Sensitivity Analysis for an Optimal Routing Policy in an Ad Hoc Wireless Network

Tara Javidi and Demosthenis Teneketzis
Department of Electrical Engineering and Computer Science
University of Michigan
Ann Arbor, MI 48109
taraj@eecs.umich.edu teneket@eecs.umich.edu

Abstract

We examine the sensitivity of optimal routing policies in ad hoc wireless networks with respect to estimation errors in channel quality. We consider an ad hoc wireless network where the wireless links from each node to its neighbors are modeled by a probability distribution describing the local broadcast nature of wireless transmissions. These probability distributions are estimated in real-time. We investigate the impact of estimation errors on the performance of a set of proposed routing policies.

I. Introduction

Due to the variability and uncertainty in the behavior of the wireless channel, wireless networks should be modeled as stochastic systems. Hence, service provisioning and resource allocation issues (such as admission control, routing, etc.) in wireless networks are best modeled as stochastic scheduling and stochastic control problems, where the wireless links are described by stochastic processes. The statistics of any wireless link depends on the physical channel (additive noise, path loss, shadowing, fading, etc.), the number of users that use the link simultaneously, and the users' transmission strategies. Generally, the overall structure and statistical behavior of the system, e.g. the marginal and joint distributions of the processes involved, is studied and modeled off-line, while the particular parameters of such models, e.g. mean and covariance, are left to be estimated in a real-time measurement-based fashion. For instance, a single-hop wireless link might be modeled as an independent identically distributed binary symmetric channel, whose transmission error probability p_e is estimated on-line. In such an approach the control strategy regulates all communications in the system, hence it can potentially provide information on the statistics of the wireless channels. Hence, even in situations where the wireless system is controlled in a centralized manner, the estimation problem combined with the control issues, should be ideally studied as a stochastic control problem with imperfect information. Stochastic control problems with imperfect information are dual control problems that address joint estimation and control issues. The information state [1] for these problems lies in an infinite dimensional space even when the state-space and action space are finite. This feature makes such dual control problems analytically and computationally difficult. An alternative approach is to decouple the estimation and control issues in dual control problems. Such an approach provides a parameter estimation algorithm which operates independently of the control decisions and feeds the estimated parameters into a controller designed under the perfect information assumption. Following such an approach, service provisioning in wireless networks can be addressed by the following three step procedure: (i) Off-line modeling of the overall statistical behavior of the wireless links; (ii) specification of the parameters associated with link models by exploiting parameter estimation algorithms, based on real-time measurements; and (iii) determination of optimal service provisioning strategies assuming that the results of steps (i) and (ii) describe the system's true stochastic behavior. As expected, there are errors associated with the estimation techniques used in (ii) and the accuracy of the estimated parameters is limited to the employed estimation algorithm's error margin. On the other hand, the optimal control strategy resulting from (iii) is guaranteed to be optimal only for the particular values of parameters given by (ii); and it generally varies when these parameters change. Hence, it is vital to quantify the loss in performance of the proposed provisioning strategy with respect to the aforementioned estimation errors.

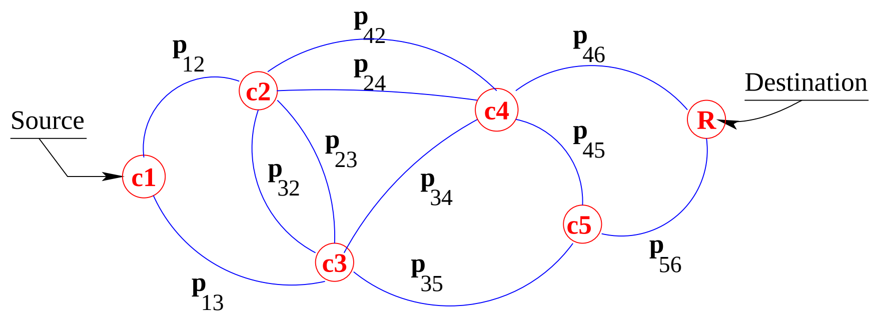

Fig. 1. Ad hoc wireless network with probabilistic local broadcast model

In this chapter we present a sensitivity analysis of a known optimal (with respect to an energy consumption criterion) routing policy in a stochastic ad hoc network. Our analysis is based on the model and results of [2]. In [2], the authors investigate a time-invariant network routing problem where a probabilistic model for wireless local broadcasts is used (see Fig. 1). Under the assumption that the transmission probabilities of the local broadcast model for each node are precisely known, the existence of an optimal priority policy with time-invariant indices is shown in [2]. As expected, these indices depend on the parameters of the local broadcast model. There exist centralized algorithms which compute these indices off-line, although such priority policies can always be implemented in a distributed fashion (see [2]). We investigate the sensitivity of this priority policy with respect to errors in the knowledge of the aforementioned transmission probabilities, and analytically determine the impact of errors in the broadcast model on the performance of the optimal policy. We quantify this impact as follows: (i) we first establish appropriate distance measures between two probabilistic broadcast models, and between two policies in terms of their performance; (ii) we construct policies $\tilde{\pi}$ and π^* that are optimal for to the true broadcast model P and the estimated broadcast model Q, respectively; and (iii) we bound the distance between the performance of the two policies $\tilde{\pi}$ and π^* by a term proportional to the distance between broadcast models P and Q (estimation error).

One key feature of an ad hoc network is that there exists no central control or computation unit to supervise the implementation and calculation of routing decisions. This feature underlines the importance of providing a distributed algorithm for computation and implementation of an optimal policy. The authors in [2] provide algorithms in which each node computes its optimal local routing actions in a distributed fashion, i.e. each node only uses the local information available to it to make routing decisions. It is shown that under a technical condition, which holds almost in all real scenarios, these algorithms all converge to an optimal stationary policy which is consistent with the optimal index policy computed centrally. Combining this result with our sensitivity analysis for the centralized control problem, we extend our sensitivity results to optimal routing strategies that are computed in a distributed fashion.

The remainder of this chapter is organized as follows. In Section II, we formulate the problem we analyze. In Section II-A we first present the formulation of the routing problem in an ad hoc networks, provide some useful notation and definitions and state the result given by [2] on the structure of the optimal routing strategy. In Section II-B, we formulate the problem of sensitivity of routing policies with respect to estimation errors and establish the desirable goals of such study. In Section II-B.2, we construct examples to illustrate that such goals may not always be achievable. In Section III we present the main result of our sensitivity analysis. In Section III-A we establish the mathematical relationships between loss of performance and the error in the estimation of the local broadcast models. In Section III-B, we discuss the essence of our sensitivity results through examples. In Section III-C, we extend our results to Model (**M4**), where the optimal routing policies are computed and constructed in a distributed fashion. In Section IV, we conclude the chapter.

II. Problem Formulation

We first revisit the following problem (Problem ($\mathbf{P1}$)) formulated in [2], and state the results proved in [2] that are necessary for our analysis. Based on Problem ($\mathbf{P1}$), we formulate Problem ($\mathbf{P2}$) which is an investigation of the sensitivity of an optimal routing policy with respect to errors in estimation of the transmission probabilities which are described by the local broadcast model.

A. Problem ($\mathbf{P1}$): Statement and Results

A.1 Model ($\mathbf{M}$), Notation, and Preliminaries

We begin by briefly defining notation and stating definitions for the system model, which we refer to as Model ($\mathbf{M}$), under consideration. As discussed in [2], Model ($\mathbf{M}$) is probabilistic, and control is centralized, meaning that the controller has access to all the information available at the network. A description of the elements of this network model is given below.

N is the number of nodes in the network.

$\Omega = \{1, \ldots, N\}$ is the set of all nodes. So $|\Omega| = N$.

$S \subseteq \Omega$ refers to a state of the system, defined as the set of nodes which have received the message. S_t refers to the state at time t.

We define $\mathcal{S} := \{S : S \subset \Omega\}$.

We write $P^i(S'|S)$ to indicate the probability of reaching state S' from state S when choosing node i for transmission, $i \in S$. We write $P^i(S|i)$ as shorthand for $P^i(S|\{i\})$.

We refer to $P = \{P^i(S'|S)\}_{i,S',S}$ as the broadcast model.

We define $P_{ij} := \sum_{S:i,j \in S} P^i(S|i)$.

We assume that transmission events at a given node are i.i.d., and transmission events are independent among all nodes.

Node j is called a *neighbor* of node i if $P_{ij} > 0$.

Given the local broadcast model P, $\mathcal{N}_P(i)$ is the set of all neighbors of i, together with i itself. Note that $P_{ij} \neq P_{ji}$ is permitted.

Definition 1 (Increasing Property) Model ($\mathbf{M}$) is said to have the *increasing property* if for any system realization under any policy we have $S_{t_2} \supseteq S_{t_1}, \;\; \forall t_1, \forall t_2 > t_1$.

Definition 2 (Decoupling Property) Model ($\mathbf{M}$) is said to have the *decoupling property* if successful transmission from a node to a set of neighbors at a given time is unaffected by which other nodes already have the message.

We assume that Model ($\mathbf{M}$) has both the increasing and decoupling properties.

$R : 2^{\Omega} \to \mathbb{R}^+$ is the reward function, and $R_i := R(\{i\})$. Also $R_{\max} := \max_{i \in \Omega} R_i$.

π is a Markov policy.

We write $\pi(S) = i$ to indicate policy π transmits at node i when in state S.

We write $\pi(S) = r$ to indicate policy π retires and receives reward $R(S)$ when in state S. For convenience we write $\pi(S) = r_i$ as shorthand that policy π retires and receives $R_i, i \in S$. In this case, we say that policy π retires and *receives the reward* of node i.

By $\pi(S) \neq i, r_i$, we mean both $\pi(S) \neq i$ and $\pi(S) \neq r_i$.

By $\pi(S) = \tilde{\pi}(S)$, we mean either $\pi(S) = \tilde{\pi}(S) = i$, or $\pi(S) = \tilde{\pi}(S) = r_i$, for some i.

Each transmission from node i incurs a cost of c_i.

We next formulate the centralized version of the stochastic routing problem with time-invariant parameters.

A.2 Statement of Problem ($\mathbf{P1}$)

Problem ($\mathbf{P1}$) We consider the transmission of a single message, from a given initial state S_0 (i.e. a given set of nodes) to a set of destination states, in a wireless ad hoc network of N nodes described by Model ($\mathbf{M}$) in which the transition probabilities are given by the broadcast model P. Transmission instances occur at discrete time points. Each transmission from a given node i incurs a fixed cost $c_i > 0$. According to Model $(\mathbf{M})$: (i) at each transmission instance the transmitting node is chosen by a central controller that always knows the current state of the system (i.e. the set of nodes that have the message); (ii) node transmissions are local broadcasts, that is, multiple neighbor nodes may all simultaneously receive the message; (iii) given the node chosen to transmit, the probability that a given set of nodes receives the message is known and fixed; (iv) The central controller is informed, without any cost, as to which nodes receive the message. Control information flow between the nodes and the controller is considered free of energy and instantaneous in time; (v) each transmission event is assumed independent of those before and after; (vi) a reward function R is specified. At any instance, the central controller can terminate the transmission process or choose to continue transmitting. The objective is to choose: (i) the node to transmit at each transmission instance, and (ii) the instance to terminate the transmission process, to maximize over all Markov policies,

$$J_P^{\pi}(S_0) = E^{\pi}\left\{R(S_f) - \sum_{t=1}^{\tau-1} c_{i(t)}\right\}, \tag{1}$$

where π is the transmission/termination policy the controller follows, τ is the time when the transmission process is terminated under policy π, S_f is the state at τ, $i(t)$ is the node chosen by the transmission policy at time t, and $J_P^{\pi}(S)$ is the expected reward when starting in state S under policy π under local broadcast model P.

