Integration of Communication and Control using Discrete Time Kuramoto Models for Multivehicle Coordination over Broadcast Networks

Daniel J. Klein, Phillip Lee, Kristi A. Morgansen, and Tara Javidi

Abstract

This paper considers the integration of communication and control with respect to the task of coordinated heading control for a group of N vehicles with the energy efficiency of communications in mind. The heading control employed on each vehicle is a discretization of the well-known Kuramoto model of nonlinearly coupled oscillators over a sequence of logical graphs. Stability for both all-to-all and random one-to-all broadcasts is shown to be dependent on the coupling strength, K, and the time discretization, ΔT. For desired system performance characteristics, ΔT imposes a tight deadline by which the state information (M bits) must be propagated through the communication network. Routing optimization with respect to minimizing energy consumption is formulated considering the ΔT deadline. Due to tight time deadline a one-to-all single-hop broadcasting scheme is shown to be more energy efficient for practical choices of $M/\Delta T$. The proposed modularization is illustrated via a set of simulations where the overall communication energy to reach alignment is optimized.

Index Terms

Coordinated Control, Discrete Time Kuramoto, Network Routing, Energy Optimal Communication

I. Introduction

A fundamental challenge in designing networked control systems is that the tasks of communication and control cannot, in general, be considered decoupled from each other without loss of optimality. In fact, the optimal solution to this coupled problem can be formulated as a decentralized stochastic problem with information constraints and imperfect observations, and the solution to such a problem is known *not* to be modular. Witsenhausen's counterexample shows, in fact, that separation of estimation and controller design fails to hold even in simple settings [1]. In addition, recently, we have seen a surge of interesting results [2], [3], [4], [5], [6], [7] addressing "old" communications questions such as channel capacity and quantization in the context of stabilization and control of linear systems (see [8] and [4] for a nice summary). Our work differs from these sets of work in that we propose a practical modularization motivated by [9] to integrate practical communication questions with coordination and control of non-linear vehicles. An important feature of our work is that we relax the power constraint on the radios while minimizing energy consumption. Availability of power guarantees a suitably large channel capacity to communicate control variables. We believe that the obtained degree of modularity, despite introducing sub-optimality, can result in solutions that give insight into the problem. Following this philosophy, the work in this paper addresses the joint tasks of communication and coordinated control of an N-vehicle system where the component designs are simultaneously modularized and coupled via a common time discretization variable. Specifically, we assume that at regular finite intervals, control messages are to be generated and transmitted reliably over a network (potentially involving power/rate adaptations as well as multiple transmission over multiple hops, etc) in which the spatial location of the communication

This work was supported in part by NSF grant CMS-0238461 and in part by AFOSR grant FO54-011-0220. D. J. Klein and K. A. Morgansen are with the Department of Aeronautics and Astronautics, University of Washington, Seattle, WA 98195-2400. {djklein,morgansen}@aa.washington.edu. P. Lee and T. Javidi are with the Department of Electrical and Computer Engineering, University of California, San Diego, La Jolla, CA 92093-0407. {pslee,tjavidi}@ucsd.edu.

A preliminary, shorter version of this paper has been submitted to the 2007 IEEE Conference on Decision and Control.

nodes is dynamically changing and strongly affects the energy required to transmit information. As a particular example where this scenario arises, consider the Acoustic Seaglider autonomous underwater vehicles [10]. These vehicles are powered by buoyancy control and an internal moving mass. Between surfacing intervals, the vehicles operate by following a descent glide path, then an ascent glide path, with the only actuation actions taking place at the beginning of the descent (to decrease buoyancy and point downward along the path), and at the beginning of the ascent (to increase buoyancy and point upward). All adjustments to heading are performed at the surface. Without the use of underwater communication (but allowed to communicate at the ocean surface with an Iridium modem), these vehicles have been energy optimized such that they can be deployed for periods up to seven months without the need for battery replacement or removal from the ocean. However, a number of current applications of interest (e.g. particulate dispersion tracking in the Monterey Bay) dictate that continual underwater communication for purposes of vehicle coordination is necessary. To this end, the vehicles have been equipped with acoustic modems. The energy required to provide acoustic communication is quite large compared to the energy used for motion control and must be expended at a significantly higher rate than energy expended for motion control.

The particular coordinated control problem considered here is heading setpoint regulation for a group of N constant speed vehicles. Each vehicle is equipped with a radio with tunable transmission power of sufficiently large magnitude. The objective is to drive all headings to either an aligned state, in which all vehicles point in the same direction, or to a balanced state, in which the average of all headings is zero (meaning the spatial centroid of the group is fixed). In this paper, we propose the following modularized approach: 1) we consider a discretized Kuramato model of coordinated control over a (potentially random) sequence of one-to-all or all-to-all logical directed graphs which must provide reliable dissemination of state information every ΔT units of time; 2) we provide an energy optimal networking scheme which delivers the appropriate state information along the edges of the given logical graphs within a ΔT deadline; and 3) these results are analyzed to determine a relationship between controller choice and optimal energy consumption in this setting.

Significant attention has lately been focused on linear consensus type algorithms for multi-agent systems [11], [12]. These algorithms are designed to bring the state of each agent to a common point in the state space. For heading control, however, it is desirable to use an algorithm that naturally incorporates angle wrapping. One established approach to heading control design that admits tractable analysis is to couple heading information from other vehicles nonlinearly via sinusoids in the style of the Kuramoto oscillators that have been studied extensively in the physics, chemistry and mathematics communities [13], [14], [15]. Combining heading alignment and heading balance control with additional spacing control terms has been demonstrated in continuous time to enable vehicles to circle a fixed beacon or align with each other [16], or to track a moving target [17], [18]. The continuous-time controller derived from the Kuramoto oscillator model implicitly assumes that all-to-all communication is available at every time instant, without delay. However, in the situation where the time constants of the system dynamics are significantly faster than the communication rate (e.g. underwater vehicles with acoustic modems), this assumption does not fit with realistic communication. A more appropriate assumption in such cases is that the communication naturally occurs at discrete time instants. To retain the desirable group properties of the controller while adhering to a more realistic communication model, a discrete-time reformulation of the Kuramoto-inspired controller will be considered here. The application of interest here, coordinated control with dynamic communication, leads to a natural time discretization of the system dynamics, however, the bulk of the analysis for Kuramoto models has been made under the assumption of continuous time dynamics. A discretized form of the Kuramoto controller was first studied in [19], where the authors were able to show that many of the properties of the continuous time Kuramoto control translate to discrete time when some sufficient conditions are satisfied. Additional studies of discrete time Kuramoto systems with (possibly dynamic) incomplete connectivity have been made in [20], [21]. The work in this paper focuses not only on sufficient stability conditions on the controller, but also on system performance and energy optimal routing. The stability of the multivehicle system will be shown here to be dependent on the product of two parameters: a control gain K and the discretization interval ΔT. One consequence of this result will be that larger values of K yield improved system performance at the cost of decreasing ΔT, thus imposing a hard deadline on communication that must be met regardless of vehicle positions. For the work here, the discretization timescale is assumed to be sufficiently different from typical clock slew rates that assuming synchronized clocks, and a constant ΔT, is reasonable.

As stated above, in order to realize state feedback among the networked vehicles, the vehicles are coupled via wireless radio transmissions. In particular, our interest is to provide dynamic and reliable communication of the control variables represented as a series of logical one-to-all graphs G_k. The realization of any given graph G_k in the communication domain can be translated into a quality of service (QoS) wireless networking problem. For example, in the case of one-to-all random broadcast, a single vehicle (node) is selected at random to have its state information delivered to all other vehicles in no more than ΔT seconds. The state update message for each vehicle is assumed to be quantized with sufficient resolution[1], have the same format, and consist of an identical number of M bits. The question of interest in the networking context is the design of an energy optimal relaying/routing scheme as a function of ΔT and G_k. For instance, Fig. 1 shows two alternative schemes delivering M bits to two nodes in ΔT seconds. In this work, we show that for almost all practical scenarios (where M is sufficiently large), a single-hop long range transmission to the farthest node consumes far less energy per ΔT than more sophisticated multi-hopping and gossiping. In fact, as shown below, the energy consumption and efficiency of communication can be introduced and analyzed as a function of G_k and $M/\Delta T$. The work here in energy optimal relaying and networking can be viewed as an extension of energy optimal multi-cast tree construction, as studied in [22], [23], [24], [25]. As an important addition to these models, this work addresses the impact of a strict time deadline on the construction of an *optimal multi-cast tree*. In this regard, the energy optimal multicast problem is combined with that of *lazy scheduling* first studied in [26]. The optimality of lazy scheduling in a point-to-point setting can then be extended to the case of multiple hops. The optimality of lazy scheduling was previously extended for a stochastic setting in [27] where concepts of minimum curves and arrival curves from network calculus were used to strengthen the result. We use similar techniques to identify an optimal choice of power and rate for each hop and transmission sessions. The optimality of single-hop (star topology) transmission under strict time-deadlines (independent of topology) obtained here can be interpreted as an important extension of lazy scheduling, as will be detailed in Section III. Using these results, the total transmission energy required for the state updates of each node under the time constraint ΔT can be analyzed for single hop broadcasting and multi-hop routing and expressed in terms of the effective information rate $M/\Delta T$.

