(Q, R, n) = \sum_{i=1}^{n} f(|C_i'|, |C_i|) Q_{C_i}.
\]

**Definition.** Consider two rank orderings \( R \) and \( R' \) and a bivariate function \( f \). We say \( R \) penalizes \( Q \) less than \( R' \) and write \( R \prec Q R' \) if

- \( \Lambda_f(Q, R, m(R, R')) < \Lambda_f(Q, R', m(R, R')) \),
- \( \Lambda_f(Q, R, m(R, R')) = \Lambda_f(Q, R', m(R, R')) \) and \( R \) is a one-step refinement of \( R' \).

Let \( D_f^+(R) \), \( R \in \mathcal{R}_c \), be a subset of \( \mathbb{R}_+^N \) such that for all \( Q \in D_f^+(R) \) and all \( R' \in \mathcal{A}_c(R) \), \( R \prec Q R' \), i.e.

\[
D_f^+(R) = \left\{ Q \in \mathbb{R}_+^N : R \prec Q R' \text{ for all } R' \in \mathcal{A}_c(R) \right\}.
\]

**Remark.** Let \( R \) and \( R' \) be two rank orderings and let \( \eta \in \mathbb{R}_+^N \) be a constant. If \( R \prec Q R' \) then \( R \prec Q R' \). In other words, \( D_f^+(R) \) is a cone in \( \mathbb{R}_+^N \).

**Remark.** Due to the linearity of \( \Lambda_f(., R, n) \) and finiteness of \( \mathcal{A}_c(R) \), the boundaries of the cone corresponding to rank ordering \( R \) consists of finitely many hyperplanes of the form

\[
\Lambda_f(Q, R, m(R', R')) = \Lambda_f(Q, R', m(R, R')), \quad \text{where } R' \in \mathcal{A}_c(R).
\]

**Lemma 1.** Let bivariate function \( f \) satisfy the following two conditions:

- (C1) For all \( m \geq 0 \) and \( n_1, n_2 > 0 \)

\[
\frac{1}{f(m, n_1 + n_2)} = \frac{1}{f(m, n_2)} + \frac{1}{f(m + n_1, n_2)}.
\]

- (C2) For all \( m \geq 0 \) and \( n_1, n_2 > 0 \)

\[
f(m, n_1) \geq f(m + n_1, n_2).
\]

Then for any \( Q \in \mathbb{R}_+^N \), there exists a unique \( R \in \mathcal{R}_c \) such that \( Q \in D_f^+(R) \).

**Proof:** The proof is given in [13].
Remark. By Lemma 1, \( \{ D_f^j(R) \}_{R \in \mathcal{R}_c} \) forms a partition of \( \mathbb{R}_+^N \). Hence, it is meaningful to define a function \( \pi_f^j : \mathbb{R}_+^N \to \mathcal{R}_c \) such that \( \pi_f^j(Q) = R \iff Q \in D_f^j(R) \).

Now we are ready to provide the precise definition of path-connected \( f \)-policy as discussed earlier.

**Definition.** Path-connected \( f \)-policy is a priority-based routing policy \( \Pi_{\{R(t)\}} \) where \( R(t) = \pi_f^j(Q(t)) \).

**Example 1.** Consider a network of three nodes as given in Fig. 1(a). Let \( \mathcal{R}_c \) be the set of all path-connected rank orderings of \( \{1, 2\} \), and \( f(m, n) = \frac{1}{3^{m(3^n-1)}} \) (it is easy to show that function \( f \) satisfies (C1) and (C2)). Fig. 1(b) shows the structure of the cones \( \{D_f^j(R)\}_{R \in \mathcal{R}_c} \).

Next we state the main results of this paper.

**D. Overview of the Results**

**Theorem 1.** Let \( f \) be a bivariate function that satisfies conditions (C1) and (C2). Then the associated path-connected \( f \)-policy is throughput optimal.

Theorem 1 introduces a new class of throughput optimal routing policies. The sketch of the proof is provided in Section III, with the details provided in the appendix.

**Definition.** Let \( \Pi_{\{R(t)\}} \) and \( \Pi'_{\{R'(t)\}} \) be two priority-based routings. We say \( \Pi'_{\{R'(t)\}} \) respects \( \Pi_{\{R(t)\}} \) if \( R'(t) \) is a refinement of \( R(t) \) for all time slots \( t \).