Restriction to Markov policies does not entail any loss of optimality because Problem $(\mathbf{P1})$ is a stochastic control problem with perfect observations [1].

Mathematically, Problem $(\mathbf{P1})$ is parameterized by a tuple $(N, P, \underline{c}, R)$.

A.3 The Transmission Control Problem

Consider an ad hoc network in which control of transmission type (in terms of power, antenna directionality, and addressing) is allowed. In such a network at each time step the central controller chooses a node for transmission, among the nodes with the message, and a transmission type, among a finite set of allowable types, is chosen for that node. To each node i and transmission type k, a transmission cost $c_{(i,k)}$ and a probability distribution, denoted by $P^{(i,k)}(S|i)$, describing the probability that a given set of nodes receive the message are assigned. The objective for the controller in such a network is to determine a policy which maximizes a total reward similar to that of equation (1). Such a policy specifies the optimal number and coverage of hops, along each realization of the operation of network.

Such a network can be modeled by Model $(\mathbf{M})$ and can be formulated as Problem $(\mathbf{P1})$ with a particular structure on the probability of successful transmission, i.e. a particular structure on the broadcast model. This is possible since in Model $(\mathbf{M})$ no particular assumption regarding the statistical correlation among the transition probabilities has been made. We seek to construct a particular network described by Model $(\mathbf{M})$, which satisfies the conditions required by the addition of multiple transition types resulting from the use of multiple power levels. In order to do so we represent each node i as a set of sister nodes with cardinality W_i, where W_i refers to the number of transition types available at node i. Each sister node in such a set represent a transmission type for node i, and is identified by the pair (i, k), where $i \in \Omega$ and $k = 1, 2, \dots, W_i$. We define Ω_p to be the collection of these sets of sister nodes. Transmissions in the Ω_p space are based on the corresponding events in Ω, as follows. Each transmission at node $i \in \Omega$ with transmission type k corresponds to a control decision u which chooses node $(i, k) \in \Omega_p$. Such transmission incurs a cost $c_{(i,k)}$. If such a transmission leads to a set of nodes, say $S_1 \in \Omega$, receiving the message, all nodes $(l, k) \in \Omega_p$ such that $l \in S_1$ receive the message. This implies that all sister nodes receive the message simultaneously, or in other words, message receptions for sister nodes in Ω_p are deterministically coupled. Finally each sister node of i receives the same reward described by the reward function.

The problem of optimal routing in the ad hoc wireless network described by Ω_p is a special case of Problem ($\mathbf{P1}$), when transmission probabilities have a particular coupled structure to accommodate different transmission types as sister nodes at which the message receptions are deterministically decoupled (for more details see [2] and [3]). Hence the sensitivity result derived in this chapter apply to a wireless ad hoc network with power control and multiple transmission types.

A.4 Critique of Model ($\mathbf{M}$) and Problem ($\mathbf{P1}$)

In this section we briefly discuss the fundamental assumptions and modeling choices in Problem ($\mathbf{P1}$). We, like the authors in [3], point out that a model's value depends on how well it captures important aspects of reality. Furthermore, ad hoc networks vary greatly in their characteristics in terms of mobility, transmission channels, etc. Hence, the validity of the model depends on these characteristics, i.e. a good model for a particular network with low mobility and heavy traffic might be of no use for a heavily dynamic network with rare messages and vice versa. In this section we discuss the fundamental assumptions used in the construction of Model ($\mathbf{M}$), and their implications on the utility of the resulting optimal routing strategy. The discussion here is qualitative aimed at providing insight into the applicability of Model ($\mathbf{M}$) to various ad hoc networks. Hence no attempt is made to precisely quantify the phenomena we address.

The first assumption is the broadcast nature of local transmissions in Model ($\mathbf{M}$). This assumption is justified by the omni-directionality of the antennas. It implies the ability of multiple neighbors to receive a message simultaneously, decode its header, and acknowledge the transmitter.

Another assumption is the time-invariant nature of the reward for the reception of a message at its destination, and a fixed consumption of transmission energy (can be extended to average reception) at each node. As a consequence of this assumption, the objective is to minimize the energy transmission, even at the expense of timeliness or system capacity.

In Model ($\mathbf{M}$), each message is considered in isolation. This implies that the result is valid mainly for systems where in spite of the presence of sufficient messages in the system to justify Route Maintenance, the neighbor interference is not the dominant channel impairment.

Finally, in Model ($\mathbf{M}$) it is assumed that transmission events at a given node are described by a time-invariant conditional distribution, which is independent among all nodes. We clarify the meaning of this assumption in terms of the channel and network models. Though this assumption implies a time homogeneity on transmission probabilities, we can satisfy this requirement with less rigid assumptions such as (1) slow change of topology and (2) restriction of channel degradation to either "slow" or "fast" channel effects. The dynamics of transmission success depend on the characteristic times of different on-going processes. By characteristic time, we mean the approximate time for a process (e.g. a change of network topology, arrival of a message to its destination, etc.) to be completed. We define the following characteristic times of fundamental network and channel processes.

$$
\begin{aligned}
\tau_n &:= \text{characteristic time of significant network topology variation} \\
\tau_t &:= \text{characteristic time of message transit from source to destination} \\
\tau_m &:= \text{characteristic time of a one node message transmission} \\
\tau_s &:= \text{characteristic time of average path loss} \\
\tau_l &:= \text{characteristic time of log-normal shadowing} \\
\tau_p &:= \text{characteristic time of multi-path fading}
\end{aligned}
$$

The assumption of iid successful transmission probabilities is reasonable and justified if one of the following two relations is true.

$$
\begin{aligned}
\tau_p &\ , \tau_l \ll \tau_m \leq \tau_t \ll \tau_n, \tau_s \\
\tau_p &\ll \tau_m \leq \tau_t \ll \tau_l, \tau_n, \tau_s
\end{aligned}
$$

A.5 Preliminary Results: Optimal Routing in Problem ($\mathbf{P1}$)

In this section, we summarize the results in [2] that are relevant to our work. For that matter we need the following definitions:

Definition 3: A Markov policy π is a priority policy if there is a strict priority ordering of the nodes s.t. $\forall i \in \Omega$ we have $\pi(S \cup \{i\}) = \pi(\{i\}) = i$ or $r_i, \forall S \subset \Omega_i^\pi$, where Ω_i^π is the set of nodes of priority lower than i.

Definition 4: For priority policy π, we write $i >_\pi j$ when i has higher priority than j under π.

Definition 5: For priority policy π and node i, we denote by $U_P^\pi(i)$ the class of higher priority subsets of neighbors of i, i.e. $U_P^\pi(i) := \{S : i \in S \subset \mathcal{N}_i(P), \pi(S) \neq i\}$. Similarly we define $L_P^\pi(i) := \{S : i \in S \subset \mathcal{N}_i(P), \pi(S) = i\}$

Now we state the following facts from [2] (for more details see [3]).

Fact. 1: For priority policy π we have $J_P^\pi(S) = J_P^\pi(\{\pi(S)\}) = J_P^\pi(\{i\})$, when $i >_\pi j$ for $\forall j \in S - \{i\}$.

This fact holds by the decoupling property and the definition of a priority policy.

Fact. 2: For priority policy π and any state S where $\pi(S) = i$ or r_i we can write the expected reward as

$$J_P^\pi(S) = J_P^\pi(\{i\}) = \begin{cases} -c_i + \sum_{S' \supset \{i\}} P^i(S'|i) J_P^\pi(\{\pi(S')\}) & \text{if } \pi \text{ selects transmission} \\ R_i & \text{if } \pi \text{ selects retirement} \end{cases} \tag{2}$$

Fact. 3: There is an optimal Markov policy π^* for Problem ($\mathbf{P1}$) which is a priority policy whose expected reward has the following property:

$$J_P^{\pi^*}(S) = \max_{i \in S} J_P^{\pi^*}(\{i\}) \tag{3}$$

Fact. 4: Under the optimal Markov policy π^* the expected reward for each node i defines an index

$$J_P^{\pi^*}(\{i\}) = \max \left\{ \frac{-c_i + \sum_{S \in U_P^{\pi^*}(i)} P^i(S|i) J_P^\pi(\{\pi(S)\})}{\sum_{S \in U_P^{\pi^*}(i)} P^i(S|i)}, R_i \right\}; \tag{4}$$

this index, in turn, defines an optimal ordering of the nodes and the actions taken at these nodes; i.e.

$$\begin{aligned} J_P^{\pi^*}(\{i\}) > J_P^{\pi^*}(\{j\}) &\implies i >_{\pi^*} j \\ i >_{\pi^*} j &\implies J_P^{\pi^*}(\{i\}) \geq J_P^{\pi^*}(\{j\}) \end{aligned} \tag{5}$$

All these facts which are proved in [2] establish that, under the assumption that the network parameters are fixed and known, there exists an optimal routing priority ordering of nodes. The authors in [2] propose a centralized algorithm (Algorithm 1 in [3]) with complexity $O(N^2)$ which computes the optimal priority listing of nodes in Problem ($\mathbf{P1}$) when all the network parameters are known and available at the same location (this algorithm is similar in nature to standard Dijkstra in OSPF [4]). Furthermore, three distributed algorithms to compute the optimal priority listing of the neighbors at each node are proposed. Note that due to the nature of the optimal priority policy, there is a natural distributed implementation of such policy; such an implementation requires, for each node, a priority list of the node itself and its neighbors. A node transmits until another node of higher priority successfully receives the message. In other words, the network layer does not dictate the route, and in fact there is no *one* route, but the actual route a message takes between source and destination is sample path dependent. This characteristic of the solution is distinctly different from that of other proposed algorithms (see [5], [6], [4], [7], [8]).

The above model and all proposed algorithms assume knowledge of the transmission probabilities P. To actually implement the algorithms, methods to estimate these probabilities should be employed (see discussion in [3]). On the other hand, such methods introduce various levels of error. In other words, in reality P is not known but has to be estimated. The presence of estimation errors raises the important issue of the sensitivity of the results in [2] with respect to (small) variations in P. This motivates the sensitivity analysis presented in this chapter.

In the remainder of this chapter, we frequently use the following lemma. Lemma 1 is a direct consequence of the results given in [2].

Lemma 1: Let π^* and $\tilde{\pi}$ be two optimal priority routing policies under broadcast models P and Q, respectively. Assume that $\pi^*(\{i\}) = i$ ($\tilde{\pi}(\{i\}) = i$), that is, policies π^* and $\tilde{\pi}$ do not retire when the state is $\{i\}$. Then

$$\sum_{S \in U_P^{\pi^*}(i)} P^i(S|i) \geq \frac{c_i}{R_{\max} - R_i} > 0 \qquad (\sum_{S \in U_Q^{\tilde{\pi}}(i)} Q^i(S|i) \geq \frac{c_i}{R_{\max} - R_i} > 0).$$

Proof. $\pi^*(\{i\}) = i$ implies that

$$R_i \leq \frac{-c_i + \sum_{S \in U_P^{\pi^*}(i)} P^i(S|i) J_P^{\pi^*}(\{\pi^*(S)\})}{\sum_{S \in U_P^{\pi^*}(i)} P^i(S|i)} \tag{6}$$

On the other hand, for $\forall S \in \mathcal{S}$ and any policy π we have $J_P^{\pi}(S) \leq R_{\max}$. Hence,

$$R_i \leq \frac{-c_i}{\sum_{S \in U_P^{\pi^*}(i)} P^i(S|i)} + R_{\max} \tag{7}$$

This implies that $\sum_{S \in U_P^{\pi^*}(i)} P^i(S|i) \geq \frac{c_i}{R_{\max} - R_i} > 0$.