The proposed modularized solution not only enables leveraging of existing and seminal works in both control theory and wireless networking, it also allows for a cross-layer rethinking of the wireless networks used in service of control applications. In other words, the above analysis can be easily extended and used to balance the performance of the controller (decreases with increasing ΔT) and energy efficiency of communication schemes (increases with increasing ΔT). Via an integrated set of simulations, the proposed design is illustrated not only in terms of the dynamical performance of formations, e.g. rate of convergence, but also in terms of energy efficiency and communication overhead.

While the work in the paper certainly does not solve all of communication and control integration, as the material is primarily theoretical rather than experimental, we feel that it is certainly a novel contribution in the right direction. The selected non-linear unicycle system is well studied in the control theory community, in part because it is sufficiently general as to cover many non-holonomic platforms (cars, boats, planes, etc). As compared to similar papers found in the control theoretic community, this paper in particular employs a much more realistic communication model in that agents exchange information at discrete time instants only, and performance is evaluated both relative to motion control and relative to communication energy.

[1]In this work, we take the quantization issues for granted in that we assume M is chosen large enough to ensure negligible distortion. Furthermore, we assume no power constraints at the vehicles, hence ensuring the feasibility of broadcasting M bits over the network in ΔT seconds.

The remainder of the paper is organized as follows. In Section II, the discrete time Kuramoto model is discussed, and its stability is analyzed for all-to-all and random one-to-all communication. Optimal network routing is considered with respect to a fixed time deadline in Section III. Simulation results are in Section IV followed by conclusions in Section V.

II. The Discrete Time Kuramoto Model

Consider a collection of identical planar unit-speed vehicles which do not have direct actuation of lateral motion (e.g. ground vehicles, surface vessels on water, or aircraft or underwater vehicles at constant altitude/depth). The dynamics of each vehicle in the group can be described with the Frenet-Serret equations of motion,

$$\dot{r}_i = \begin{bmatrix}\cos\theta_i & \sin\theta_i\end{bmatrix}^T, \quad \dot{\theta}_i = u_i, \tag{1}$$

where, $r_i \in \mathbb{R}^2$ and $\theta_i \in \mathbb{T}$ are the position and orientation of the i^{th} agent. The control input, u_i is simply the curvature of the path taken by the vehicle, and in multivehicle applications provides state feedback coupling between the vehicles. These nonlinear dynamics are typically quite difficult to analyze, even in the case of linear control laws as typically studied in most consensus problems. One approach to control synthesis in such settings which has proved amenable to analysis is to use oscillatory coupling of the headings such as in the Kuramoto model. The classic continuous-time Kuramoto model of N nonlinearly coupled oscillators is given by

$$\dot{\theta}_i(t) = \omega_i - \frac{K}{|\mathcal{N}_i| + 1}\sum_{j\in\mathcal{N}_i}\sin\big(\theta_j(t) - \theta_i(t)\big), \tag{2}$$

where θ_i and ω_i are the phase (or heading) and natural frequency of the i^{th} agent ($i \in \mathcal{I} \equiv \{1, 2, \ldots, N\}$), $\mathcal{N}_i \subseteq \mathcal{I}\backslash i$ is the set of neighbors of agent i, and K is the coupling strength. The coupling is typically assumed to be all-to-all in which $\mathcal{N}_i = \mathcal{I}\backslash i,\ \forall i \in \mathcal{I}$. A useful metric in the study of phase coupled oscillator models is the phase centroid,

$$\mathcal{R}(\boldsymbol{\theta}(t)) \equiv \frac{1}{N}\sum_{i=1}^{N}\begin{bmatrix}\cos\theta_i(t)\\ \sin\theta_i(t)\end{bmatrix}. \tag{3}$$

The magnitude of the phase centroid, $\rho(t) \equiv \|\mathcal{R}(\boldsymbol{\theta}(t))\| \in [0, 1]$, roughly measures the order in the system. Choosing (2) as the controller for (1) with $\omega_i = 0$ for each $i \in \mathcal{I}$ results in a multi-vehicle system that will converge either to a state in which all vehicles are heading in the same direction (aligned) or to a state in which the average of all headings is zero and the centroid of the group remains fixed (balanced) [16], [28]:

$$\begin{aligned}\text{Aligned Set:}\quad & \mathcal{A} = \{\boldsymbol{\theta}(t) \mid \|\mathcal{R}(\boldsymbol{\theta}(t))\| = 1\}\\ \text{Balanced Set:}\quad & \mathcal{B} = \{\boldsymbol{\theta}(t) \mid \|\mathcal{R}(\boldsymbol{\theta}(t))\| = 0\}.\end{aligned}$$

These desirable group properties can be preserved with a first order (zero-order hold) discrete time approximation of (2) with $\omega_i = 0$:

$$\theta_i(h+1) = \theta_i(h) - \frac{K\Delta T}{|\mathcal{N}_i| + 1}\sum_{j\in\mathcal{N}_i}\sin\big(\theta_j(h) - \theta_i(h)\big), \tag{4}$$

where ΔT is the discretization period and h is the time step. The parameters K and ΔT play a critical role in the stability of the N vehicle discrete time system. For very small ΔT, the discrete time system (4) behaves much like its continuous time counterpart (2). However, the system becomes unstable as $K\Delta T$ becomes too large. The control theoretic results in this paper focus on finding sufficient conditions on $K\Delta T$ to guarantee stability to either the balanced or aligned group state.

A. All-to-All Broadcast Network

In previous work [19] for $N = 2$, the discrete time Kuramoto model was shown to converge to an aligned state for $-2 < K\Delta T < 0$ and to a balanced state for $0 < K\Delta T < 2$. Further, by linearization for $N > 2$ and $-2 < K\Delta T < 0$, a non-zero region of attraction about the aligned set (i.e. the set of all aligned states) was found. Here, a proof of asymptotic stability to the aligned set that is independent of N and does not require linearization is used to strengthen this earlier result.

Theorem 2.1 (All-to-All Aligned State Stability): For a system of N all-to-all coupled oscillators in discrete time (6), an aligned state will be approached for almost all initial conditions if $-2 < K\Delta T < 0$.

Proof: For the case of *all-to-all* coupling, the phase of the phase centroid, $\bar{\theta}(h) \equiv \angle\mathcal{R}(\boldsymbol{\theta}(h))$, can be used to rewrite the sum of sines of angle differences in (2), (4) in mean field coupling form ([13]):

$$\frac{1}{N}\sum_{j=1}^{N}\sin(\theta_j(h)-\theta_i(h)) = \rho(h)\sin(\bar{\theta}(h)-\theta_i(h)), \quad i\in\mathcal{I}. \tag{5}$$

With this result in mind, the all-to-all discrete time Kuramoto model (4) can be rewritten as

$$\theta_i(h+1) = \theta_i(h) - K\Delta T\rho(h)\sin\big(\bar{\theta}(h)-\theta_i(h)\big). \tag{6}$$

The remainder of the proof makes use of LaSalle's Invariance Principle for discrete time systems [29]. Take as a Lyapunov candidate $V(h) = 1-\rho(h)$. This function achieves a minimum value of zero only when all vehicles are aligned and is otherwise positive. An equivalent expression for the Lyapunov candidate at time step h is

$$V(h) = 1 - \mathcal{R}(h)^T e_{\bar{\theta}(h)}, \tag{7}$$

where $e_{\bar{\theta}(h)} \equiv \begin{bmatrix}\cos\bar{\theta}(h) & \sin\bar{\theta}(h)\end{bmatrix}^T$ is a unit vector in the direction of $\bar{\theta}(h)$, and for brevity $\mathcal{R}(h) = \mathcal{R}(\boldsymbol{\theta}(h))$. The difference in V between two consecutive time steps can be bounded above:

$$\begin{aligned}\Delta V(h) &= \mathcal{R}(h)^T e_{\bar{\theta}(h)} - \mathcal{R}(h+1)^T e_{\bar{\theta}(h+1)} &(8)\\ &\leq \mathcal{R}(h)^T e_{\bar{\theta}(h)} - \mathcal{R}(h+1)^T e_{\bar{\theta}(h)}. &(9)\end{aligned}$$