**Theorem 2.** Suppose \( \Pi_{\{R(t)\}} \) is a priority-based routing policy that is throughput optimal. Any priority-based routing policy that respects \( \Pi_{\{R(t)\}} \) is also throughput optimal.

Note that Theorem 2 enables the proof of throughput optimality of specific routing policies. For example, in Section IV, Theorems 1 and 2 are used to prove the throughput optimality of a known routing policy, ORCD [7]. The proof of Theorem 2 is fairly straightforward and is given in Appendix D.

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III. THROUGHPUT OPTIMALITY OF PATH-CONNECTED \( f \)-POLICY

In this section, we assume that routing decisions \( \{ \mu_{ij}^*(t) \}_{i, j \in \mathcal{O}} \) are made under a path-connected \( f \)-policy for which \( f \) is a bivariate function satisfying conditions (C1) and (C2). In this setting we prove that path-connected \( f \)-policy is throughput optimal. The proof is based on the following corollary to Foster-Lyapunov Theorem.

**Fact 2** (Lemma 4.1 in [14]). Let \( L^* : \mathbb{R}_+^N \to \mathbb{R}_+ \) be a Lyapunov function. If there exist constants \( B > 0 \), \( \epsilon > 0 \), such that for all time slots \( t \) we have:

\[
\mathbb{E} [L^*(Q(t+1)) - L^*(Q(t))] \leq B - \epsilon \sum_{k=1}^N Q_k(t),
\]

then the network is stable, i.e. the time average queue backlog of all nodes remain finite.

To prove Theorem 1, we identify a class of Lyapunov functions that under the corresponding path-connected \( f \)-policy satisfy the conditions of Fact 2 for all arrival rate vectors \( \lambda \in \text{int}(\mathcal{S}) \). In particular, we construct a piecewise Lyapunov function, \( L_f^* : \mathbb{R}_+^N \to \mathbb{R}_+ \), by assigning to each cone \( D_f^j(R) \), \( R = (C_1, C_2, \ldots, C_M) \in \mathcal{R}_c \), a quadratic function of the queue backlogs:

\[
L_f(Q, R) = \sum_{i=1}^M f(|C_i|)Q_i^2.
\]

Since the collection of cones forms a partition of \( \mathbb{R}_+^N \), we can
combine the above quadratic functions to arrive at a piecewise quadratic function
\[ L_f^*(Q) = L_f(Q, \pi_f^*(Q)) = \sum_{R \in \mathcal{R}_e} L_f(Q, R) 1_{Q \in D_f^*(R)}. \] (4)

**Lemma 2.** $L_f^*(\cdot)$ is continuous and differentiable.

Next we provide the main steps in showing $L_f^*$ has a negative expected drift.

Let us consider the Lyapunov drift when $Q(t) \in D_f^*(R)$ for some $R = (C_1, C_2, \ldots, C_M) \in \mathcal{R}_e$. By Lemma 2, $L_f^*(\cdot)$ is continuous and differentiable. Thus, we can write $L_f^*(Q(t+1))$ in terms of its first-order Taylor expansion around $L_f^*(Q(t))$ and we obtain
\[
L_f^*(Q(t+1)) - L_f^*(Q(t)) = (Q(t+1) - Q(t)) \cdot \nabla L_f^*(Q(t)) + o(\|Q(t+1) - Q(t)\|)
\]
\[
= \sum_{i=1}^{M} f(|C_i|^1, |C_i|)2Q_{C_i}(t)(Q_{C_i}(t+1) - Q_{C_i}(t))
+ o(\|Q(t+1) - Q(t)\|)
\]
\[
= \sum_{i=1}^{M} f(|C_i|^1, |C_i|) [Q_{C_i}(t+1) - Q_{C_i}(t)]^2
+ o(\|Q(t+1) - Q(t)\|)
\]
\[
= \sum_{i=1}^{M} f(|C_i|^1, |C_i|) [Q_{C_i}(t+1) - Q_{C_i}(t)]
+ o(\|Q(t+1) - Q(t)\|).
\] (5)

**Lemma 3.** Let $R = (C_1, C_2, \ldots, C_M) \in \mathcal{R}_e$ and $Q(t) \in D_f^*(R)$. We have
\[
Q_{C_i}(t+1) - Q_{C_i}(t) \leq \beta f - 2QC_i(t)(\mu_{\ast C_i, out}(t) - \mu_{\ast C_i, in}(t) - AC_i(t)),
\]
where $\beta$ is a constant bounded real number.