Similarly, if $\tilde{\pi}(\{i\}) = i$, $\sum_{S \in U_Q^{\tilde{\pi}}(i)} Q^i(S|i) \geq \frac{c_i}{R_{\max} - R_i} > 0$. The proof of Lemma 1 is now complete.

■

Based on Model ($\mathbf{M}$) and the above preliminary results we proceed with the sensitivity analysis of Problem ($\mathbf{P1}$).

B. Sensitivity Analysis: Problem Formulation

Consider Problem ($\mathbf{P1}$) associated with two sets of system parameters, $(N, P, \underline{\mathbf{c}}, R)$ and $(N, Q, \underline{\mathbf{c}}, R)$, describing the true and estimated models of the system, respectively. According to the results given in Section II-A.5 there exists an index policy π^* which is an optimal routing policy for Problem ($\mathbf{P1}$) with parameters $(N, P, \underline{\mathbf{c}}, R)$. At the same time the optimal solution to the estimated model, $(N, Q, \underline{\mathbf{c}}, R)$, is an index policy $\tilde{\pi}$ that is not optimal for the true model $(N, P, \underline{\mathbf{c}}, R)$, in general. Policy $\tilde{\pi}$ is applied to the system with the true broadcast model (distribution) P. We are interested in: (i) Determining/quantifying the difference between the performance of policy $\tilde{\pi}$ in such a system and the best possible performance, achieved by π^*. (ii) Relating the aforementioned difference to a quantity describing the estimation error in the (true) broadcast model.

To quantify the difference specified in (i) we define an appropriate metric on the space of all routing policies. We define *the distance between policies* π_1 *and* π_2 *at state* S *in the context of the distribution* P as

$$d_P(\pi_1, \pi_2, S) := |J_P^{\pi_1}(S) - J_P^{\pi_2}(S)|. \tag{8}$$

We define *the distance between policies* π_1 *and* π_2 *in the context of distribution* P as

$$d_P(\pi_1, \pi_2) := \max_S |J_P^{\pi_1}(S) - J_P^{\pi_2}(S)| \tag{9}$$

To relate the difference specified in (i) to the estimation error in the (true) broadcast model we first quantify this error by defining a distance measure between the true broadcast model P and the estimated model Q. We use the total variation metric for this purpose (see [9], [10]). The total variation distance between two local broadcast models, P and Q, describing the probabilities of transmission success for node i, is defined as

$$\sigma(P_i, Q_i) = \sup_{A \subset \mathcal{S}} \left| \sum_{S' \in A} \left(P^i(S'|i) - Q^i(S'|i) \right) \right|. \tag{10}$$

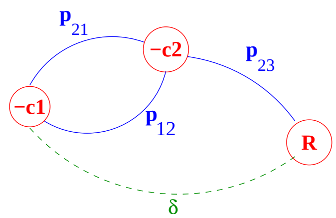

Fig. 2. Example 1.

We extend this measure to define the total variation distance between broadcast models P and Q as

$$\sigma(P, Q) = \max_i \sigma(P_i, Q_i). \tag{11}$$

Based on the above we reformulate the following sensitivity analysis problem.

Problem (**P2**) Consider Problem (**P1**) for two sets of parameters, $(N, P, \underline{\mathbf{c}}, R)$ and $(N, Q, \underline{\mathbf{c}}, R)$, describing the true and estimated models of the system, respectively. Let π^* be an optimal routing policy for Problem (**P1**) with parameters $(N, P, \underline{\mathbf{c}}, R)$, and $\tilde{\pi}$ be an optimal routing policy for Problem (**P1**) with parameters $(N, Q, \underline{\mathbf{c}}, R)$. The objective is to determine the distance between policies $\tilde{\pi}$ and π^* in the context of P, and relate this distance to the total variation distance between the estimated model Q and the true system model P, i.e. $\sigma(P, Q)$.

B.1 Basic Assumption on Convergence Rate of Estimation Algorithm

As mentioned before, the estimated local broadcast model Q is constructed through an on-line parameter estimation algorithm. Denote by τ_e the characteristic time of the estimation algorithm, that is the time required for the estimation algorithm to converge to some estimate Q.

We assume throughout the following analysis that $\tau_e + \tau_c \ll \tau_n$, where τ_c is the characteristic time for the computation and dissemination of such calculation through the network, and τ_n is the characteristic time of significant network topology variation, defined in Section II-A.4.

The following example shows that, in general, optimal routing can be extremely sensitive to estimation errors, i.e. there exist scenarios where a small error in estimation can unboundedly deteriorate the performance of the constructed priority policy.

B.2 Example

Consider the simple network given by Fig. 2. We assume that transmission success probabilities are given by the true model P and the estimated values of these probabilities are given by model Q. The value of these transmission probabilities are $P^1(\{1,2,3\}|1) = Q^1(\{1,2,3\}|1) = 0$, $P^1(\{1,2\}|1) = Q^1(\{1,2\}|1) = p$, $P^2(\{1,2\}|2) = Q^2(\{1,2\}|2) = p$, $P^2(\{1,2,3\}|2) = Q^2(\{1,2,3\}|2) = 0$, and finally $P^2(\{2,3\}|2) = Q^2(\{2,3\}|2) = p$. Furthermore, we assume $P^1(\{1,3\}|1) = 0$ while $Q^1(\{1,3\}|1) = \delta$. Assume that node i has transmission costs c_i; $i = 1, 2$. Rewards are zero for the first two nodes, and it is equal to $R > \frac{c_i}{\delta}$ at the destination. The cost of transmission at node 2 is much larger than the cost of transmission at node 1, i.e. $c_2 \gg c_1$.

In this example the total variation distance between the two broadcast models P and Q, i.e. $\max_i \sigma(P_i, Q_i)$ is δ. There are two priority policies π^* and $\tilde{\pi}$ possible in this network. Under these policies we have $1 <_{\pi^*} 2$ and $2 <_{\tilde{\pi}} 1$. The distance between these two policies in the context of P is infinite, since $J_P^{\tilde{\pi}}(\{1\}) = -\infty$. We show that even for a small distance between models P and Q, i.e. small δ, policy $\tilde{\pi}$ can be selected as the optimal policy (due to its optimality in the context of Q). To prove this, we write the expected rewards:

$$\begin{aligned} J_Q^{\pi^*}(\{2\}) &= R - \frac{c_2}{p} \\ J_Q^{\pi^*}(\{1\}) &= R - \frac{c_1 + c_2}{\delta + p} \\ J_Q^{\tilde{\pi}}(\{1\}) &= R - \frac{c_1}{\delta} \\ J_Q^{\tilde{\pi}}(\{2\}) &= R - \frac{c_2}{2p} - \frac{c_1}{2\delta} \end{aligned} \tag{12}$$

Since $c_2 \gg c_1$, there exists a (small) δ for which $\tilde{\pi}$ is selected as the optimal priority policy when the estimated distribution Q is assumed to describe the transmission success. On the other hand, $J_P^{\tilde{\pi}}(\{1\}) = -\infty$. Therefore we have $|J_P^{\pi_1}(1) - J_P^{\pi_2}(1)| = \infty$ even though $\max_i \sigma(P_i, Q_i) = \delta$.

This example illustrates that in general a small error in channel estimation can cause an unbounded decrease in performance.

III. Sensitivity Analysis

A. Analysis of Problem ($\mathbf{P2}$)

In this section our goal is to bound the distance between two policies $\tilde{\pi}$ and π^* by a term proportional to the distance between the broadcast models P and Q. As illustrated in Example II-B.2, this is not possible in general. Thus, to obtain a positive result, we make the following assumption on the nature of the estimation error.

Assumption 1: For any node i such that $\tilde{\pi}(i) = i$, there exists $\Delta_i \leq 1$ such that

$$\frac{\sigma(P_i, Q_i)}{\sum_{S \in U_Q^{\tilde{\pi}}(\eta_i)} Q^{\eta_i}(S|\eta_i)} \leq \Delta_i.$$

Intuitively Assumption 1 implies that the magnitude of the estimation error at each node can not be larger than the total estimated probability of routing the message "closer" to the destination. In other words, transmission at each node has a true positive probability of reaching a better node.

Under the above assumption we seek to bound the loss of performance by a term proportional to the estimation error. To do so we first present the following definition which will simplify our notation.

Definition 6: For any measurable function $\nu : \mathcal{S} \longmapsto \mathbb{R}$ which is $P^i(.|S)$-integrable, we define the *Markov operator*

$$P\nu(S, i) := \sum_{S' \in \mathcal{S}} P^i(S'|S)\nu(S') \tag{13}$$

Using this definition we can write the expected reward for priority policy π at state S, assuming $\pi(S) = i$ or r_i, as

$$J_P^{\pi}(S) = J_P^{\pi}(\{i\}) = \begin{cases} -c_i + PJ_P^{\pi}(S, i) & \text{if } \pi \text{ selects transmission} \\ R_i & \text{if } \pi \text{ selects retirement} \end{cases} \tag{14}$$

We begin by establishing Lemma 2 which relates the overall distance between two policies $\tilde{\pi}$ and π^* to the distance between the policies at each singleton $\{j\}$.

Lemma 2: $d_P(\pi^*, \tilde{\pi}) \leq \max_j \left\{ J_P^{\pi^*}(\{j\}) - J_Q^{\tilde{\pi}}(\{j\}) \right\} + \max_j \left| J_Q^{\tilde{\pi}}(\{j\}) - J_P^{\tilde{\pi}}(\{j\}) \right|$.

Proof. Let $i = \pi^*(S)$ and $j = \tilde{\pi}(S)$. Then,

$$\begin{aligned} d_P(\pi^*, \tilde{\pi}, S) &= \left| J_P^{\pi^*}(S) - J_P^{\tilde{\pi}}(S) \right| \\ &= J_P^{\pi^*}(S) - J_P^{\tilde{\pi}}(S) \\ &= J_P^{\pi^*}(\{\pi^*(S)\}) - J_P^{\tilde{\pi}}(\{\tilde{\pi}(S)\}) \\ &= J_P^{\pi^*}(\{i\}) - J_P^{\tilde{\pi}}(\{j\}) \\ &\leq J_P^{\pi^*}(\{i\}) - J_Q^{\tilde{\pi}}(\{i\}) + J_Q^{\tilde{\pi}}(\{j\}) - J_P^{\tilde{\pi}}(\{j\}) \\ &\leq J_P^{\pi^*}(\{i\}) - J_Q^{\tilde{\pi}}(\{i\}) + \left| J_Q^{\tilde{\pi}}(\{j\}) - J_P^{\tilde{\pi}}(\{j\}) \right| \end{aligned} \tag{15}$$

The second equality in equation (15) is true since π^* is optimal in the context of P, and the first inequality holds because policy $\tilde{\pi}$ is optimal in the context of Q.