Thus, the result will follow from examining $-\Delta\mathcal{R}(h)^T e_{\bar{\theta}(h)}$, where $\Delta\mathcal{R}(h) = \mathcal{R}(h+1)-\mathcal{R}(h)$. Starting from (9) and defining $\delta\bar{\theta}_i(h) \equiv \bar{\theta}(h)-\theta_i(h)$,

$$\begin{aligned}\Delta V(h) &\leq -\Delta\mathcal{R}(h)^T \begin{bmatrix}\cos\bar{\theta}(h)\\ \sin\bar{\theta}(h)\end{bmatrix} &(10)\\ &= \frac{1}{N}\sum_{i=1}^{N}\left\{\begin{bmatrix}\cos\theta_i(h)\\ \sin\theta_i(h)\end{bmatrix} - \begin{bmatrix}\cos\theta_i(h+1)\\ \sin\theta_i(h+1)\end{bmatrix}\right\}^T \begin{bmatrix}\cos\bar{\theta}(h)\\ \sin\bar{\theta}(h)\end{bmatrix} &(11)\\ &= \frac{1}{N}\sum_{i=1}^{N}\left\{\cos(\delta\bar{\theta}_i(h)) - \cos(\bar{\theta}(h)-\theta_i(h+1))\right\} &(12)\\ &= \frac{1}{N}\sum_{i=1}^{N}\left\{\cos(\delta\bar{\theta}_i(h)) - \cos\left(\delta\bar{\theta}_i(h) + K\Delta T\rho(h)\sin(\delta\bar{\theta}_i(h))\right)\right\}. &(13)\end{aligned}$$

Provided $\rho(h) \neq 0$, each term of the above sum is negative when $\sin(\delta\bar{\theta}_i(h)) \neq 0$, and

$$-2(\delta\bar{\theta}_i(h)) < K\Delta T\rho\sin(\delta\bar{\theta}_i(h)) < 0. \tag{14}$$

A sufficient condition is then $-2 < K\Delta T < 0$, because

$$-2 \leq -\frac{2\delta\bar{\theta}_i(h)}{\rho\sin(\delta\bar{\theta}_i(h))} < K\Delta T < 0. \tag{15}$$

Thus, $V(h)$ is a valid Lyapunov function for this system and the stated range of $K\Delta T$.

LaSalle's Invariance Principle states that the solution of a dynamical system will approach the largest positively invariant set contained in $E = \{\boldsymbol{\theta} \in \mathbb{T}^N | \Delta V = 0\} \cap \bar{D}$, where $\bar{D}$ is the closure of the domain. From (13), ΔV is zero when $\rho = 0$ and/or when $\sin(\delta\bar{\theta}_i(h)) = 0,\ \forall i \in \mathcal{I}$. The heading rate (6) is zero at each of these points, thus rendering them positively invariant. However, not all of these equilibria are stable. When $\rho = 0$, the system is in an unstable balanced state ($V(h)$ from (7) attains its maximum value at these points). A small perturbation of $\boldsymbol{\theta}$ resulting in $\rho > 0$ will allow the system to move towards alignment. When $\sin(\delta\bar{\theta}_i(h)) = 0,\ \forall i \in \mathcal{I}$, all vehicle headings are parallel, but do not necessarily point in the same direction, as desired. However, these states are also unstable because any small perturbation will make $\Delta V < 0$ for $-2 < K\Delta T < 0$. Thus, the only stable equilibria are those in which all vehicle headings are aligned. ■

Stability to a balanced state is more difficult to show because it is not sufficient to examine the projection of the change in phase centroid onto $e_{\bar{\theta}(h)}$ as in (9). However, prior work with the continuous-time model has shown that it is stable to the balanced set for positive values of the coupling gain. Previous work [19] and informal simulation results permit the following conjecture to be stated with confidence.

Conjecture 2.2: For a system of N all-to-all coupled oscillators in discrete time (6), a balanced state will be reached for almost all initial conditions if $0 < K\Delta T < 2$.

The main difference between Conjecture 2.2 and Theorem 2.1 is the sign of the coupling gain. Thus if the conjecture is true, the main results of this paper can be easily extended to the case of balanced set stability. Ongoing research is focused on a proof of this conjecture.

B. Random One-to-All Broadcast Network

For the purposes here, we define a random broadcast network to be one in which at each time step h, one vehicle is chosen at random from a uniform distribution to broadcast its heading to all other vehicles. To implement this controller on a real system, the sequence of random broadcasters can be selected ahead of time. The random broadcast network simplifies the discrete-time Kuramoto model as the summation in (4) reduces to a single term:

$$\theta_i(h+1) = \theta_i(h) - \widetilde{K}\Delta T \sin(\theta_{b(h)}(h) - \theta_i(h)),\ \ i \in \mathcal{I}. \tag{16}$$

Here, $b(h) \in \mathcal{I}$ denotes the index of the randomly selected broadcasting agent at time step h and $\widetilde{K} \equiv K/2$.

Theorem 2.3: For a system of N oscillators coupled by random one-to-all broadcasts in discrete time (16), an aligned state will be reached in probability for $-2 < \widetilde{K}\Delta T < 0$, from almost all initial conditions.

Proof: Because each agent has an equal probability of being selected as the broadcaster, the expected value of the next heading of each agent $i \in \mathcal{I}$,

$$E\{\theta_i(h+1)\} = \theta_i(h) - \frac{\tilde{K}\Delta T}{N} \sum_{j=1}^{N} \sin(\theta_j(h) - \theta_i(h)), \tag{17}$$

is exactly the update given by all-to-all communication. Therefore, the reasoning in Theorem 2.1 can be used to conclude that (7) is a supermartingale [30], and thus the state will approach the aligned set *in probability*, from almost all initial conditions. This approach works in part because the aligned set is positively invariant with respect to the broadcast update (16). ■

On each broadcast, the state is expected to become more aligned than it was previously. Although a sequence of broadcasters for which alignment is never reached can be constructed, the probability of such a sequence being randomly selected approaches zero as the length of the sequence goes to infinity. Stability to the balanced set does not work with one-to-all broadcast communications because the states in the balanced set are not positively invariant with respect to the broadcast update (16), for any $K\Delta T$. For coherency in the remainder of this paper, the coupling gain will be denoted K for both one-to-all and all-to-all communications.

III. Network Routing Optimization

In this section, energy efficient communication schemes are addressed for realizing the sequence of logical graphs corresponding to the one-to-all communication needed for the control application. From a wireless networking perspective, one can obtain results on the energy optimal realization of each broadcast tree which transfers M bits of information (representing the heading $\theta_b(h)$) from the broadcaster to all others in no more than ΔT seconds. This result is a generalization of energy-efficient multicast trees as it includes a hard deadline of ΔT for a multicast session. For notional simplicity, we denote the effective transmission rate with $\bar{R} = M/\Delta T$.

Note that the simplest routing/relaying strategy is a single-hop wireless broadcast, while other options include multi-hop routing and relaying (also known as gossiping) (see Fig. 1). As will be shown below, the delay constraint imposed on the broadcast transmission has a significant impact on the solution to the minimum energy routing problem. The tradeoff between single hop broadcasting and multi-hop transmission is, in principle, the tradeoff between energy saving in transmission rate and transmission distance. When the message is broadcast to all nodes in the network in a single hop, the source node can use the entire time interval ΔT and transmit at a lower rate; but the transmission has to reach the farthest node in the network. On the other hand, when the message is relayed via intermediate nodes, the distance of each hop is smaller but the effective transmission rate for each node is larger than the single hop case.

To better describe this tradeoff, we define $f(R_i(t))$ as the expected SNR required for reliable communication at rate $R(t)$, $t \in [h\Delta T, (h+1)\Delta T]$. Suppose $f(R_i(t))$ grows faster than polynomial order with an increase in transmission rate. Taking into account that the typical power loss along a transmission path is only polynomial order, this trend intuitively suggests that savings in transmission rate would be more crucial as transmissions are constrained by a tight deadline. This result is formalized in this section: when the effective information rate ($\bar{R} = M/\Delta T$) is above some threshold R_c, single-hop broadcasting (a star topology) minimizes energy while meeting the strict delay deadline ΔT. Note that this result is a direct multi-hop result of lazy scheduling in [27] and [26]. The result will first be proved for the case of a linear network with three nodes and a path loss exponent of four; the results are easily generalized to a network with an arbitrary path loss exponent, general topology, admitting a network of more than three users .

A. Energy Efficient Routing for Broadcast Networks

The path loss exponent is generally assumed to be $2 \sim 4$ for wireless settings. We will assume the path loss exponent a to be four to examine the case when the tradeoff in energy saving between transmission distance and transmission rate is the largest. As Remark 2 shows, the main result holds for any $a < 4$.[2] Considering a network with N nodes, the problem of topology choice reduces to finding the optimal transmission SNR $P_i(t),\ i = 1, 2...N$, at time $t \in [h\Delta T, (h+1)\Delta T]$ (without loss of generality, the noise power is normalized at the receiver to value 1). We make the following assumptions regarding the operation of a network in service of coordinated control.