Now taking expectation from both sides of (5) and using Lemma 3 we obtain,
\[
\mathbb{E}[L_f^*(Q(t+1)) - L_f^*(Q(t))|Q(t)]
\leq B_f - 2 \sum_{i=1}^{M} f(|C_i|^1, |C_i|)Q_{C_i}(t)E[\mu_{\ast C_i, out}(t) - \mu_{\ast C_i, in}(t) - AC_i(t)]
+ o(\|Q(t+1) - Q(t)\|),
\] (6)
where $B_f$ is a constant bounded real number.

Lemma 4 below shows that under a path-connected $f$-policy the negative drift term in (6) is bounded by the negative drift under any other set of routing decisions, including the stabilizing randomized rule $\{\tilde{\mu}_{ij}(t)\}_{i,j \in \Omega}$ given in Fact 1.

**Lemma 4.** Let $R = (C_1, C_2, \ldots, C_M) \in \mathcal{R}_e$ and $Q(t) \in D_f^*(R)$. For any collection of routing decisions $\{\mu_{ij}(t)\}_{i,j \in \Omega}$, we have
\[
\sum_{i=1}^{M} f(|C_i|^1, |C_i|)Q_{C_i}(t)(\mu_{\ast C_i, out}(t) - \mu_{\ast C_i, in}(t))
\geq \sum_{i=1}^{M} f(|C_i|^1, |C_i|)Q_{C_i}(t)(\mu_{\ast C_i, out}(t) - \mu_{\ast C_i, in}(t)).
\] (7)

However, since $\lambda \in int(\mathbb{S})$, there exists a positive vector $\epsilon$ (vector of length $N$ with all elements equal to $\epsilon$, $\epsilon > 0$) such that $\lambda + \epsilon \in \mathbb{S}$. Thus, from Fact 1
\[
\mathbb{E}[\tilde{\mu}_{C_i, out}(t) - \mu_{C_i, in}(t) - AC_i(t)|Q(t)] \geq \epsilon.
\] (8)

Combining (7) and (8) with (6), we have
\[
\mathbb{E}[L_f^*(Q(t+1)) - L_f^*(Q(t))|Q(t)]
\leq B_f - 2 \epsilon \sum_{i=1}^{M} f(|C_i|^1, |C_i|)Q_{C_i}(t) + o(\|Q(t+1) - Q(t)\|).
\]

Now the assertion of the theorem follows from $f(0, |C_i|) \geq f(|C_i|^1, |C_i|) \geq \cdots \geq f(|C_M|^1, |C_M|) \geq f(N, 1)$, and by letting $B'_f = B_f + o(\|Q(t+1) - Q(t)\|)$ and $\epsilon' = 2\epsilon f(N, 1)$.

**IV. PATH-CONNECTED $f$-POLICY AND THE DESIGN OF THROUGHPUT OPTIMAL ROUTING POLICIES**

In this section, we first give a brief description of a congestion-based routing policy, known as opportunistic routing with congestion diversity (ORCD) [7]. Then we establish the throughput optimality of ORCD using Theorems 1 and 2.

In [7], ORCD was introduced as an alternative to backpressure routing to improve the delay performance. However, the throughput optimality of ORCD was left as a conjecture. ORCD is a priority-based routing policy $\Pi_{\text{ORCD}}(t)$ in which nodes are ordered according to a cost measure of congestion “down the stream” from each node $i$ denoted by $V_i(t)$. In other words, $i \prec_{\text{ORCD}} j$ if $V_i(t) < V_j(t)$. The congestion cost measures for nodes $i \in \Omega$ at time $t$, $V_i(t)$’s, form a vector $[V_0(t), V_1(t), \ldots, V_N(t)]$ that satisfies the following fixed point equation:
\[
V_0(t) = 0,
V_i(t) = Q_i(t) + \sum_{S \subseteq \Omega} P(S|i) \min_{j \in S} V_j(t).
\] (9)