Hence,

$$\begin{aligned} d_P(\pi^*, \tilde{\pi}) &= \max_S d_P(\pi^*, \tilde{\pi}, S) \\ &\leq \max_j \left\{ J_P^{\pi^*}(\{j\}) - J_Q^{\tilde{\pi}}(\{j\}) \right\} + \max_j \left| J_Q^{\tilde{\pi}}(\{j\}) - J_P^{\tilde{\pi}}(\{j\}) \right|. \end{aligned} \tag{16}$$

■

To bound the performance loss, we develop upper bounds on each of the maxima that appear in Lemma 2 (right hand side of equation (16)). Bounds on the first term, i.e. $\max_j \left\{ J_P^{\pi^*}(\{j\}) - J_Q^{\tilde{\pi}}(\{j\}) \right\}$, are obtained via Lemma 3 and Corollary 1. Bounds on the second term, i.e. $\max_j \left| J_Q^{\tilde{\pi}}(\{j\}) - J_P^{\tilde{\pi}}(\{j\}) \right|$, are obtained via Lemmas 4 and 5 and Corollary 2. The proofs of Lemmas 3 and 4 are given in Appendix C.

Lemma 3 below establishes at each node j a relationship between $J_P^{\pi^*}(\{j\}) - J_Q^{\tilde{\pi}}(\{j\})$ and the distance between broadcast models P and Q.

Lemma 3: Assume that $\{\xi_1, \xi_2, \cdots, \xi_N\}$ is the nodes' strict priority ordering under policy π^*, i.e. $\xi_j >_{\pi^*} \xi_{j+1}$. Then, we have

$$J_P^{\pi^*}(\{\xi_j\}) - J_Q^{\tilde{\pi}}(\{\xi_j\}) \leq \alpha(\xi_j) \max_{k \leq j} \sigma(P_{\xi_k}, Q_{\xi_k}), \tag{17}$$

where $\alpha(\xi_j)$ is increasing in j and satisfies the recursion

$$\alpha(\xi_1) = 0, \qquad \alpha(\xi_i) = \begin{cases} \frac{R_{\max}}{\sum_{S \in U_P^{\pi^*}(\xi_i)} P^{\xi_i}(S|\xi_i)} + \alpha(\xi_{i-1}) & \text{if } \pi^*(\xi_i) = \xi_i \\ \alpha(\xi_{i-1}) & \text{if } \pi^*(\xi_i) = r_{\xi_i} \end{cases}$$

Lemma 4 below establishes, at each node η_j, a relationship between $|J_P^{\tilde{\pi}}(\{\eta_j\}) - J_Q^{\tilde{\pi}}(\{\eta_j\})|$ and the distance between the broadcast models P and Q.

Lemma 4: Assume that $\{\eta_1, \eta_2, \cdots, \eta_N\}$ is the nodes' strict priority ordering under policy $\tilde{\pi}$, i.e. $\eta_j >_{\tilde{\pi}} \eta_{j+1}$. We have

$$|J_P^{\tilde{\pi}}(\{\eta_j\}) - J_Q^{\tilde{\pi}}(\{\eta_j\})| \leq \gamma(\eta_j) \max_{k \leq j} \sigma(P_{\eta_k}, Q_{\eta_k}), \tag{18}$$

where $\gamma(\eta_j)$ is increasing in j and satisfies the recursion

$$\gamma(\eta_1) = 0, \qquad \gamma(\eta_i) = \begin{cases} \frac{R_{\max}}{\sum_{S \in U_P^{\tilde{\pi}}(\eta_i)} P^{\eta_i}(S|\eta_i)} + \gamma(\eta_{i-1}) & \text{if } \tilde{\pi}(\eta_i) = \eta_i \\ \gamma(\eta_{i-1}) & \text{if } \tilde{\pi}(\eta_i) = r_{\eta_i} \end{cases}$$

Remark: In Example II-B.2 we have $\sum_{S\in U_P^{\tilde{\pi}}(3)} P^3(S|3) = 0$ which implies the unboundedness of the right hand side of equation (18). This is consistent with the result provided in Section II-B.2, establishing that $|J_P^{\tilde{\pi}}(\{3\}) - J_Q^{\tilde{\pi}}(\{3\})| = \infty$. Note that Lemmas 3 and 4 do not assume the validity of Assumption 1.

Lemma 5: Under Assumption 1, we have

$$|J_P^{\tilde{\pi}}(\{\eta_j\}) - J_Q^{\tilde{\pi}}(\{\eta_j\})| \le \phi(\eta_j) \max_{k\le j} \sigma(P_{\eta_k}, Q_{\eta_k}), \tag{19}$$

where $\phi(\eta_j)$ is increasing in j and satisfies the recursion

$$\phi(\eta_1) = 0, \qquad \phi(\eta_i) = \begin{cases} \frac{R_{\max}}{(1-\Delta_{\eta_i})\sum_{S\in U_Q^{\tilde{\pi}}(\eta_i)} Q^{\eta_i}(S|\eta_i)} + \phi(\eta_{i-1}) & \text{if } \tilde{\pi}(\eta_i) = \eta_i \\ \phi(\eta_{i-1}) & \text{if } \tilde{\pi}(\eta_i) = r_{\eta_i} \end{cases}$$

Proof.

Case 1: If $\tilde{\pi}(\eta_j) = r_{\eta_j}$, then $J_P^{\tilde{\pi}}(\{\eta_j\}) = J_Q^{\tilde{\pi}}(\eta_j) = R_{\eta_j}$. Hence,

$$|J_P^{\tilde{\pi}}(\{\eta_j\}) - J_Q^{\tilde{\pi}}(\{\eta_j\})| = 0. \tag{20}$$

On the other hand, the right hand side of equation (5) is always a positive number. Hence,

$$|J_P^{\tilde{\pi}}(\{\eta_j\}) - J_Q^{\tilde{\pi}}(\{\eta_j\})| \le \phi(\eta_j) \max_{k\le j} \sigma(P_{\eta_k}, Q_{\eta_k}) \tag{21}$$

This completes the proof in Case 1.

Case 2: If $\tilde{\pi}(\eta_j) = \eta_j$, then Assumption 1 implies that:

$$\frac{\sigma(P_{\eta_j}, Q_{\eta_j})}{\sum_{S\in U_Q^{\tilde{\pi}}(\eta_j)} Q^{\eta_j}(S|\eta_j)} \le \Delta_{\eta_j}. \tag{22}$$

On the other hand we have

$$\begin{aligned}
&\sum_{S\in U_P^{\tilde{\pi}}(\eta_i)} P^{\eta_i}(S|\eta_i) \\
&= \sum_{S\in U_P^{\tilde{\pi}}(\eta_i)\cup U_Q^{\tilde{\pi}}(\eta_i)} P^{\eta_i}(S|\eta_i) \ge \sum_{S\in L_Q^{\tilde{\pi}}(\eta_i)} P^{\eta_i}(S|\eta_i) \\
&= \sum_{S\in L_Q^{\tilde{\pi}}(\eta_i)} P^{\eta_i}(S|\eta_i) - \sum_{S\in U_Q^{\tilde{\pi}}(\eta_i)} Q^{\eta_i}(S|\eta_i) + \sum_{S\in U_Q^{\tilde{\pi}}(\eta_i)} Q^{\eta_i}(S|\eta_i) \\
&\ge \sum_{S\in U_Q^{\tilde{\pi}}(\eta_i)} Q^{\eta_i}(S|\eta_i) - |\sum_{S\in L_Q^{\tilde{\pi}}(\eta_i)} P^{\eta_i}(S|\eta_i) - \sum_{S\in U_Q^{\tilde{\pi}}(\eta_i)} Q^{\eta_i}(S|\eta_i)| \\
&\ge \sum_{S\in U_Q^{\tilde{\pi}}(\eta_i)} Q^{\eta_i}(S|\eta_i) - \sigma(P_{\eta_i}, Q_{\eta_i}) \\
&= \sum_{S\in U_Q^{\tilde{\pi}}(\eta_i)} Q^{\eta_i}(S|\eta_i) \left(1 - \frac{\sigma(P_{\eta_j}, Q_{\eta_j})}{\sum_{S\in U_Q^{\tilde{\pi}}(\eta_j)} Q^{\eta_j}(S|\eta_j)}\right) \\
&\ge \sum_{S\in U_Q^{\tilde{\pi}}(\eta_i)} Q^{\eta_i}(S|\eta_i)(1-\Delta_{\eta_i}) \\
&= (1-\Delta_{\eta_i}) \sum_{S\in U_Q^{\tilde{\pi}}(\eta_i)} Q^{\eta_i}(S|\eta_i)
\end{aligned} \tag{23}$$

where the first equality holds since for every $S \not\subset \mathcal{N}_P(i)$, $P^i(S|i) = 0$, the third inequality follows the definition of total variation metric, and the last inequality holds because of equation (22). The assertion of the lemma follows from equation (23) and Lemma 4

■

Combining Lemmas 2, 3, and 5 we establish the following theorem, which summarizes one of the two main results of this chapter.

Theorem 1: Under Assumption 1, we have

$$d_P(\pi^*, \tilde{\pi}) \leq R_{\max}(\Upsilon(P, \underline{0}) + \Upsilon(Q, \underline{\Delta})) \max_j \sigma(P_j, Q_j), \tag{24}$$

where the function $\Upsilon(A, \underline{\epsilon})$ denotes the optimal probabilistic connectivity for a broadcast model A, its corresponding optimal priority policy π, and the vector of bounds on estimation error denoted by $\underline{\epsilon} = [\epsilon_1, \epsilon_2, \dots, \epsilon_N]$ and is defined as

$$\Upsilon(A, \underline{\epsilon}) = \sum_{j \in \Omega, \pi(j) = j} \frac{1}{(1 - \epsilon_j) \sum_{S \in U_P^{\pi}(j)} A^j(S|j)} \tag{25}$$

The term $\Upsilon(P, \underline{0}) + \Upsilon(Q, \underline{\Delta})$ in equation (24) depends on the topology of the network under the true and the estimated broadcast models. This dependency provides insight into the study of sensitivity of the optimal routing policies with respect to the estimation error under various topological structures. It can be seen that loss of performance for networks where each transmission reaches a large set of higher priority nodes is smaller than for networks with "hot links," where the condition of a few links is critical in determining the ability of the routing policy to transfer the message to the destination. This is an important feature of the proposed optimal priority routing policy. We will elaborate on this in Section III-B.

On the other hand, sensitivity analysis might be used to provide guidelines in designing on-line estimation algorithms with acceptable margin of error. In such applications, the dependency of our bounds on the true and estimated structures and topology of the network can create difficulty. In general and in a practical setting, such models are not known and cannot be exploited to design appropriate algorithms. Furthermore, in an ad hoc network it is undesirable to assume any particular topological structure. For these applications, we provide Theorem 2 below to eliminate the dependency of our bound on the particular (and unknown) topology of the network. To do so, we need Corollaries 1 and 2.

Corollary 1 (of Lemma 3) Assume that $\{\xi_1, \xi_2, \cdots, \xi_N\}$ is the nodes' strict priority ordering under policy π^*. Then, we have

$$J_P^{\pi^*}(\{\xi_j\}) - J_Q^{\tilde{\pi}}(\{\xi_j\}) \leq \beta(\xi_j) \max_{k \leq j} \sigma(P_{\xi_k}, Q_{\xi_k}), \tag{26}$$

where $\beta(\xi_j)$ is increasing in j and satisfies the recursion

$$\beta(\xi_1) = 0, \qquad \beta(\xi_i) = \frac{R_{\max}(R_{\max} - R_{\xi_i})}{c_{\xi_i}} + \beta(\xi_{i-1}). \tag{27}$$

Proof. The assertion of the corollary follows directly from Lemmas 1 and 3.