Technical Assumptions A

A1. Each node can transmit with high enough power for a wireless broadcast transmission to reach the farthest node in a single hop;

A2. The interference model is the protocol model in [31], where the guard zone is as large as the diameter of the network[3];

A3. The received signal power at a distance d from the transmitting node varies as d^{-a}, where the path loss exponent a depends on the characteristics of the transmitting medium;

A4. Cooperative relaying, or network coding, is not considered, hence, the work follows a simple packet switching model with separation of layer functionalities; and

A5. Each update message has the same format and the same size of M bits.

[2]Similar techniques can be used to extend the main result for any arbitrary $a > 4$.

[3]In other words, we assume that only one link can be active during each transmission session.

Technical Assumptions A1 and A2 are relevant for a scenario with a relatively small number of vehicles present in the network.

Mathematically, this reduces to the following optimization problem:

$$\min_{\{P_i(\cdot)\}_{i=1}^N \in \mathcal{P}} \sum_{i=1}^{N} \int_0^{\Delta T} P_i(t)dt, \tag{18}$$

subject to

$$\begin{aligned} P_i(t) &\geq 0, i = 1, 2...N \quad &(19)\\ P_i(t)P_j(t) &= 0, \text{ for all } i \neq j, \quad &(20)\\ P_i(t) &= f(R_i(t))d^a_{ij_t(i)}, \quad &(21)\\ M &= \int_0^{\Delta T} R_i(t)dt, \qquad \forall i; \quad &(22) \end{aligned}$$

where $\mathcal{P}$ is the class of collective power allocation policies $\{P_i(\cdot)\}_{i=1}^N$ satisfying (18)-(22), and $j_t(i)$ is the farthest node to whom, at time t, node i attempts to deliver its M bits reliably. In other words, one can interpret $f(\bar{R})$ to be the transmit power required to transmit 1 bit of information with rate $\bar{R}$ to a node unit distance away. We make the following technical assumptions on $f(\bar{R})$[4].

<u>Technical Assumptions B</u>

B1. $f(\bar{R}) \geq 0$ with equality if and only if $\bar{R} = 0$;
B2. $f(\bar{R})$ is a convex, monotonic increasing function in $\bar{R}$;
B3. $f(\bar{R})$ is analytic in the interval [-$\infty, +\infty$]; and
B4. $\frac{d^n f(0)}{d\bar{R}^n} > 0, \ \forall n \in \mathbb{N}$.

The main result of this section is the following theorem:

Theorem 3.1: If M and ΔT are chosen such that

$$\frac{M}{\Delta T} > \max_{n \in \{1...8\}} \left\{ (n+1) \frac{f^{(n)}(0)}{f^{(n+1)}(0)} \right\},$$

a single hop broadcasting scheme (see Fig. 1) is more energy efficient than any multi-hop relaying scheme.

To prove this theorem, we first identify the optimal transmission policy within each transmission session and over every single hop. In particular, a *fixed rate transmission session* is shown to be optimal in terms of minimizing total energy consumption. In [27], Zafer and Modiano have used the concept of a *minimum departure curve* and proposed constructive algorithms to build the optimal departure curve for given delay constraints with single hop point-to-point communications. Graphically, their results suggest that the *optimal* departure curve is the one that has the *shortest length* among all feasible curves constrained between the minimum departure and arrival curves. This result is readily applicable to the setting here when considering a single transmission session over a single-hop for any transmitting node (one contiguous interval during which $P_i(t) > 0$) . The following lemma establishes this fact.

Lemma 3.2: A necessary condition for optimality is that each node has to transmit at a fixed rate within each transmission session.

Proof: Assume that the optimal transmission policy $F_{jk}(t)$ for node j does not transmit at a fixed rate in its kth transmission session, and denote the corresponding energy consumption to be W_k. Based on the result in [27], we can always find another feasible policy that is more energy efficient than $F_{jk}(t)$. The existence of this policy can be simply shown by using the same argument in [27]: for any feasible departure curve, we can replace a small portion of it by a straight line (corresponding to a fixed rate policy); the new transmission policy is still feasible but the total transmission energy corresponding to the new policy would be smaller than the original one. Therefore, we can always find another feasible fixed

[4]As we show later, the power-rate relationship over an AWGN channel satisfy these assumptions.

rate policy with energy consumption W'_k such that $W'_k < W_k$, which contradicts the optimality assumption of the original policy. This argument shows that fixed rate transmission within each session is a necessary condition for any active node's optimal transmission policy. ■

To prove the main result, we first consider a linear network with three nodes and path loss exponent $a = 4$. Later we see that this is easily extended to the general setting. The energy required for relaying $\bar{R}\Delta T$ bits of information in ΔT second is given by

$$E_m(\alpha, \beta, \bar{R}) = \Delta T D^4 \left(\beta^4 \alpha f\left(\frac{\bar{R}}{\alpha}\right) + (1-\beta)^4 (1-\alpha) f\left(\frac{\bar{R}}{1-\alpha}\right)\right), \tag{23}$$

where $\alpha\Delta T$ denotes the time fraction spent on transmission between the source node and the intermediate node, and βD denotes the distance fraction between the source node and the intermediate node. By construction, $\alpha, \beta \in (0, 1)$. On the other hand, the energy required for single-hop broadcasting is written as

$$E_s(\bar{R}) = \Delta T D^4 f(\bar{R}). \tag{24}$$

Remark 1: Without loss of generality, and for simplicity of notation, we normalize $\Delta T D^4$ to one.

Lemma 3.3 below establishes the optimality of the single-hop broadcasting scheme for a three node linear network.

Lemma 3.3: Consider a linear network of 3 nodes. If $M = \bar{R}\Delta T$ and ΔT are such that

$$\frac{M}{\Delta T} = \bar{R} > \max_{n\in\{1\ldots 8\}} \left\{ (n+1)\frac{f^{(n)}(0)}{f^{(n+1)}(0)} \right\}, \tag{25}$$

the energy of broadcasting $M = \bar{R}\Delta T$ bits in ΔT seconds in a single hop fashion is less than any relaying scheme independent of ratios of the node's distances, $\beta \in (0, 1)$, and the times allocated to each hop, $\alpha \in (0, 1)$.

From Technical Assumptions B, $f(\bar{R})$ is an analytic function in $[-\infty, \infty]$. Using a Taylor series expansion at $\bar{R}$, we can then rewrite $E_m(\alpha, \beta, \bar{R})$, and $E_s(\bar{R})$ as follows:

$$E_m(\alpha, \beta, \bar{R}) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(\bar{R})^n \left(\frac{\beta^4}{\alpha^{n-1}} + \frac{(1-\beta)^4}{(1-\alpha)^{n-1}}\right), \tag{26}$$

$$E_s(\bar{R}) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(\bar{R})^n \tag{27}$$

where

$$a_n(\bar{R}) := \frac{f^{(n)}(0)}{n!}(\bar{R})^n \tag{28}$$

$$b_n(\alpha, \beta) := \frac{\beta^4}{\alpha^{n-1}} + \frac{(1-\beta)^4}{(1-\alpha)^{n-1}}. \tag{29}$$

Before proceeding with the proof of the above lemma, the properties of the sequences $a_n(\bar{R})$, $b_n(\alpha, \beta)$ must be established.

Lemma 3.4: Properties of $b_n(\alpha, \beta)$.

- For a given $\alpha, \beta \in (0, 1)$, $b_n(\alpha, \beta)$ is a monotonically increasing function in n.
- For all $\alpha, \beta \in (0, 1)$, and $\forall n \geq 4$, $b_n(\alpha, \beta) \geq 1$.
- For all $\alpha, \beta \in (0, 1)$, and $\forall n \geq 7$, $b_n(\alpha, \beta) \geq \frac{1}{b_1(\alpha,\beta)}$.

Lemma 3.5: If $\bar{R} > (n+1)\frac{f^{(n)}(0)}{f^{(n+1)}(0)}$ then, $a_{n+1}(\bar{R}) > a_n(\bar{R})$.

Lemma 3.6: If there exists R_c such that $E_m(\alpha, \beta, R_c) \geq E_s(R_c)$ for some α and β, then $E_m(\alpha, \beta, \bar{R}) \geq E_s(\bar{R})$ is also true for any $\bar{R} \geq R_c$.

The proofs of lemmas 3.4-3.6 are included in the appendix. The next lemma provides the last step in proving Lemma 3.3.