Here, we prove the throughput optimality of ORCD by showing that ORCD respects path-connected $f$-policy corresponding to any bivariate function $f$ that satisfies condition (C1) and for all $m \geq 0$ and $n_1, n_2 > 0$
\[
\frac{f(m+n_1, n_2)}{f(m+n_2, n_1)} \geq \frac{1}{p_{min}},
\] (11)
where $p_{min} = \min \{P(S|i) : i \in \Omega, S \subseteq \Omega, P(S|i) > 0\}$. Note that, for instance, function $f(m,n) = K = \frac{1}{(m+n_1)(m+n_2)}$.

K $\geq 1 + \frac{1}{p_{min}}$, is such a function. In other words, we show that ORCD respects the path-connected $f$-policy for all such $f$. Mathematically, for all $j, k \in \Omega$ such that $j \prec \pi_f^*(Q(t)) k$, then $j \prec_{\text{ORCD}} k$ as well. Let $\pi_f^*(Q(t)) = (C_1, C_2, \ldots, C_M) \in \mathcal{R}_e$, and let $i \in C_i$ and $j \in C_{i}$. We consider two cases:

**Case I.** Node $k$ reaches a node in $C_{i-1} \cup \{0\}$. In such a case, we need to show
\[
V_k(t) \geq Q_k(t) > \frac{Q_{C_i-1}(t)}{p_{min}} \geq V_j(t).
\] (12)
The first inequality in (12) is immediate from (36). The second and third inequality follow from the arguments below.

Lemma 7 in the appendix implies the second inequality in (12), i.e.

\[ Q_k(t) > \frac{f(0, |C^{-1}|)Q_{C_i}}{f(|C^{-1}|, 1)} Q_{C_{i-1}}(t) \geq \frac{Q_{C_{i-1}}(t)}{p_{\min}}. \quad (13) \]

On the other hand, since \( \pi_t^f(Q(t)) \) is path-connected, there exist distinct intermediate nodes \( j_1, j_2, \ldots, j_l \in C^{-1} \) such that \( j \rightarrow j_1 \rightarrow j_2 \rightarrow \ldots \rightarrow j_l \rightarrow 0 \). Using Lemma 8 in the appendix recursively and noting that \( V_0(t) = 0 \), we have the following upper bound of \( V_j(t) \).

\[ V_j(t) \leq \frac{Q_j(t)}{p_{\min}} + \frac{Q_{j_1}(t)}{p_{\min}} + \ldots + \frac{Q_{j_l}(t)}{p_{\min}} \leq \frac{Q_{C_{i-1}}(t)}{p_{\min}}. \quad (14) \]

which gives the last inequality in (12).

**Case II.** Node \( k \) does not reach any node in \( C^{i-1} \cup \{0\} \). Let \( C_i \) be the set of nodes in \( C_i \) that reach a node in \( C^{i-1} \cup \{0\} \). All the paths from node \( k \) to the destination are through the nodes in \( C_i \), hence, \( V_k(t) \geq \min_{m \in C_i} V_m(t) \). However, from Case I, for each node \( j \in C^{i-1} \) and \( m \in C_i \), \( V_m(t) \geq V_j(t) \). This completes the proof.

V. DISCUSSION AND FUTURE WORK

In this paper, we provided a large class of throughput optimal policies by considering a class of piece-wise quadratic Lyapunov functions. We also specialized our result to prove the throughput optimality of a known routing policy, ORCD. The delay performance improvements of ORCD, reported in [7], shed light on the importance of the path-connected structure of \( f \)-policy. In a parallel area of research, we have used the insight obtained by considering a path-connected \( f \)-policy to design throughput optimal policies with low overhead and complexity [15]. For instance, an interesting research question involves the throughput and delay performance of distributed and low-complexity variants of ORCD [15].

In this paper we considered a single destination scenario and we assumed that the network topology as well as probability of successful transmissions were fixed. The generalization to the case of 1) multi-destination scenario and 2) ergodic time-varying network topology and transmission probabilities are believed to be also straightforward but remain as future areas of work.