■

Corollary 2 (of Lemma 5) Under Assumption 1, we have

$$|J_P^{\tilde{\pi}}(\{\eta_j\}) - J_Q^{\tilde{\pi}}(\{\eta_j\})| \leq \theta(\eta_j) \max_{k \leq j} \sigma(P_{\eta_k}, Q_{\eta_k}), \tag{28}$$

where $\theta(\eta_j)$ is increasing in j and satisfies the recursion

$$\theta(\eta_1) = 0, \qquad \theta(\eta_i) = \frac{R_{\max}(R_{\max} - R_{\eta_i})}{c_{\eta_i}(1 - \Delta_{\eta_i})} + \theta(\eta_{i-1}). \tag{29}$$

Proof. The assertion of the corollary follows directly from Lemmas 1 and 5.

■

We now use Corollaries 1 and 2 to prove Theorem 2, which provides a bound on the error independently of the topology.

Theorem 2: Under Assumption 1, we have

$$d_P(\pi^*, \tilde{\pi}) \leq K \max_j \sigma(P_j, Q_j), \tag{30}$$

where

$$K = \sum_{j=1}^{N} \frac{R_{\max}(R_{\max} - R_j)(2 - \Delta_j)}{c_j(1 - \Delta_j)}.$$

Proof. Combining Lemma 2 and Corollaries 3 and 4, we have

$$\begin{aligned}
d_P(\pi^*, \tilde{\pi}) &\leq \max_j \left\{ J_P^{\pi^*}(\{j\}) - J_Q^{\tilde{\pi}}(\{j\}) \right\} + \max_j \left| J_Q^{\tilde{\pi}}(\{j\}) - J_P^{\tilde{\pi}}(\{j\}) \right| \\
&\leq \max_j \left\{ \beta(\xi_j) \max_{k \leq j} \sigma(P_{\xi_k}, Q_{\xi_k}) \right\} + \max_j \left\{ \theta(\eta_j) \max_{k \leq j} \sigma(P_{\eta_k}, Q_{\eta_k}) \right\} \\
&= (\beta(\xi_N) + \theta(\eta_N)) \max_k \sigma(P_k, Q_k) \\
&\leq (\sum_{i=1}^{N} \frac{R_{\max}(R_{\max} - R_{\xi_i})}{c_{\xi_i}} + \sum_{i=1}^{N} \frac{R_{\max}(R_{\max} - R_{\xi_i})}{c_{\xi_i}(1 - \Delta_{\eta_j})}) \max_k \sigma(P_k, Q_k) \\
&= (\sum_{i=1}^{N} \frac{R_{\max}(R_{\max} - R_i)(2 - \Delta_i)}{c_i(1 - \Delta_i)}) \max_k \sigma(P_k, Q_k) \\
&= K \max_k \sigma(P_k, Q_k),
\end{aligned} \tag{31}$$

where the first inequality holds because of Lemma 2, the second inequality is a result of corollaries 3 and 4, and the first and second equalities result from the definition of functions β_i and θ_i and their monotonicity in the index i. ■

A.1 Sufficient Conditions for Assumption 1 to Hold

Assumption 1 is the minimum requirement needed to guarantee and obtain a finite bound on the sensitivity of the optimal index routing policy to the estimation error in the broadcast model. This assumption depends on the nature of the optimal policies (of both true model P and estimated model Q), hence the topology of the network. Nevertheless, there are stronger conditions which are independent of topology, are sufficient to guarantee Assumption 1, are easy to verify, and are given below.

Condition 1: For any node i, there exists $M_i < \infty$ such that

$$Q^i(S|i) - P^i(S|i) \leq M_i P^i(S|i).$$

Condition 1 guarantees that there exists $\Delta_j = \frac{M_j}{1+M_j}$ for which Assumption 1 is satisfied. Intuitively, Condition 1 has two significant implications. First, it implies that the network topology under the estimated broadcast model Q does not contain links which do not really exist, i.e. $\forall i \in \{1, 2, \cdots, N\}$, $\mathcal{N}_Q(i) \subset \mathcal{N}_P(i)$. Second, the

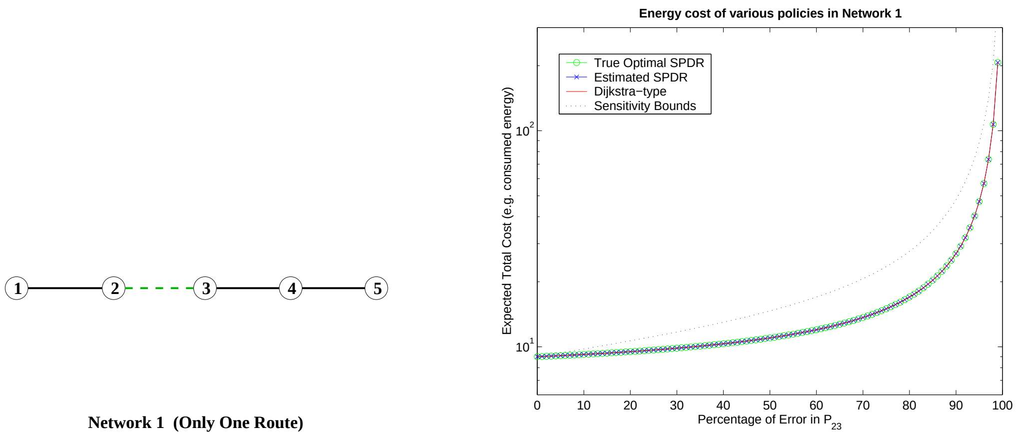

Fig. 3. Network Ω_1

condition implies that, whenever there is a link between nodes i and j, i.e. $P_{ij} > 0$, there is a finite bound M_i on the percentage of error in over-estimation of the quality of the link, i.e. $\frac{Q_{ij}-P_{ij}}{P_{ij}} \leq M_i$. In other words, M_i specifies the maximum error percentage in the estimation of the quality of links connected to node i.

Condition 2: The estimation error is bounded by the following expression:

$$\sigma(P_k, Q_k) \leq \Delta_k \frac{c_k}{R_{\max} - R_k}$$

From Lemma 1 it follows that Condition 2 guarantees that Assumption 1 holds. Although Condition 2 is too restrictive, it can be checked without any knowledge of the probabilistic topology of the network and/or the structure of the optimal policies. Unlike Condition 1, Condition 2 bounds the absolute value of error in estimation rather than the error percentage.

B. *Examples and Discussion*

In this section we present three networks ($\Omega_1, \Omega_2, \Omega_3$) given by figures III-B-III-A to illustrate how topology affects the sensitivity of the optimal routing policy with respect to the estimation error. Consider these three networks. We assume that successful transmissions along different links are independent. Hence Model ($\mathbf{M}$) for network Ω_k can be defined by $(N_k, Q_k, \underline{\mathbf{c}}_k, R_k)$ where Q_i is a transition matrix whose (i, j)th element represents the probability of successful transmission from node i to j. We assume $R_1 = R_2 = R_3 = 300$, $N_1 = 5$, $N_2 = 7$, $N_3 = 9$, $\underline{\mathbf{c}}_1 = [1, 1, 1, 1, 1]$, $\underline{\mathbf{c}}_2 = [\underline{\mathbf{c}}_1, 1.5, 3]$, and $\underline{\mathbf{c}}_2 = [\underline{\mathbf{c}}_2, 2.2, 4]$. We assume that the estimated value of the transmission probabilities are

$$Q_1 = \begin{pmatrix} 1 & 0.3 & 0 & 0 & 0 \\ 0.1 & 1 & 0.5 & 0 & 0 \\ 0 & 0.1 & 1 & 0.6 & 0 \\ 0 & 0 & 0.2 & 1 & 0.5 \\ 0 & 0 & 0 & 0.2 & 1 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} & & & & & 0.6 & 0.5 \\ & & & & & 0 & 0 \\ & & Q_1 & & & 0.1 & 0.1 \\ & & & & & 0 & 0 \\ & & & & & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0.25 & 0 & 0 & 0 & 1.0 \end{pmatrix}, \quad \text{and}$$

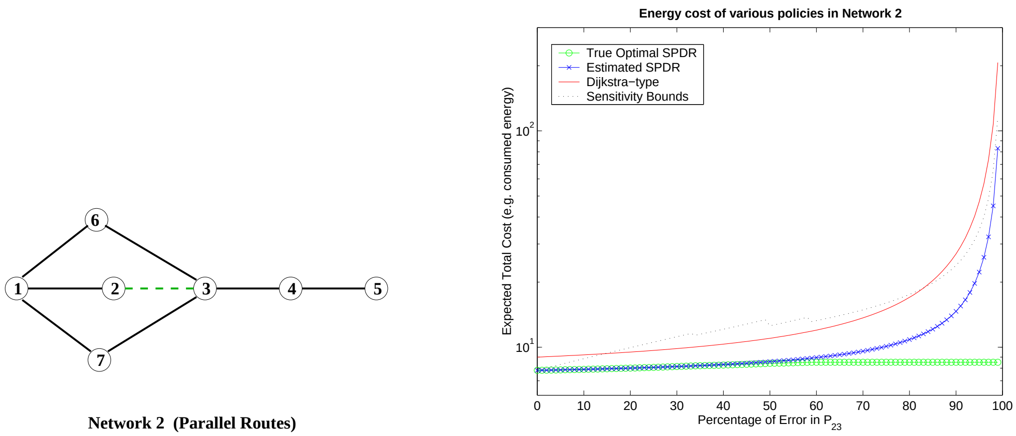

Fig. 4. Network Ω_2

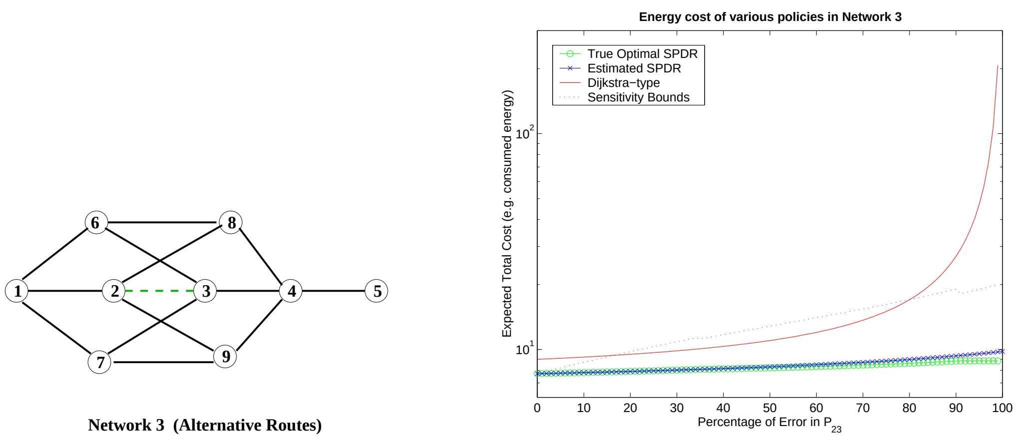

Fig. 5. Network Ω_3

$$Q_3 = \begin{pmatrix} & & & & & & & 0 & 0 \\ & & & & & & & 0.25 & 0.5 \\ & & & & & & & 0 & 0 \\ & & & Q_2 & & & & 0 & 0 \\ & & & & & & & 0 & 0 \\ & & & & & & & 0.45 & 0 \\ & & & & & & & 0 & 0.45 \\ 0 & 0 & 0 & 0.6 & 0 & 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0 & 0 & 0 & 1.0 \end{pmatrix}.$$

Furthermore, we assume that the probabilities of successful transmission between each pair of nodes are known except for the link between nodes 2 and 3 in each network where there is an ϵ error in the estimation of the quality of the link. Hence, for any error value ϵ and $k = 1, 2, 3$, we have $P_k(\epsilon) = Q_k - \epsilon \mathbf{1}_{23}$, where P_i represents the true broadcast model for Network i, and $\mathbf{1}_{ij}$ is a matrix whose only non-zero element is the (i, j)th element which is equal to 1. We vary the error ϵ from 0 to 100 percent of the estimated value of the link's transmission probability $Q(2, 3)$, and compare (1) the loss of performance in the three different networks, and (2) the bounds provided by Theorem 1 in each case. Furthermore, for comparison purposes, we provide the performance of a OSPF-type routing algorithm (see [5], [6], [4], [7], [8]) where some form of shortest route is established as the minimum energy route. In this routing algorithm a full route with minimum expected energy is identified and set up, and all information between source and destination is transmitted on this fixed route.