Lemma 3.7: Consider a linear network consisting of 3 nodes. There exists

$$R_c = \max_{n\in\{1\dots8\}} \left\{(n+1)\frac{f^{(n)}(0)}{f^{(n+1)}(0)}\right\}, \tag{30}$$

such that for all $\alpha, \beta \in (0,1)$, $E_m(\alpha,\beta,R_c) \geq E_s(R_c)$.

Proof: Define $d_n(\alpha,\beta,R_c)$ as

$$d_n(\alpha,\beta,R_c) := \frac{f^{(n)}(0)}{n!}(R_c)^n\left(\frac{\beta^4}{\alpha^{n-1}} + \frac{(1-\beta)^4}{(1-\alpha)^{n-1}} - 1\right) = a_n(\bar{R})\left(b_n(\alpha,\beta)-1\right). \tag{31}$$

In other words,

$$E_m(\alpha,\beta,R_c) - E_s(R_c) = \sum_{n=0}^{\infty} d_n(\alpha,\beta,R_c). \tag{32}$$

For simplicity, write $d_n(\alpha,\beta,R_c)$, $b_n(\alpha,\beta)$, and $a_n(R_c)$ as d_n, b_n and a_n, respectively.

We first bound $\sum_{n=0}^{\infty} d_n$ from below:

$$\sum_{n=0}^{\infty} d_n \;>\; \sum_{n=1}^{3} d_n + \sum_{n=7}^{\infty} d_n \tag{33}$$

$$>\; (b_1-1)\sum_{n=1}^{3} a_n + \left(\frac{1}{b_1}-1\right)\sum_{n=7}^{\infty} a_n \tag{34}$$

$$=\; (1-b_1)\left(\frac{1}{b_1}\sum_{n=7}^{\infty} a_n - \sum_{n=1}^{3} a_n\right), \tag{35}$$

where the first inequality is from the fact that $a_0 = f(0) = 0$, and $d_n \geq 0$ for $n \geq 4$, while the second inequality results from Lemma 3.4.

Next, we show that the lower bound is non-negative. From the definition (30), we know that for $1 \leq n \leq 8$, $R_c > (n+1)\frac{f^{(n)}(0)}{f^{(n+1)}(0)}$. This relation implies that for all $1 \leq n \leq 8$, $a_n(R_c)$ is monotonically increasing in n. In other words,

$$a_1 + a_2 + a_3 \leq a_7 + a_8 + a_9 < \sum_{n=7}^{\infty} a_n. \tag{36}$$

On the other hand, $b_1 = \beta^4 + (1-\beta^4) \leq 1$, and the assertion of the lemma follows. ■

The above work can be extended to the case of the general network with multiple users in the following steps.

Corollary 3.8: Single hop broadcasting is more energy efficient than multi-hopping in any network with three users when

$$\bar{R} > \max_{n\in\{1\dots8\}} \left\{(n+1)\frac{f^{(n)}(0)}{f^{(n+1)}(0)}\right\}. \tag{37}$$

Proof: Consider a network with three users where S denotes the source node, U_{near} the closest node to the source node, and U_{far} the farthest node from the source node. Let D be the distance between the source node and the farthest node. By the triangle inequality,

$$D = d(S,U_{far}) \leq d(S,U_{near}) + d(U_{near},U_{far}). \tag{38}$$

Defining $D_{new} = d(S,U_{near}) + d(U_{near},U_{far})$, the energy required for multi-hopping is

$$E_m(\alpha,\beta,\bar{R},D_{new}) \;=\; \Delta T D_{new}^4\left(\beta^4\alpha f\left(\frac{\bar{R}}{\alpha}\right) + (1-\beta)^4(1-\alpha) f\left(\frac{\bar{R}}{1-\alpha}\right)\right). \tag{39}$$

However, from Lemma 3.3, when $\bar{R} > \max_{n\in\{1\ldots8\}}\left\{(n+1)\frac{f^{(n)}(0)}{f^{(n+1)}(0)}\right\}$, the above function is always greater than $E_s(\bar{R}, D_{new}) = \Delta T D_{new}^4 f(\bar{R})$. Because $D_{new} \geq D$ from the triangle inequality, the following inequality is true for all $\alpha, \beta \in (0, 1)$:

$$E_m(\alpha, \beta, \bar{R}, D_{new}) > E_s(\bar{R}, D_{new}) \geq E_s(\bar{R}, D). \tag{40}$$

Therefore, in a network with three users, single hop broadcasting is always more energy efficient than multi-hopping when $\bar{R} > \max_{n\in\{1\ldots8\}}\left\{(n+1)\frac{f^{(n)}(0)}{f^{(n+1)}(0)}\right\}$. ■

Now consider a network with N users, where links are activated one at a time to avoid interference, i.e. the interference model is the protocol model in [31]. In this setting, we can, inductively, arrive at our main result as a direct consequence of Corollary 3.8.

Theorem 3.1: Single hop broadcasting is more energy efficient than multi-hopping in any network when $\bar{R} > \max_{n\in\{1\ldots8\}}\left\{(n+1)\frac{f^{(n)}(0)}{f^{(n+1)}(0)}\right\}$.

Remark 2: Theorem 3.1 holds for any path loss exponent $0 < a < 4$. This can easily seen: If $\bar{R} > \max_{n\in\{1\ldots8\}}\left\{(n+1)\frac{f^{(n)}(0)}{f^{(n+1)}(0)}\right\}$, then we have

$$\beta^4 \alpha f\left(\frac{\bar{R}}{\alpha}\right) + (1-\beta)^4(1-\alpha) f\left(\frac{\bar{R}}{1-\alpha}\right) > f(\bar{R}) \ \forall\ \alpha, \beta \in (0,1). \tag{41}$$

However, since $\beta \in (0, 1)$, for $0 < a < 4$,

$$\beta^a \alpha f\left(\frac{\bar{R}}{\alpha}\right) + (1-\beta)^a(1-\alpha) f\left(\frac{\bar{R}}{1-\alpha}\right) > \beta^4 \alpha f\left(\frac{\bar{R}}{\alpha}\right) + (1-\beta)^4(1-\alpha) f\left(\frac{\bar{R}}{1-\alpha}\right), \tag{42}$$

for a given $\alpha, \beta \in (0, 1)$. Combining the two inequalities, for $0 < a < 4$,

$$\Delta T D^a(\beta^a \alpha f\left(\frac{\bar{R}}{\alpha}\right) + (1-\beta)^a(1-\alpha) f\left(\frac{\bar{R}}{1-\alpha}\right)) > \Delta T D^a(f(\bar{R})) \ \forall\ \alpha, \beta \in (0,1). \tag{43}$$

In other words, Lemma 3.3 holds for any $0 < a < 4$.

Remark 3: Variations of Theorem 3.1 can be easily developed in an identical manner for all cases where the path loss is of a polynomial form.

B. Example: Broadcast Network over the AWGN Channel

In this section, we consider a case where transmissions over links follow Shannon capacity formula for an AWGN channel, i.e. the expected transmit SNR required for reliable communication at rate $R(t)$ to a node at distance d is given by

$$P_i(t)/\sigma = d^a(2^{2R(t)} - 1), \tag{44}$$

where σ is the noise level at the receiver.

Because $f(\bar{R}) = 2^{2\bar{R}} - 1$ satisfies Technical Assumptions B, we can directly invoke Theorem 3.1 to show that when

$$\frac{M}{\Delta T} > \max_{n\in\{1\ldots8\}}\left\{(n+1)\frac{f^{(n)}(0)}{f^{(n+1)}(0)}\right\} = \frac{9}{2\ln 2} \approx 6.5, \tag{45}$$

single hop broadcasting is more energy efficient than multi-hop relaying. On the other hand, one can numerically test that $E_m(\alpha, \beta, 4) - E_s(4) = \sum_{n=0}^{\infty} d_n(\alpha, \beta, 4) > 0$, arriving at a tighter bound.

IV. Simulation Results

This section contains simulation results to support the theoretical findings in this paper. In our simulations, we have used $N = 5$ vehicles, starting from arbitrary initial states in two dimensions, a path loss exponent of $a = 4$, and quantization of state variables into $M = 10$ bits[5]. To demonstrate the multivehicle control being used here, Fig. 2 shows an example of the trajectories of all vehicles using the discrete time Kuramoto controller model under all-to-all and random one-to-all communication with $K\Delta T$ equal to -0.5. As Theorem 2.3 suggests, both the one-to-all and all-to-all communication topologies produce stability to a state in the aligned set for the given parameter values. Note, however, that while an aligned state is reached in each case, the heading to which the vehicles align is not identical as this value was not part of the control design.