APPENDIX

A. Preliminary Lemmas

In this appendix, we provide some preliminary lemmas. These lemmas are technical and only helpful in proving the main lemmas of the paper, i.e. Lemmas 2-4.

**Lemma 5.** Let \( R = (C_1, C_2, \ldots, C_M) \) and \( R' = (C_1, C_{i-1}, C_i \cup C_{i+1}, C_{i+2}, \ldots, C_M) \).

- If \( R \leq Q \), then
  \[ f(|C^{-1}|, |C_i|)Q_{C_i} \leq f(|C^{-1}|, |C_i| + |C_{i+1}|)(Q_{C_i} + Q_{C_{i+1}}) \leq f(|C^{-1}|, |C_{i+1}|)(Q_{C_{i+1}}) \]

- If \( R' \leq Q \), then
  \[ \frac{f(|C^{-1}|, |C_i|)Q_{C_i}}{f(|C^{-1}|, 1)} Q_{C_{i-1}} \]

**Proof:** Suppose \( R \leq Q \), then

\[ f(|C^{-1}|, |C_i|)Q_{C_i} > f(|C^{-1}|, |C_i| + |C_{i+1}|)(Q_{C_i} + Q_{C_{i+1}}) > f(|C^{-1}|, |C_{i+1}|)Q_{C_{i+1}}. \]

**Proof:** Suppose \( R \leq Q \), then

\[ \frac{f(|C^{-1}|, |C_i|)Q_{C_i}}{f(|C^{-1}|, 1)} Q_{C_{i-1}} \leq \frac{f(|C^{-1}|, |C_i| + |C_{i+1}|)(Q_{C_i} + Q_{C_{i+1}})}{f(|C^{-1}|, 1)}. \]

Using (16) and property (C1) of function \( f \), however,

\[ \frac{1}{f(|C^{-1}|, |C_i| + |C_{i+1}|)Q_{C_{i+1}}} \]

\[ = \frac{1}{f(|C^{-1}|, |C_i|)Q_{C_{i+1}} + \frac{1}{f(|C^{-1}|, |C_i| + |C_{i+1}|)Q_{C_{i+1}}}} \]

\[ \geq \left( \frac{1}{f(|C^{-1}|, |C_i| + |C_{i+1}|)} - \frac{1}{f(|C^{-1}|, |C_i|)} \right) Q_{C_i} \]

\[ + \frac{1}{f(|C^{-1}|, |C_i| + |C_{i+1}|)Q_{C_{i+1}}} \]

\[ = \frac{1}{f(|C^{-1}|, |C_i| + |C_{i+1}|)}(Q_{C_i} + Q_{C_{i+1}}). \]

Combining (16) and (17) completes the proof for the case \( R \leq Q \). The proof for the second case follows (16) and (17) identically.

**Lemma 6.** Let \( R = (C_1, C_2, \ldots, C_M) \in R_e \) and \( Q \in D_f(R) \). Then for \( i = 1, 2, \ldots, M - 1 \),

\[ f(|C^{-1}|, |C_i|)Q_{C_i} \leq f(|C^{-1}|, |C_{i+1}|)Q_{C_{i+1}}. \]

**Proof:** For all \( 1 \leq i \leq M - 1 \), \( R'_i = (C_1, \ldots, C_{i-1}, C_i \cup C_{i+1}, C_{i+2}, \ldots, C_M) \) is a one-step confinement of \( R \). Note that if \( R \in R_e \), then \( R'_i \in R_e \). Now, since \( Q \in D_f(R) \), we have \( R \leq Q \), and from Lemma 5, we have the assertion of the Lemma.