Before discussing the sensitivity of routing policies in these examples, we would like to point out a known advantage of using the sample path dependent routing policy (SPDR) proposed by [2]. As mentioned before, an important feature of the proposed optimal routing policy given in [2] is the fact that the route a message takes between source and destination is sample path dependent, i.e. it depends on the particular realizations of channels and transmission success. In other words, when implementing the proposed SPDR, the network layer avoids establishing a fixed route. It is known (see [2]) that due to this property, in well-connected networks the optimal SPDR policy shows an advantage in achieving lower expected cost over strategies which set up fixed routes. This is mainly because in well-connected networks, due to the presence of multiple alternative routes between source and destination, the optimal SPDR strategy avoids unnecessary retransmissions by choosing the best neighbor among those which have received the message. The graphs provided by Fig. 3-5 illustrate this fact in an error-free system.

We use networks Ω_1-Ω_3 to discuss the nature of our sensitivity analysis. We have set up these examples to illustrate the effect of topology of the network on the sensitivity of the performance of the optimal SPDR policy (network Ω_1 has the lowest connectivity, while Ω_3 is the most connected). In each case, we study the performance of the optimal SPDR strategy when the estimated probabilistic model includes an ϵ% estimation error over a particular link on the minimum energy route. In addition, we assume that such error occurs in the form of over-estimation. For each network Ω_k we plot the performance of policy $\tilde{\pi}_k$, which is the optimal (SPDR) priority policy associated with the estimated model $(N_k, Q_k, \underline{\mathbf{c}}_k, R_k)$. We provide the optimal cost, had the true model been known, as a bench-mark. Furthermore, we plot the bounds provided by Theorem 2.

The results can be summarized as follows. Network Ω_1 consists of a single route between the source and the destination, hence it is expected for all policies to demonstrate identical performance and robustness. In such a scenario, all routing policies rely on the failing link for their routing decisions. As the quality of the link decreases all policies fail to successfully transport the message from the source to the destination. Notice that due to the retiring option the optimal cost, had the true model been known, is slightly below the cost of other policies. In network Ω_2 there are more than one route from source to destination. This feature of the network has no effect on the performance of OSPF-type routing strategies, since such strategies always select a fixed route. On the other hand, the optimal priority (SPDR) policy shows better performance even in the presence of estimation error; Notice there is still only one route from node 2 to the destination, hence as the error goes to 100%, Δ_2 goes to zero, hence the right hand side of equation (24) becomes unbounded. In network Ω_3 where there exist alternative routes around the failing link, the optimal SPDR policy shows a high degree of robustness. In addition to capturing the effect of topology on the sensitivity of the optimal policy, these examples provide a rough idea on the tightness of the bounds provided by the right hand side of equation (24). It can be seen that the obtained bound becomes tighter as $\max_i \Delta_i$ increases.

C. Distributed Computation of the Optimal Policy

As mentioned in Section 4.1, the key feature of an ad hoc network is the absence of a central control or computation unit, and this underlines the importance of providing distributed algorithms for computation and implementation of an optimal policy. The authors in [2] provide three algorithms in which each node uses its local information in order to compute a local priority list of its neighbors. The priority list for each node is determined based on the estimates of the optimal expected reward of the node and its neighbors. The local information available at each node consists of the local broadcast model for that node and the updated estimates of its neighbors' optimal expected rewards. These estimates are received regularly by the node from its neighbors. Each node computes a new estimate of its own optimal expected reward according to an update equation specified by the distributed computation algorithms and it regularly communicates this estimate to its neighbors. To a great extent these algorithms are, in their information structure, similar to the Distributed Bellman-Ford algorithm [11], [12]), and can be thought of the sample-path dependent extensions of such policy. In all the distributed algorithms presented in [2], each node uses information only about its own local broadcast model and not the global network model. This implies that, in practice, it is sufficient for each node to estimate its broadcast model locally. We are interested in studying the loss in performance of these algorithms when there are estimation errors at the local broadcast model.

In [2] it is shown that under the proposed distributed algorithms the estimates of the optimal expected reward of each node converge to their true values. Furthermore, it is proved that almost in all practical scenarios this convergence occurs in finite time. In other words, in almost all practical cases an optimal local index policy which is consistent with the optimal index policy described in Section II-A.5 can be constructed in finite time. This implies that, given a sufficiently long time horizon, all the algorithms of [2] demonstrate identical performance loss in the presence of estimation errors in the broadcast model. More specifically, assume that the true system model is $(N, P, \underline{\mathbf{c}}, R)$ as before, and the vector $\Pi_t^a = \left\{\pi_t^1 \pi_t^2 \dots \pi_t^N\right\}$ represents a collection of optimal local routing decisions at nodes $1, 2, \cdots, N$ at time t under distributed algorithm a(one of the three algorithm provided in [2]). Now assume that each node i estimates its local broadcast model, $\hat{P}(S/i)$, for each $S \subset \mathcal{N}(i)$ in a local fashion based on all its communications, both control signals and messages, as well as its channel measurements. Construct an overall local broadcast model Q such that for $\forall i \in \{1, 2, \cdots, N\}$ and $\forall S \subset \Omega$ we have

$$Q^i(S/i) = \begin{cases} \hat{P}(S/i) & \text{if } S \subset \mathcal{N}_{\hat{P}}(i) \\ 0 & \text{otherwise} \end{cases}$$

Assume that vector $\hat{\Pi}_t^a = \left\{\hat{\pi}_t^1 \hat{\pi}_t^2 \dots \hat{\pi}_t^N\right\}$, represents a collection of local routing decisions at nodes $1, 2, \cdots, N$ at time t under distributed algorithm a and under the estimated model $\hat{P}$. For almost all practical cases there exists a large horizon T_1 such that for $\forall t' > T_1$ $\hat{\Pi}_{t'}^a = \tilde{\pi}$, where $\tilde{\pi}$ is a stationary policy that is optimal for the centralized problem with model $(N, Q, \underline{\mathbf{c}}, R)$. This implies that, in the context of true model P, the performance of policy $\left\{\hat{\Pi}_{t'}^a\right\}_{t'=T_1}^{\infty}$, constructed by distributed algorithm a is independent of a and is equal to the performance of the stationary policy $\tilde{\pi}$. On the other hand, due to the optimality of the distributed algorithm a, there exists a finite horizon T_2 such that policy $\{\Pi_t^a\}_{t=T_2}^{\infty}$ is optimal in the context of the true model P; i.e. its performance is identical to that of the stationary policy π^*, where π^* is an optimal index policy for the centralized problem with the true model $(N, P, \underline{\mathbf{c}}, R)$. Hence past horizon $T = \max\{T_1, T_2\}$, under any of the distributed algorithms of [2], the loss in performance due to estimation errors in the broadcast model is equal to the loss in performance of policy $\tilde{\pi}$. Such a loss has been calculated in Section III-A and is bounded by the estimation error, i.e. the distance between the true model P and estimated model Q.

Theorem 3 summarizes this discussion in a precise fashion under the following assumptions.

Assumption 2: For any node i such that $\tilde{\pi}(i) = i$, there exists $\Delta_i \leq 1$ such that

$$\frac{\sigma(P_i, Q_i)}{\sum_{S \in U_Q^{\tilde{\pi}}(\eta_i)} Q^{\eta_i}(S|\eta_i)} \leq \Delta_i.$$

This assumption is identical to Assumption 1.

Assumption 3: For any pair of nodes i and j such that $i \in \mathcal{N}_{\hat{P}}(j)$, $J_{\hat{P}}^{\tilde{\pi}}(\{i\}) \neq J_{\hat{P}}^{\tilde{\pi}}(\{j\})$.

This assumption implies that no two neighboring nodes have the same expected reward (i.e. optimal index) to route a message to the destination. It holds in almost all practical scenarios, since two neighboring nodes have identical indexes when the network has strong symmetry properties which never hold in an ad hoc wireless environment. Under this assumption, the policies constructed by any of the three distributed algorithms in [2] converge to a stationary optimal index policy in finite time, i.e. $\hat{\Pi}_t^a = \tilde{\pi}$ for all t sufficiently large.

Theorem 3: If Assumptions 2 and 3 hold and T is sufficiently large, then for $\forall t > T$ and for any of the distributed algorithms of [2], say a,

$$d_P(\Pi_t^a, \hat{\Pi}_t^a) \leq R_{\max}(\Upsilon(P, \underline{0}) + \Upsilon(Q, \underline{\Delta})) \max_j \sigma(P_j, \hat{P}_j) \tag{32}$$

where the function $\Upsilon(A, \underline{\epsilon})$ is defined by equation (25).

Proof. Assumption 3 implies that for every $t > T \geq T_1$, we have $\hat{\Pi}^a_{t'} = \tilde{\pi}$ (see the above discussion and [2]), where $\tilde{\pi}$ is an optimal index policy for the estimated model Q, described in Section III-A.

On the other hand, for all $t > T \geq T_2$ we have $J_P^{\Pi^a}(\{i\}) = J_P^{\pi^*}(\{i\})$, where π^* is an optimal index policy for the true broadcast model P described in Section III-A (see [2]).

Hence, for $t > \max(T_1, T_2)$

$$d_P(\Pi^a_t, \hat{\Pi}^a_t) = d_P(\pi^*, \tilde{\pi}) \leq R_{\max}(\Upsilon(P, \underline{0}) + \Upsilon(Q, \underline{\Delta})) \max_j \sigma(P_j, \hat{P}_j)$$

where the inequality follows from Theorem 1.

■

Notice that T in Theorem 3 is the time to compute and disseminate the local optimal policies. In other words $T \approx \tau_c$ where τ_c is defined in Section II-B.1

IV. Conclusion

In this chapter we examined the sensitivity of optimal routing policies in ad hoc wireless networks with respect to estimation errors in channel quality. We considered an ad hoc wireless network where the wireless links from each node to its neighbors are modeled by a probability distribution describing the local broadcast nature of wireless transmissions. These probability distributions are estimated in real-time. We investigated the impact of estimation errors on the performance of a set of proposed routing policies. Our results can be used as a guideline to design on-line estimation algorithms with acceptable margin of error. At the same time our results provide a method to study the effect of the network topology on the robustness of a routing strategy with respect to errors in channel estimation. We provided a few numerical examples to illustrate such robustness issues. Furthermore, we discussed the implication of estimation errors on the distributed computation of optimal index routing policies.