As the product $K\Delta T$ is varied within the allowable bounds, the time required for the headings to converge within an ϵ-ball (settling time) will vary. The general trend of this variation is depicted in Fig. 3 where the number of iterations needed for the headings to converge within an ϵ-ball is plotted against $K\Delta T$ for the case of all-to-all communication (indicating the result in (16)) and a particular case of one-to-all communication with $\epsilon = 10^{-7}$. Convergence rate is similar for all-to-all and one-to-all broadcast, but keep in mind the factor of two gain difference. The best performance is obtained with $K\Delta T$ near one because for lesser values of $K\Delta T$ the system lacks control authority, and for greater values of $K\Delta T$ ringing occurs. To determine actual time to alignment, the steps to alignment must be multiplied by ΔT, so smaller ΔT means faster convergence.

Because a decreased settling time for the multivehicle system requires decreased step size ΔT but decreased communication energy requires increased ΔT, an optimal choice of ΔT can be found as a function of K for the coupled communication and control problem. The analytical result of this optimization problem is the subject of ongoing research, however a numerical result is shown in Fig. 4. In this figure, total communication energy for convergence of the headings to an ϵ-ball of size $\epsilon = 10^{-7}$ is shown for three values of K and as a function of ΔT. The total communication energy is dependent on the distances of the vehicles from one another, however, the heading control was not designed to minimize the maximal distance between the vehicles. In order to realistically represent the communication energy required for each set of parameters, a Monte Carlo approach was used where for each K and ΔT, 100 simulations were run from initial vehicle positions drawn uniformly from $[-15, 15] \times [-15, 15]$, initial headings drawn uniformly from $[-\pi, \pi]$, and with random one-to-all broadcast until convergence. Each randomly selected vehicle was assumed to be transmitting $M = 10$ bits per ΔT. The consumed communication energy was then averaged and plotted. At $\Delta T \approx 0$, both communication energy per ΔT and the number of steps to alignment are large. As $\Delta T \to -\frac{1}{K}$, communication energy per ΔT and the number of steps to alignment decrease sharply, while as $\Delta T \to -\frac{2}{K}$, the increase in convergence time dominates the decrease in communication energy per ΔT, resulting in an asymptote.

V. Conclusions

The work in this paper has advanced previous work with the Kuramoto model in discrete time for multiple vehicle coordination. In addition to extending theoretical results from previous work, network routing optimization was considered, and the single-hop broadcast communication topology was shown to be optimal in certain situations. It was shown that the discretization period, ΔT, not only enables an integrated approach to the problem of coordinated control, but also can be chosen so as to optimally balance the controller performance with the communication performance using total communication energy as a metric.

Future work will focus on a proof of Conjecture 2.2 in addition to determining whether these conditions are necessary in addition to being sufficient. Additionally, an analytical result for the optimal relation

[5]We have intentionally picked the number of bits small to allow for the possibility of strong and effective quantization of the state of variable, i.e. single-hop broadcasting is more energy efficient, even if we assume that the state variable in $\mathbb{T}^N$ can be quantized without loss using a small number of bits (node id+ number of quantized bits=10). Note that increasing M only favors single-hopping.

between total communication energy, discretization step size and coupling gain will be pursued. Another avenue for future work is to look at communication topologies other than all-to-all and random one-to-all broadcasts. The results presented here should extend to a regular transmission pattern instead of a random broadcast. Finally, it would be interesting to examine the effect of heterogeneous delay on this system (homogeneous delay was studied in [19]).

Interesting questions also remain for the minimum energy routing problem. First, the interference model was taken to be the protocol model [31] with the guard zone as large as the diameter of the network. This choice in principle corresponds to a network of vehicles in relatively small numbers. In networks with many vehicles, it is not hard to imagine cases where multiple links can be active during each transmission session. In such settings, multi-hop routing benefits from spatial reuse, by dividing the message into small pieces to allow concurrent transmissions. Extension of the study here to such scenarios is an interesting area of future study. The extension of current work to fading channel models (i.e. channel quality, hence transmissions are stochastic) is another important area of research. The results from [32] will be a good instrument for this research.

Acknowledgment

The authors would like to thank Ben Triplett.

Appendix

A. Proof of Lemma 3.4

Lemma 3.4: Properties of $b_n(\alpha,\beta)$.

1) For a given $\alpha,\beta \in (0,1)$, $b_n(\alpha,\beta)$ is a monotonically increasing function in n.
2) For all $\alpha,\beta \in (0,1)$, and $\forall n \geq 4$, $b_n(\alpha,\beta) \geq 1$.
3) For all $\alpha,\beta \in (0,1)$, and $\forall n \geq 7$, $b_n(\alpha,\beta) \geq \frac{1}{b_1(\alpha,\beta)}$.

Proof:

1) For a given $\alpha,\beta \in (0,1)$, $b_n(\alpha,\beta)$ is a monotonically increasing function in n:

$$b_{n+1}(\alpha,\beta) = \left(\frac{\beta^4}{\alpha^{n-1}}\right)\frac{1}{\alpha} + \frac{(1-\beta)^4}{(1-\alpha)^{n-1}}\frac{1}{(1-\alpha)}. \tag{46}$$

Because $0 < \alpha < 1$,

$$\left(\frac{\beta^4}{\alpha^{n-1}}\right)\frac{1}{\alpha} > \left(\frac{\beta^4}{\alpha^{n-1}}\right), \tag{47}$$

and

$$\frac{(1-\beta)^4}{(1-\alpha)^{n-1}}\frac{1}{(1-\alpha)} > \frac{(1-\beta)^4}{(1-\alpha)^{n-1}} \tag{48}$$

In other words, $b_{n+1}(\alpha,\beta) > b_n(\alpha,\beta)$.

2) Because $0 < \alpha < 1$, $b_{n+1}(\alpha,\beta) > b_n(\alpha,\beta)$. Also, from definition of $b_n(\alpha,\beta)$,

$$b_4(\alpha,\beta) = \frac{\beta^4}{\alpha^3} + \frac{(1-\beta)^4}{(1-\alpha)^3}. \tag{49}$$

It is easy to show that $b_4(\alpha,\beta)$ is a convex function in α that achieves its global minimum of value 1 when $\alpha = \beta$. On the other hand, b_n is monotonically increasing in n, establishing the assertion of the lemma; i.e. $b_n(\alpha,\beta) \geq 1$ for $n \geq 4$.

3) Define

$$r_n(\alpha,\beta) := \frac{b_{n+1}(\alpha,\beta)}{b_n(\alpha,\beta)}. \tag{50}$$

We will show that $r_{n+1}(\alpha,\beta) - r_n(\alpha,\beta) \geq 0$. First,

$$r_{n+1}(\alpha,\beta) - r_n(\alpha,\beta) = \frac{b_{n+2}(\alpha,\beta)b_n(\alpha,\beta) - (b_{n+1}(\alpha,\beta))^2}{b_n(\alpha,\beta)b_{n+1}(\alpha,\beta)}. \tag{51}$$

Since $b_n(\alpha,\beta)b_{n+1}(\alpha,\beta) > 0$, we only need to examine the numerator:

$$b_{n+2}(\alpha,\beta)b_n(\alpha,\beta) - (b_{n+1}(\alpha,\beta))^2 = \frac{\beta^4(1-\beta)^4}{\alpha^{n-1}(1-\alpha)^{n-1}}\left(\frac{\alpha}{1-\alpha} + \frac{1-\alpha}{\alpha} - 2\right) \geq 0. \tag{52}$$

The inequality holds because $\frac{\alpha}{1-\alpha} + \frac{1-\alpha}{\alpha}$ is a convex function that achieves its global minimum of 2 when $\alpha = 0.5$. Therefore, $r_{n+1}(\alpha,\beta) \geq r_n(\alpha,\beta)$.
Because $r_n(\alpha,\beta)$ is a monotonically non-decreasing function in n,

$$b_1(\alpha,\beta)(r_4(\alpha,\beta))^3 \geq b_4(\alpha,\beta) = b_1(\alpha,\beta)r_1(\alpha,\beta)r_2(\alpha,\beta)r_3(\alpha,\beta) \geq 1, \tag{53}$$

which leads to

$$r_4(\alpha,\beta) \geq (\frac{1}{b_1(\alpha,\beta)})^{1/3}, \tag{54}$$

$$b_7(\alpha,\beta) \geq b_4(r_4)^3 \geq \frac{1}{b_1(\alpha,\beta)}. \tag{55}$$

Because $b_n(\alpha,\beta)$ is monotonically increasing in n, we have $b_n(\alpha,\beta) \geq \frac{1}{b_1(\alpha,\beta)}$ when $n \geq 7$.

■

B. *Proof of Lemma 3.5*

Lemma 3.5: If $\bar{R} > (n+1)\frac{f^{(n)}(0)}{f^{(n+1)}(0)}$ then, $a_{n+1}(\bar{R}) > a_n(\bar{R})$.