**Lemma 7.** Let \( R = (C_1, C_2, \ldots, C_M) \in R_e \) and \( Q \in D_f(R) \). For any node \( k \) in ranking class \( C_i \) that reaches a node in \( C^{i-1} \cup \{0\} \),

\[ Q_k > \frac{f(0, |C^{-1}|)}{f(|C^{-1}|, 1)} Q_{C^{i-1}}. \]

**Proof:** Consider \( R' = (C_1, \ldots, C_{i-1}, \{k\}, C_i - \{k\}, C_{i+1}, \ldots, C_M) \). Note that \( R' \) is path-connected since \( R \) is path-connected and node \( k \) reaches a node in \( C^{i-1} \cup \{0\} \). Since \( Q \in D_f(R) \), we have \( R \leq Q \), which together with Lemma 5 gives,

\[ f(|C^{-1}|, |C_i|)Q_{C_i} > f(|C^{-1}|, 1)Q_{C_{i+1}}. \]

On the other hand and since \( Q \in D_f(R) \), Lemma 6 implies that

\[ \frac{f(|C^{-1}|, |C_i|)Q_{C_i}}{f(|C^{-1}|, |C_{i+1}|)} \geq Q_{C_i} \quad j = 1, 2, \ldots, i - 1. \]
Summing over \( j = 1, 2, \ldots, i - 1 \) yields
\[
 f(|C^{i-1}|, |C_i|) \left( \sum_{j=1}^{i-1} \frac{1}{f(|C^{j-1}|, |C_j|)} \right) Q_{C_i} \geq Q_{C^{i-1}}. \tag{20}
\]
However, condition (C1) implies that
\[
 \sum_{j=1}^{i-1} \frac{1}{f(|C^{j-1}|, |C_j|)} = \sum_{j=1}^{i-1} \frac{1}{f(|C^{j-1}|, |C_j|)} = \frac{1}{f(0, |C^{i-1}|)}. \tag{21}
\]
Combining (20) and (21), we obtain
\[
 Q_{C_i} \geq \frac{f(0, |C^{i-1}|)}{f(|C^{i-1}|, |C_i|)} Q_{C^{i-1}},
\]
which together with (18) completes the proof. \( \blacksquare \)

B. Proof of Lemma 2

**Lemma 2.** \( L_f^*(\cdot) \) is continuous and differentiable.

**Proof:** For all \( R \in \mathcal{R}_c \), \( L_f(\cdot, R) \) is a simple quadratic function in \( Q \). Hence, to prove continuity and differentiability of \( L_f^*(\cdot) \), it suffices to show that \( L_f^*(\cdot) \) is continuous and differentiable at any \( Q \) on the hyperplanes separating routing decision cones \( \{ D_f(R) \}_{R \in \mathcal{R}_c} \).

Let \( R = (C_1, \ldots, C_i, C_{i+1}, \ldots, C_M) \in \mathcal{R}_c \) and \( R' = (C_1, \ldots, C_{i-1}, C_i \cup C_{i+1}, C_{i+2}, \ldots, C_M) \in \mathcal{A}_c(R) \). The hyperplane separating \( D_f^i(R) \) and \( D_f^i(R') \) is given by \( \Lambda_f(Q, R, i) = \Lambda_f(Q, R', i) \). From Lemma 5, this hyperplane can be written as
\[
 f(|C^{i-1}|, |C_i|)Q_{C_i} = f(|C^{i-1}|, |C_i|+|C_{i+1}|)(Q_{C_i} + Q_{C_{i+1}}) = f(|C^i|, |C_{i+1}|)Q_{C_{i+1}}. \tag{23}
\]
On one side of this hyperplane, \( L_f^*(\cdot) = L_f(\cdot, R) \), and on the other side, \( L_f^*(\cdot) = L_f(\cdot, R') \). For any \( Q \) on this hyperplane,
\[
 L_f(Q, R) - L_f(Q, R') = f(|C^{i-1}|, |C_i|)Q_{C_i}^* + f(|C^i|, |C_{i+1}|)Q_{C_{i+1}}^* - f(|C^{i-1}|, |C_i|+|C_{i+1}|)(Q_{C_i} + Q_{C_{i+1}})^2 = 0, \tag{24}
\]
where the last equality follows from (23). Equation (24) implies that \( L_f^*(\cdot) \) is continuous on the hyperplane separating \( D_f^i(R) \) and \( D_f^i(R') \).