References

[1] R. Kumar and P. Varaiya, *Stochastic Control*, Prentice-Hall, 1986.

[2] C. G. Lott and D. Teneketzis, “Stochastic routing in ad hoc wireless networks,” in *Proceedings of the IEEE Conference on Decision and Control*, 2000, vol. 3, pp. 2302–2307.

[3] C. Lott and D. Teneketzis, “Stochastic routing in ad hoc wireless networks,” Control Group Report CGR-01-01, University of Michigan, EECS Department, February 2001.

[4] J. Kurose and K. Ross, *Computer Networking*, Addison-Wesley, 2000.

[5] S. Benarjee and A. Misra, “Minimum energy paths for reliable communication in multi-hop wireless networks,” Technical Report CS-TR 4315, University of Maryland at College Park, Department of Computer Science, December 2001.

[6] J. Gomez, M. Naghshineh A. Campbell, and C. Bisdikian, “Paro: A power-aware routing optimization scheme for mobile ad hoc networks,” Work in Progress draft-gomez-paro-manet-00.txt, IETF, February 2001.

[7] A. Misra and S. Banerjee, “Mrpc: Maximizing network lifetime for reliable routing in wireless environment,” in *Proceedings of 2002 IEEE Wireless Communication and Networking Conference (WCNC02)*, 2002, vol. 2, pp. 800–806.

[8] C. E. Perkins, *Ad Hoc Networking*, Addison Wesley, 2001.

[9] A. Müller, “How does the value function of a Markov Decision Process depend on the transition probabilities?,” *Mathematics of Operaions Research*, vol. 22, no. 4, pp. 872–885, Nov 1997.

[10] G. B. Folland, *Real Analysis*, John Wiley & Sons, 1984.

[11] D. T. Bertsekas and R. Gallager, *Data Networks*, Prentice-Hall, Englewood Cliffs, N.J., 1987.

[12] J. M. Jaffe and F. Moss, “A responsive distributed routing algorithm for computer networks,” *IEEE Transactions on Communications*, vol. 30, pp. 1758–1762, July 1982.

[13] S. T. Rachev, *Probability Metrics and the Stability of Stochastic Models*, John Wiley & Sons, 1991.

[14] M. Shaked and J. G. Shanthikumar, *Stochastic Orders and their Applications*, Academic Press, 1994.

Appendices

I. Integral Probability Metric

In this appendix we provide the background material needed to introduce the appropriate distance measure between two local broadcast models. Furthermore, we provide a result which is fundamental in our sensitivity analysis. As discussed in Chapter 4, we use the total variation metric as the distance measure between two broadcast models P (the true model) and Q (the estimated model). Total variation metric is an instance of a larger class of distance measures called integral probability metrics. As we will show in this appendix, this class of measures is closely related to Markov operators (see Definition 12), which makes them appropriate for our sensitivity analysis. In the interest of breivity, we only state the basic definitions related to the topic, and the rsults most useful for our analysis. For more detailed discussion on this topic, see [9], [13], and [14].

Definition 7: Consider a subset $\mathcal{D}$ of bounded functionals on $\mathcal{S}$. Consider the space of $\mathcal{P}$ of probability distributions defined on the σ-field $2^{\mathcal{S}}$. We define the *integral probability metric associated with* $\mathcal{D}$ on the space $\mathcal{P}$ by the distance $d_{\mathcal{D}}$

$$\begin{aligned} d_{\mathcal{D}}(P,Q) \quad &:= \quad \sup_{\nu\in\mathcal{D}}\left|\int_{\mathcal{S}}\nu(S')dP-\int_{\mathcal{S}}\nu(S')dQ\right| \qquad \forall P,Q\in\mathcal{P} \\ \text{or} \quad &:= \quad \sup_{\nu\in\mathcal{D}}\left|\sum_{S'\in\mathcal{S}}\nu(S')P(S')-\sum_{S'\in\mathcal{S}}\nu(S')Q(S')\right| \quad \forall P,Q\in\mathcal{P} \end{aligned} \tag{33}$$

where the second eqality holds for finite set $\mathcal{S}$ and the corresponding discrete σ-field $2^{\mathcal{S}}$. We call $\mathcal{D}$ a generator of the integral probability metic $d_{\mathcal{D}}$.

Example. We can define an integral probability distance associated with set $\mathcal{D}$ between two local broadcast models describing the transmission probabilities for node i as

$$d_{\mathcal{D}}(P_i,Q_i)=\sup_{\nu\in\mathcal{D}}\left|\sum_{S'\in\mathcal{S}}\nu(S')P^i(S'|i)-\sum_{S'\in\mathcal{S}}\nu(S')Q^i(S'|i)\right|=\sup_{\nu\in\mathcal{D}}|P\nu(\{i\},i)-Q\nu(\{i\},i)| \tag{34}$$

Note that the term $\int_{\mathcal{S}}\nu(S')dP$ $(\sum_{S'\in\mathcal{S}}\nu(S')P(S'))$ can be written as a Markov operator operating on a bounded functional ν. On the other hand Markov operators and their difference are used to determine the distance between the performance of Markov policies in an MDP, hence they are vital to our sensitivity analysis. Unfortunately, they are not sufficient to establish the sensitivity analysis since they only relate the integral metric and the markov operator on the particular functions $\nu\in\mathcal{D}$. It can happen that the value functions and/or expected rewards of a certain policy do not belong in the bound set $\mathcal{D}$. To relate the difference between two Markov operators to the integral probability metric for functions $\nu\notin\mathcal{D}$ we need the following definitions and facts (for details and proof, see [9]).

Definition 8: A set $\mathcal{V}$ of functionals on $\mathcal{S}$ is *balanced*, iff for $\forall\nu\in\mathcal{V}$ and $\forall\alpha$, $|\alpha|\leq 1$, the functional $\alpha\nu$ also belongs to $\mathcal{V}$.

Fact. 5 (Theorem 3.2 in [9]) If $\mathcal{D}$ is an arbitrary generator of an integral probability metric, then the balanced convex hull spanned by $\mathcal{D}$ generates the same probability metric as $\mathcal{D}$.

A balance convex subset $\mathcal{D}$ of functionals can be used to define the Minkowski functional associaciated with $\mathcal{D}$.

Definition 9: Consider a balanced and convex subset $\mathcal{D}$ of the set $\mathcal{B}$ of bounded functionals on the space $\mathcal{S}$. We define *the Minkowski functional associated with* $\mathcal{D}$, $\mu_{\mathcal{D}}:\mathcal{B}\longmapsto\mathbb{R}$, as

$$\mu_{\mathcal{D}}(\nu):=\inf\left\{t>0:t^{-1}\nu\in\mathcal{D}\right\} \tag{35}$$

Intuitively, the Minkowski functional "down-scales" a function $\nu\notin\mathcal{D}$ by the smallest amount t so as to result in a scaled vector $t^-1\nu$ which belongs to the balanced, bounded, convex set $\mathcal{D}$.

From Definition 9 and Fact 5 we conclude that there can be a Minkowski functional associated with any integral probability. Furthermore, these facts and definitions provide a relationship between the difference of two Markov operators (closely related to the distance between policies) on a function, its Minkowski functional, and the integral probability metric between the corresponding probability distributions (e.g. local broadcast models). This relationship is given by the following fact. For the proof see [9].

Fact. 6: $\left|\int_{\mathcal{S}} \nu(S')dP - \int_{\mathcal{S}} \nu(S')dQ\right| \leq \mu_{\mathcal{D}}(\nu).d_{\mathcal{D}}(P,Q)$

II. Total Variation Metric

In this appendix we use the notions introduced in Appendix C and illustrate how to construct the total variation metric as an integral probability metric. After such construction we identify Fact 7 as a special case of Fact 6, where the integral metric is the total variation. Fact 7 is the basis of the proof of Lemmas 3 and 4.

Define $\mathcal{D}_1$ as the set of all indicator functions of subsets of $\mathcal{S}$, i.e. $\mathcal{D}_1 = \{\psi_A : A \subset \mathcal{S}\}$, where the indicator function ψ_A is defined as $\psi_A(S) = \begin{cases} 1 & \text{if } S \in A \\ 0 & \text{if } S \in \mathcal{S} - A \end{cases}$

Definition 10: The *total variation metric* on the set $\mathcal{P}$ of probability distributions defined on the σ-field $2^{\mathcal{S}}$ is defined as the integral probability metric associated with $\mathcal{D}_1$ on $\mathcal{P}$.

For the purpose of sensitivity analysis of an optimal routing strategy in ad hoc networks, we focus our attention on the total variation distance on the set $\mathcal{P}_\rangle$ of all local broadcast models for node i. Note that a local broadcast model for node i fully describes the probabilities of successful transmission success from node i to its neighbors. Using the mechanism introduced in Appendix A, we can find the total variation distance between two local broadcast models around node i, P_i and Q_i

$$\sigma(P_i, Q_i) = \sup_{A \subset \mathcal{S}} \left| \sum_{S' \in A} \left(P^i(S'|i) - Q^i(S'|i) \right) \right|. \tag{36}$$

Fact. 7: Consider a bounded functional $\nu : \mathcal{S} \longmapsto \mathbb{R}$. We have

$$|P\nu(S,i) - Q\nu(S,i)| \leq (\sup_S \nu(S) - \inf_S \nu(S)).\sigma(P_i, Q_i) \tag{37}$$

Proof. We prove Fact 7 as a special case of Fact 6. In order to construct the Minkowski functonal associated with the total variation metric we first use Fact 5 to construct the balanced covex hall of $\mathcal{D}_1$, $\mathcal{D}_{tot-var}$. In other words,

$$\mathcal{D}_{tot-var} = \{\nu : 0 \leq \nu(S) \leq 1 \text{ for } \forall S \in \mathcal{S}, \text{ or } -1 \leq \nu(S) \leq 0 \text{ for } \forall S \in \mathcal{S}\}. \tag{38}$$

Since $\mathcal{D}_{tot-var}$ is a balanced convex hull of $\mathcal{D}_1$ it generates the same integral probability metric on the local broadcast modes, i.e. it generates the total variation metric.

For any bounded function f defined on space $\mathcal{S}$, i.e. $f \in \mathcal{B}$, we can easily construct the Minkowski functional for the total variation metric as

$$\mu_\sigma(f) = \begin{cases} 0 & \text{if } f \in \mathcal{D} \\ \sup(f) - \inf(f) & \text{if } f \in \mathcal{B} - \mathcal{D} \end{cases} \tag{39}$$

Using Fact 6 we have

$$\begin{aligned} |P\nu(S,i) - Q\nu(S,i)| &\leq (\sup_S \nu(S) - \inf_S \nu(S)). \sup_{A \subset \mathcal{S}} \left| \sum_{S' \in A, S \subset S'} \left(P^i(S'|S) - Q^i(S'|S) \right) \right| \\ &= (\sup_S \nu(S) - \inf_S \nu(S)). \sup_{A \subset \mathcal{S}} \left| \sum_{S' \in A, \{i\} \subset S'} \left(P^i(S'|i) - Q^i(S'|i) \right) \right| \\ &= (\sup_S \nu(S) - \inf_S \nu(S)).\sigma(P_i, Q_i) \end{aligned} \tag{40}$$

where the first equality holds because of the decoupling property.

Hence, the assertion of Fact 7 is true.