Proof: We first note that

$$a_{n+1}(\bar{R}) - a_n(\bar{R}) = \frac{\bar{R}^n}{n!}\left(\frac{f^{(n+1)}(0)}{n+1}\bar{R} - f^{(n)}(0)\right) \tag{56}$$

but this result indicates that

$$\bar{R} > (n+1)\frac{f^{(n)}(0)}{f^{(n+1)}(0)} \tag{57}$$

is a sufficient condition to guarantee that $a_{n+1}(\bar{R}) > a_n(\bar{R})$. ■

C. *Proof of Lemma 3.6*

Lemma 3.6: If there exists R_c such that $E_m(\alpha,\beta,R_c) \geq E_s(R_c)$ for some α and β, then $E_m(\alpha,\beta,\bar{R}) \geq E_s(\bar{R})$ is also true for any $\bar{R} \geq R_c$.

Proof: As seen before in equation (31), define $d_n(\alpha,\beta,R_c)$ as

$$d_n(\alpha,\beta,R_c) := \frac{f^{(n)}(0)}{n!}(R_c)^n\left(\frac{\beta^4}{\alpha^{n-1}} + \frac{(1-\beta)^4}{(1-\alpha)^{n-1}} - 1\right) = a_n(\bar{R})\left(b_n(\alpha,\beta) - 1\right). \tag{58}$$

Because $\sum_{n=0}^{\infty} d_n(\alpha,\beta,R_c) \geq 0$ there exists an n such that $d_n(\alpha,\beta,R_c) \geq 0$. Define $J = \min\{n : d_n(\alpha,\beta,R_c) \geq 0\}$. From monotonicity of $b_n(\alpha,\beta)$ in n, and $a_n(\bar{R}) \geq 0$, we have $d_m(\alpha,\beta,R_c) < 0 \leq d_n$

for all $m < J \leq n$. Thus

$$E_m(\alpha,\beta,\bar{R}) - E_s(\bar{R}) = \sum_{n=1}^{\infty} d_n(\alpha,\beta,\bar{R}) = \sum_{n=1}^{\infty} \left(\frac{\bar{R}}{R_c}\right)^n d_n(\alpha,\beta,R_c) \quad (59)$$

$$= \sum_{n=1}^{J-1} \left(\frac{\bar{R}}{R_c}\right)^n d_n(\alpha,\beta,R_c) + \sum_{n=J}^{\infty} \left(\frac{\bar{R}}{R_c}\right)^n d_n(\alpha,\beta,R_c) \quad (60)$$

$$\geq \left(\frac{\bar{R}}{R_c}\right)^J (\sum_{n=1}^{J-1} d_n(\alpha,\beta,R_c) + \sum_{n=J}^{\infty} d_n(\alpha,\beta,R_c)) \quad (61)$$

$$= \left(\frac{\bar{R}}{R_c}\right)^J \sum_{n=1}^{\infty} d_n(\alpha,\beta,R_c) \geq 0, \quad (62)$$

where the first inequality comes from the fact that $\bar{R}/R_c \geq 1$, and $d_i < 0 \ \forall \ 1 \leq i \leq J$, while the second inequality is from the assumption. Combining the first and the last terms, we arrive at the assertion of the lemma:

$$E_m(\alpha,\beta,\bar{R}) \geq E_s(\bar{R}) \quad (63)$$

■

References

[1] H. Witsenhausen, "A counterexample in stochastic optimum control," *SIAM Journal on Control*, vol. 6, 1968.

[2] S. Tatikonda and S. Mitter, "Control under communication constraints," *IEEE Transactions on Automatic Control*, vol. 49, Issue: 7, pp. 1056– 1068, July 2004.

[3] D. Liberzon and J. P.Hespanha, "Stabilization of nonlinear systems with limited information feedback," *IEEE Transactions on Automatic Control*, vol. 50, Issue: 6, pp. 910–915, January 2005.

[4] G. N. Nair, F. Fagnani, S. Zampieri, and R. J. Evans, "Feedback control under data rate constraints: An overview," *Proceedings of the IEEE*, vol. 95, Issue: 1, pp. 108–137, January 2007.

[5] A. Sahai and S. Mitter, "The necessity and sufficiency of anytime capacity for control over a noisy communication link," *IEEE Transactions on Information Theory*, 2005.

[6] N.C.Martins, M.A.Dahleh, and N.Elia, "Feedback stabilization of uncertain systems in the presence of a direct link," *IEEE Transactions on Automatic Control*, vol. 51, pp. 438–447, Mar. 2006.

[7] V.Gupta, B.Hassibia, and R.Murray, "Optimal lqg control across packet-dropping links," *Systems and Control Letters*, vol. 56, pp. 439–446, June. 2007.

[8] L. Schenato, B. Sinopoli, M. Franceschetti, K. Poolla, and S. S. Sastry, "Foundations of control and estimation over lossy networks," *Proceedings of the IEEE*, vol. 95, Issue: 1, pp. 163–187, January 2007.

[9] X. Liu and A. Goldsmith, "Wireless network design for distributed control," *IEEE Conference on Decision and Control*, pp. 2823–2829 Vol.3, December 2004.

[10] B. M. Howe, "Acoustic seaglider (a)," *The Journal of the Acoustical Society of America*, vol. 120, Issue:5, p. 3048, 2006.

[11] R. Olfati-Saber, J. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proceedings of the IEEE*, vol. 95, no. 1, January 2007.

[12] J. Fax and R. Murray, "Information flow and cooperative control of vehicle formations," in *IFAC World Congress*, Barcelona, Spain, 2002.

[13] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence*. Springer-Verlag, 1984.

[14] M. S. Yeung and S. H. Strogatz, "Time delay in the kuramoto model of coupled oscillators," *Physical Review Letters*, vol. 82, no. 3, pp. 648–651, January 1999.

[15] A. Jadbabaie, N. Motee, and M. Barahona, "On the stability of the kuramoto model of coupled nonlinear oscillators," in *Proceedings of the 2006 American Control Conference*, 2004, pp. 988–1001.

[16] N. Leonard, D. Paley, and R. Sepulchre, "Oscillator models and collective motion: Splay state stabilization of self-propelled particles," in *Proc. 44th IEEE Conf. Decision and Control*, Seville, Spain, December 2005.

[17] D. J. Klein and K. A. Morgansen, "Controlled collective motion for trajectory tracking," in *Proc. of the 2006 American Control Conference*, Minneapolis, MN, June 2006.

[18] D. J. Klein, C. Matlack, and K. A. Morgansen, "Cooperative target tracking using oscillator models in three dimensions," in *Proc. of the 2007 American Control Conference*, New York, NY, June 2007.

[19] B. I. Triplett, D. J. Klein, and K. A. Morgansen, "Discrete time kuramoto models with delay," in *Networked Embedded Sensing and Control, Workshop NESC'05: University of Notre Dame, USA, October 2005, Proceedings*, ser. Lecture Notes in Control and Information Sciences, P. J. Antsaklis and P. Tabuada, Eds. University of Notre Dame, USA: Springer, October 2005, pp. 9–23.

[20] A. Sarlette, R. Sepulchre, and N. E. Leonard, "Discrete-time synchronization on the N-torus," in *Proceedings of the IEEE Conference on Decision and Control*, 2006.

[21] L. Scardovi, A. Sarlette, and R. Sepulchre, "Synchronization and balancing on the N-torus," 2007, in review.
[22] J. E. Wieselthier, G. D. Nguyen, and A. Ephremides, "On the construction of energy-efficient broadcast and multicast trees in wireless networks," in *INFOCOM 2000*, vol. 2, Mar. 2000, pp. 585–594.
[23] A. S. Ahluwalia and E. H. Modiano, "On the complexity and distributed construction of energy-efficient broadcast trees in wireless ad hoc networks," *IEEE Transaction on Wireless Communication*, vol. 4, pp. 2136–2147, Sep. 2005.
[24] D. Yuan, "Computing optimal or near-optimal trees for minimum-energy broadcasting in wireless networks," in *WIOPT 2005*, Apr. 2005, pp. 323–331.
[25] S. Guo and O. W. Yang, "Minimum-energy multicast routing in static wireless ad hoc networks," in *Vehicular Technology Conference 2004*, vol. 6, Sep. 2004, pp. 3989–3993.
[26] E. Uysal-Biyikoglu and A. E. Gamal, "On adaptive transmission for energy-efficiency in wireless data networks," *IEEE Transaction on Information Theory*, December 2004.
[27] M. A. Zafer and E. Modiano, "A calculus approach to minimum energy transmission policies with quality of service guarantees," *INFOCOM 2005*, vol. 1, pp. 548–559, Mar. 2005.
[28] E. Justh and P. Krishnaprasad, "Equilibria and steering laws for planar formations," *Systems and Control Letters*, vol. 52, pp. 25–38, 2004.
[29] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Prentice Hall, 2002, 168-174.
[30] H. Kushner, *Stochastic Stability and Control*. New York: Academic Press, 1967.
[31] P. Gupta and P. R. Kumar, "The capacity of wireless networks," *IEEE Trans. Info. Theory*, vol. IT-46, pp. 388–404, March 2000.
[32] A. Fu, E. Modiano, and J. N. Tsitsiklis, "Optimal transmission scheduling over a fading channel with energy and deadline constraints," *IEEE Transactions on Wireless Communications*, vol. 5, Issue:3, pp. 630– 641, March 2006.