Similarly, to prove the differentiability of \( L_f^*(\cdot) \), we have to show that \( L_f(\cdot, R) \) and \( L_f(\cdot, R') \) have same partial derivatives at any \( Q \) on the hyperplane separating \( D_f^i(R) \) and \( D_f^i(R') \). We have:

- For \( k \in C_j, j = 1, 2, \ldots, M, \)
  \[
  \frac{\partial L_f(Q, R)}{\partial k} = 2f(|C^{j-1}|, |C_j|)Q_{C_j}, \tag{25}
  \]
- for \( k \in C_j, j \neq i, i+1, \)
  \[
  \frac{\partial L_f(Q, R')}{\partial k} = 2f(|C^{j-1}|, |C_j|)Q_{C_j}, \tag{26}
  \]
- and for \( k \in C_i \cup C_{i+1}, \)
  \[
  \frac{\partial L_f(Q, R')}{\partial k} = 2f(|C^{j-1}|, |C_i|+|C_{i+1}|)(Q_{C_i} + Q_{C_{i+1}}). \tag{27}
  \]

From (23) and (25)-(27), we have
\[
 \nabla L_f(Q, R) = \nabla L_f(Q, R'). \tag{28}
\]

C. Proof of Lemmas 3 and 4

In this appendix we prove the main steps in establishing the negative expected drift in \( L_f \) under the path-connected \( f \)-policy.

**Lemma 3.** Let \( R = (C_1, C_2, \ldots, C_M) \in \mathcal{R}_c \) and \( Q(t) \in D_f^i(R) \). We have
\[
 Q_{C_i}^2(t+1) - Q_{C_i}^2(t) \leq \beta_f - 2Q_{C_i}(t)(\mu_{C_i, out}^*(t) - \mu_{C_i, in}^*(t) - A_{C_i}(t)),
\]
where \( \beta_f \) is a constant bounded real number.

**Proof:** For all \( C_i \), if \( Q_{C_i} \geq \frac{1}{f(|C^{i-1}|, |C_i|)} \), then (18) implies that for any node \( k \in C_i \) such that it reaches a node in \( C^{i-1} \), \( Q_k \geq \frac{f(|C^{i-1}|, |C_i|)}{f(|C^{i-1}|, |C_i|)} Q_{C_i} \geq 1. \)

Let
\[
 \alpha = \max_{0 \leq m \leq N} \max_{0 \leq n \leq N} \frac{f(m,1)}{f(m,m)},
\]
If \( Q_{C_i}(t) \geq \alpha \), then using (2) we obtain
\[
 Q_{C_i}(t+1) \leq Q_{C_i}(t) - \mu_{C_i, out}^*(t) + \mu_{C_i, in}^*(t) + A_{C_i}(t). \tag{29}
\]

The expression above is an inequality rather than an equality because the actual number of packets routed to \( C_i \) from other ranking classes may be less than \( \mu_{C_i, in}^*(t) \) if there are no actual packets transmitted from the nodes in those ranking classes. Note that under path-connected \( f \)-policy, the nodes in \( C_i \) that do not reach a node in \( C^{i-1} \cup \{0\} \) may route their packets only to nodes in \( C_i \).

After taking the square of both sides of (29) and appropriate arrangements of terms, we have
\[
 Q_{C_i}^2(t+1) - Q_{C_i}^2(t) \leq (\mu_{C_i, out}^*(t) - \mu_{C_i, in}^*(t) - A_{C_i}(t))^2
\]
\[
 \leq N^2 + 2N(1 + A_{\mu})^2
\]
\[
 \leq N^2 + 2N^2(1 + A_{\mu})^2
\]
When \( Q_{C_i}(t) < \alpha \), then again using (2), we have
\[
 Q_{C_i}(t+1) \leq Q_{C_i}(t) + \mu_{C_i, in}^*(t) + A_{C_i}(t). \tag{30}
\]
This implies that
\[
 Q_{C_i}^2(t+1) - Q_{C_i}^2(t) \leq (\mu_{C_i, in}^*(t) + A_{C_i}(t))^2 + 2Q_{C_i}(t)\mu_{C_i, out}^*(t)
\]
\[
 \leq N^2 + 2N(1 + A_{\mu})^2 + 2N^2
\]
Denoting \( \beta_f := N^2 + 2N(1 + A_{\mu})^2 + 2N^2 \), and (30) and (31) result in the assertion of the lemma. \( \blacksquare \)

**Lemma 4.** Let \( R = (C_1, C_2, \ldots, C_M) \in \mathcal{R}_c \) and \( Q(t) \in D_f^i(R) \). For any collection of routing decisions \( \{\mu_{ij}(t)\}_{i,j \in \Omega} \) we have
\[
 \nabla L_f(Q, R) = \nabla L_f(Q, R'). \tag{28}
\]
\[
\sum_{i=1}^{M} f(|C^{i-1}|, |C_i|)Q_{C_i}(t)(\mu_{C_i,\text{out}}(t) - \mu_{C_i,\text{in}}(t)) \\
\geq \sum_{i=1}^{M} f(|C^{i-1}|, |C_i|)Q_{C_i}(t)(\mu_{C_i,\text{out}}(t) - \mu_{C_i,\text{in}}(t)).
\] (32)

Proof: Switching the sums in the right-hand side of (32) and using (1), we have
\[
\sum_{i=1}^{M} \sum_{k \in C_i} \left( \max_{1 \leq j \leq M} \max_{t \in S_k} \left[ f(|C^{i-1}|, |C_i|)Q_{C_i}(t) \right] \right) \\
\times \left[ f(|C^{i-1}|, |C_i|)Q_{C_i}(t) - f(|C^{j-1}|, |C_j|)Q_{C_j}(t) \right] \\
+ \left[ f(|C^{i-1}|, |C_i|)Q_{C_i}(t) \right].
\] (33)

Since \( Q(t) \in D_j(R) \), by Lemma 6, for \( i = 1, 2, \ldots, M - 1 \), we have
\[
f(|C^{i-1}|, |C_i|)Q_{C_i}(t) \leq f(|C^i|, |C_{i+1}|)Q_{C_{i+1}}(t)
\] (34)

However, from (34), the upper bound in (33) is achieved under the path-connected \( f \)-policy, i.e., \( \mu_{C_i}(t) = 1 \) only when \( l \in S_k(t) \) and \( l \leq \frac{|C^i|}{m} \) for all \( m \in S_k(t) \).

\section*{D. Proof of Theorem 2}

This short appendix establishes the following:

\textbf{Theorem 2.} Suppose \( \Pi_{(R(t))} \) is a priority-based routing policy that is throughput optimal. Any priority-based routing policy that respects \( \Pi_{(R(t))} \) is also throughput optimal.

Proof: Suppose \( \Pi_{(R(t))} \) is a priority-based routing policy that respects \( \Pi_{(R(t))} \). Let \( S_1^*(t) = \{k \in S_1(t) : k \not\in R(t) \} \) and \( S_2^*(t) = \{k \in S_1(t) : k \in R(t) \} \). Since \( R(t) \) is a refinement of \( R(t) \), \( S_2^*(t) \) is a subset of \( S_1^*(t) \). By definition of the priority-based routing, \( \Pi_{(R(t))} \) selects one of the nodes in \( S_2^*(t) \) as the next forwarder. Since \( S_2^*(t) \subseteq S_1^*(t) \), this routing decision is consistent with \( \Pi_{(R(t))} \), hence, guarantees throughput optimality.

\section*{E. Proof of Lemma 8}

\textbf{Lemma 8.} For any two nodes \( a \) and \( b \), if \( a \rightarrow b \), then
\[
V_a(t) \leq \frac{Q_a(t)}{p_{\min}} + V_b(t).
\] (35)

Proof: If \( V_a(t) \leq V_b(t) \), then (35) follows trivially. Now suppose \( b \in U_a(t) \) and \( V_a(t) < V_b(t) \). Without loss of generality, let \( U_a(t) = \{a_1, a_2, \ldots, a_K\} \) such that \( V_{a_i}(t) \leq V_{a_{i+1}}(t) \) for all \( i \leq K \). We can rewrite (10) as:
\[
V_a(t) = Q_a(t) + \sum_{i=1}^{K} V_{a_i}(t) \left( \sum_{S : i = \text{min}(l, a_i \in S)} P(S|a) \right) \\
+ V_b(t) \left( \sum_{S : S \cap U_a(t) = \emptyset} P(S|a) \right).
\] (36)

where the last inequality holds because \( b \in U_a(t) \).

\section*{REFERENCES}