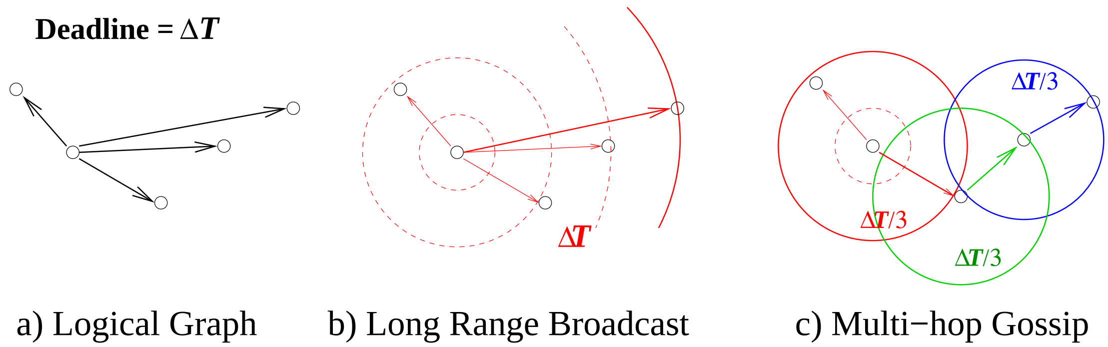

Fig. 1. A logical one-to-all graph (a) can be realized via simple wireless broadcasting (b) or gossip schemes (c).

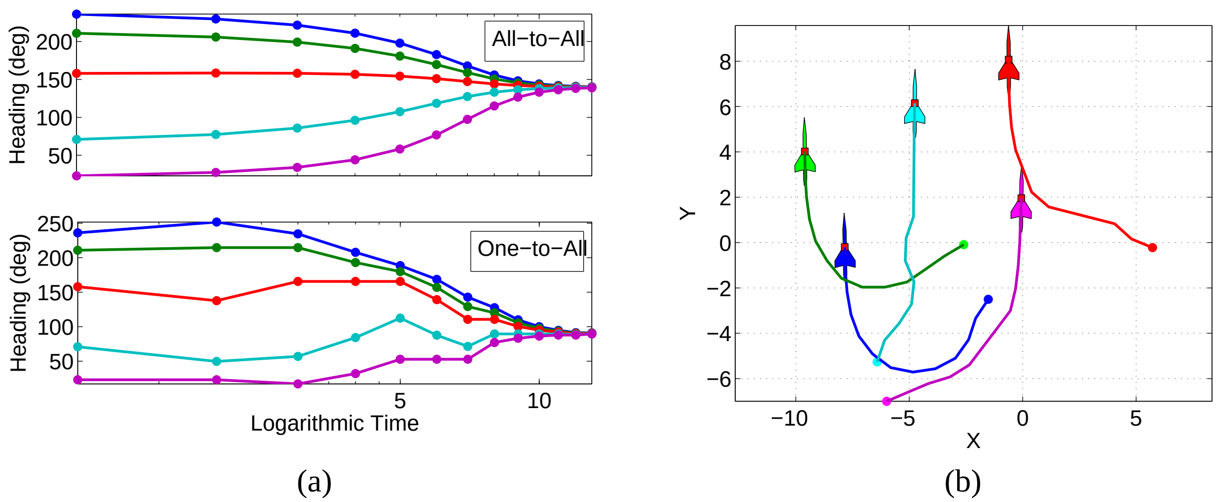

Fig. 2. The discrete time Kuramoto model (4) is shown for a group of five vehicles. (a) The vehicle headings using all-to-all (top) and one-to-all random broadcast (bottom) communication topologies for $K = -0.5$ and $\Delta T = 1$. (b) Vehicle trajectories corresponding to the one-to-all random broadcast heading control.

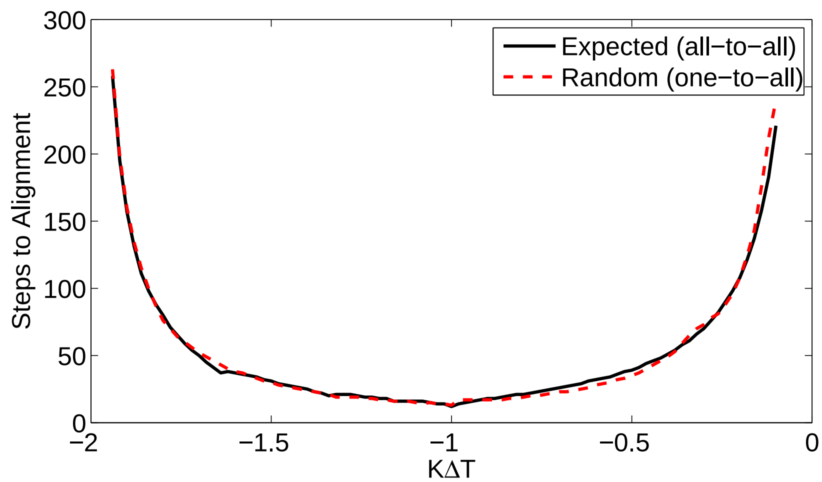

Fig. 3. The general trend of number of iterations to convergence of headings to an ϵ-ball versus $K\Delta T$ for both an expected broadcast (i.e. all-to-all) and an actual one-to-all broadcast sequence (16) is shown. The same randomly selected broadcast sequence was used for each value of $K\Delta T$.

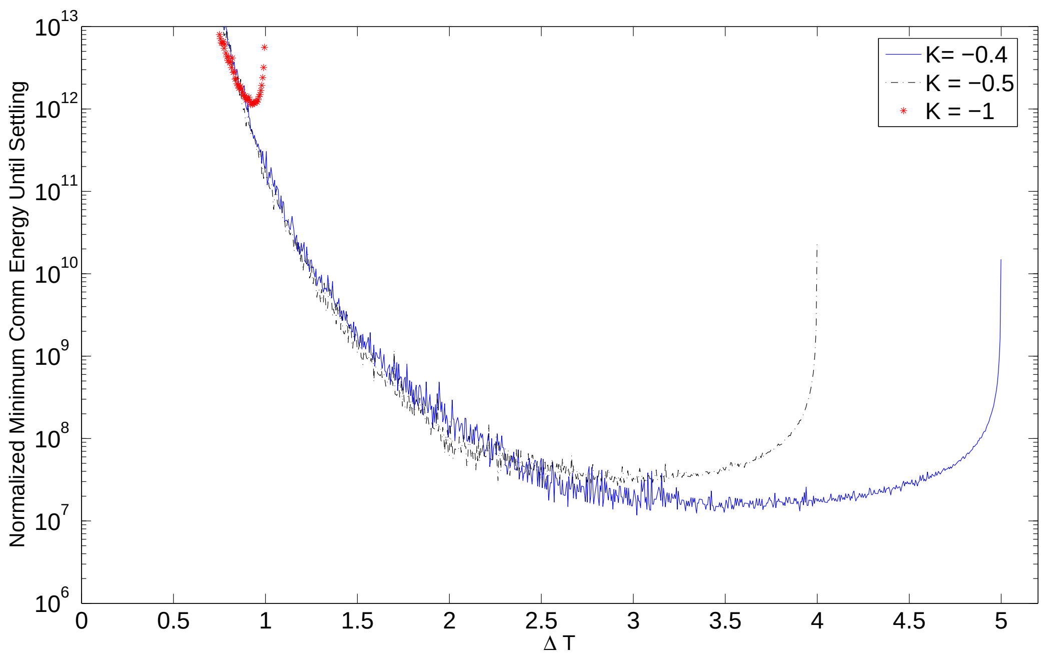

Fig. 4. Total communication energy required for the alignment of a system of vehicles to reach an ϵ-ball as a function of ΔT and for particular values of K. Each data point corresponds to the average of total energy for 100 simulations with initial vehicle positions drawn uniformly from $[-15, 15] \times [-15, 15]$, initial headings drawn uniformly from $[-\pi, \pi]$ and random one-to-all broadcast.