■

III. Proof of Lemmas 3 and 4

Now we provide the full proof for Lemmas 3 and 4.

Lemma 15: Assume that $\{\xi_1, \xi_2, \cdots, \xi_N\}$ is the strict priority ordering under policy π^*, i.e. $\xi_j >_{\pi^*} \xi_{j+1}$. Then, we have

$$J_P^{\pi^*}(\{\xi_j\}) - J_Q^{\tilde{\pi}}(\{\xi_j\}) \leq \alpha(\xi_j) \max_{k \leq j} \sigma(P_{\xi_k}, Q_{\xi_k}), \tag{41}$$

where $\alpha(\xi_j)$ is increasing in j and satisfies the recursion

$$\alpha(\xi_1) = 0, \qquad \alpha(\xi_i) = \begin{cases} \frac{R_{\max}}{\sum_{S \in U_P^{\pi^*}(\xi_i)} P^{\xi_i}(S|\xi_i)} + \alpha(\xi_{i-1}) & \text{if } \pi^*(\xi_i) = \xi_i \\ \alpha(\xi_{i-1}) & \text{if } \pi^*(\xi_i) = r_{\xi_i} \end{cases}$$

Proof. We prove the lemma by induction on the index i of the strict ordering $\{\xi_1, \xi_2, \cdots, \xi_N\}$.

Basis of Induction: Since π^* is an optimal routing policy for a problem associated with the tuple $(N, P, \underline{\mathbf{c}}, R)$ we have $J_P^{\pi^*}(\{\xi_1\}) = R_{\max}$. Similarly, $\tilde{\pi}$ is an optimal routing under policy$Q$ and $J_Q^{\tilde{\pi}}(\{\xi_1\}) = R_{\max}$ (see Section 3.3.2 in [2]). Hence we have $J_P^{\pi^*}(\{\xi_1\}) - J_Q^{\tilde{\pi}}(\{\xi_1\}) = 0$

Induction Step: We assume that

$$J_P^{\pi^*}(\{\xi_{j-1}\}) - J_Q^{\tilde{\pi}}(\{\xi_{j-1}\}) \leq \alpha(\xi_{j-1}) \max_{k \leq j-1} \sigma(P_{\xi_k}, Q_{\xi_k}). \tag{42}$$

We need to show that

$$J_P^{\pi^*}(\{\xi_j\}) - J_Q^{\tilde{\pi}}(\{\xi_j\}) \leq \alpha(\xi_j) \max_{k \leq j} \sigma(P_{\xi_k}, Q_{\xi_k}), \tag{43}$$

where $\alpha(\xi_j) = \frac{R_{\max}}{\sum_{S \in U_P^{\pi^*}(\xi_j)} P^{\xi_j}(S|\xi_j)} + \alpha(\xi_{j-1})$.

Case 1: If $\pi^*(\xi_j) = r_{\xi_j}$, then $J_P^{\pi^*}(\{\xi_j\}) = R_{\xi_j}$. Hence,

$$J_P^{\pi^*}(\{\xi_j\}) - J_Q^{\tilde{\pi}}(\{\xi_j\}) = R_{\xi_j} - \max\left\{R_{\xi_j}, -c_{\xi_j} + QJ_Q^{\tilde{\pi}}(\{\xi_j\}, \xi_j)\right\} \leq 0 \tag{44}$$

On the other hand, the right hand side of equation (43) is always a positive number. Hence,

$$J_P^{\pi^*}(\{\xi_j\}) - J_Q^{\tilde{\pi}}(\{\xi_j\}) \leq \alpha(\xi_j) \max_{k \leq j} \sigma(P_{\xi_k}, Q_{\xi_k}), \tag{45}$$

This completes the induction step in Case 1.

Case 2: If $\pi^*(\xi_j) = \xi_j$, then $J_P^{\pi^*}(\{\xi_j\}) = -c_{\xi_j} + PJ_P^{\pi^*}(\{\xi_j\}, \xi_j)$. Hence,

$$\begin{aligned}
& J_P^{\pi^*}(\{\xi_j\}) - J_Q^{\tilde{\pi}}(\{\xi_j\}) \\
&= -c_{\xi_j} + PJ_P^{\pi^*}(\{\xi_j\}, \xi_j) - \max\left\{R_{\xi_j}, -c_{\xi_j} + QJ_Q^{\tilde{\pi}}(\{\xi_j\}, \xi_j)\right\} \\
&\leq PJ_P^{\pi^*}(\{\xi_j\}, \xi_j) - QJ_Q^{\tilde{\pi}}(\{\xi_j\}, \xi_j) \\
&= PJ_P^{\pi^*}(\{\xi_j\}, \xi_j) - PJ_Q^{\tilde{\pi}}(\{\xi_j\}, \xi_j) + PJ_Q^{\tilde{\pi}}(\{\xi_j\}, \xi_j) - QJ_Q^{\tilde{\pi}}(\{\xi_j\}, \xi_j) \\
&\leq (\sup_S J_Q^{\tilde{\pi}}(S) - \inf_S J_Q^{\tilde{\pi}}(S))\sigma(P_{\xi_j}, Q_{\xi_j}) + PJ_P^{\pi^*}(\{\xi_j\}, \xi_j) - PJ_Q^{\tilde{\pi}}(\{\xi_j\}, \xi_j) \\
&\leq R_{\max}\sigma(P_{\xi_j}, Q_{\xi_j}) + P\left(J_P^{\pi^*}(\{\xi_j\}, \xi_j) - J_Q^{\tilde{\pi}}(\{\xi_j\}, \xi_j)\right) \\
&= R_{\max}\sigma(P_{\xi_j}, Q_{\xi_j}) + \sum_{S \in L_P^{\pi^*}(\xi_j)} P^{\xi_j}(S|\xi_j)\left(J_P^{\pi^*}(\{\pi^*(S)\}) - J_Q^{\tilde{\pi}}(\{\tilde{\pi}(S)\})\right) \\
&+ \sum_{S \in U_P^{\pi^*}(\xi_j)} P^{\xi_j}(S|\xi_j)(J_P^{\pi^*}(\{\pi^*(S)\}) - J_Q^{\tilde{\pi}}(\{\tilde{\pi}(S)\}))
\end{aligned}$$

$$
\begin{aligned}
&\leq R_{\max}\sigma(P_{\xi_j},Q_{\xi_j}) + \sum_{S\in L_P^{\pi^*}(\xi_j)} P^{\xi_j}(S|\xi_j)\left(J_P^{\pi^*}(\{\pi^*(S)\}) - J_Q^{\tilde{\pi}}(\{\pi^*(S)\})\right)\\
&+ \sum_{S\in U_P^{\pi^*}(\xi_j)} P^{\xi_j}(S|\xi_j)(J_P^{\pi^*}(\{\pi^*(S)\}) - J_Q^{\tilde{\pi}}(\{\pi^*(S)\}))\\
&= R_{\max}\sigma(P_{\xi_j},Q_{\xi_j}) + \sum_{S\in L_P^{\pi^*}(\xi_j)} P^{\xi_j}(S|\xi_j)(J_P^{\pi^*}(\{\xi_j\}) - J_Q^{\tilde{\pi}}(\{\xi_j\}))\\
&+ \sum_{S\in U_P^{\pi^*}(\xi_j)} P^{\xi_j}(S|\xi_j)(J_P^{\pi^*}(\{\pi^*(S)\}) - J_Q^{\tilde{\pi}}(\{\pi^*(S)\}))\\
&\leq R_{\max}\sigma(P_{\xi_j},Q_{\xi_j}) + (J_P^{\pi^*}(\{\xi_j\}) - J_Q^{\tilde{\pi}}(\{\xi_j\}))\sum_{S\in L_P^{\pi^*}(\xi_j)} P^{\xi_j}(S|\xi_j)\\
&+ \sum_{S\in U_P^{\pi^*}(\xi_j)} P^{\xi_j}(S|\xi_j)\alpha(\pi^*(S)).\max_{\{k:\xi_k>_{\pi^*}\pi^*(S)\}}\sigma(P_{\xi_k},Q_{\xi_k})\\
&\leq R_{\max}\sigma(P_{\xi_j},Q_{\xi_j}) + (J_P^{\pi^*}(\{\xi_j\}) - J_Q^{\tilde{\pi}}(\{\xi_j\}))\sum_{S\in L_P^{\pi^*}(\xi_j)} P^{\xi_j}(S|\xi_j)\\
&+ \sum_{S\in U_P^{\pi^*}(\xi_j)} P^{\xi_j}(S|\xi_j)\alpha(\xi_{j-1}).\max_{k\leq j-1}\sigma(P_{\xi_k},Q_{\xi_k})\\
&\leq R_{\max}\sigma(P_{\xi_j},Q_{\xi_j}) + (J_P^{\pi^*}(\{\xi_j\}) - J_Q^{\tilde{\pi}}(\{\xi_j\}))\sum_{S\in L_P^{\pi^*}(\xi_j)} P^{\xi_j}(S|\xi_j)\\
&+ \alpha(\xi_{j-1})\max_{k\leq j-1}\sigma(P_{\xi_k},Q_{\xi_k})\sum_{S\in U_P^{\pi^*}(\xi_j)} P^{\xi_j}(S|\xi_j) \qquad (46)
\end{aligned}
$$

where the second inequality holds because of Fact 7 (Appendix D), the third inequality holds since an the expected reward of an optimal policy at each node is bounded in the interval $[0, R_{\max}]$ and we assume that $\tilde{\pi}$ is an optimal policy in the contex of Q, the fourth inequality is true since $\tilde{\pi}$ is an optimal routing policy associated with the tuple $(N, Q, \underline{\mathbf{c}}, R)$ (i.e. $J_Q^{\tilde{\pi}}(\{\tilde{\pi}(S)\}) \geq J_Q^{\tilde{\pi}}(\{\pi^*(S)\})$), the fifth inequality holds due to the induction hypothesis and the fact that for all S such that $\xi_j \in S$ and $\pi^*(S) \neq \xi_j$ we have $\pi^*(S) \in \{\xi_1, \xi_2, \ldots, \xi_{j-1}\}$, and the last inequality holds since $\alpha(\xi_l)$ is increasing in l.

Now we solve equation (46) for $J_P^{\pi^*}(\{\xi_j\}) - J_Q^{\tilde{\pi}}(\{\xi_j\})$. We have,

$$
\begin{aligned}
&(1 - \sum_{S\in L_P^{\pi^*}(\xi_j)} P^{\xi_j}(S|\xi_j))(J_P^{\pi^*}(\{\xi_j\}) - J_Q^{\tilde{\pi}}(\{\xi_j\}))\\
&\quad\leq R_{\max}\sigma(P_{\xi_j},Q_{\xi_j}) + \alpha(\xi_{j-1})\max_{k\leq j-1}\sigma(P_{\xi_k},Q_{\xi_k})\sum_{S\in U_P^{\pi^*}(\xi_j)} P^{\xi_j}(S|\xi_j) \qquad (47)
\end{aligned}
$$

or

$$
\begin{aligned}
&(J_P^{\pi^*}(\{\xi_j\}) - J_Q^{\tilde{\pi}}(\{\xi_j\}))\sum_{S\in U_P^{\pi^*}(\xi_j)} P^{\xi_j}(S|\xi_j)\\
&\quad\leq R_{\max}\max_{k\leq j}\sigma(P_{\xi_k},Q_{\xi_k}) + \alpha(\xi_{j-1})\max_{k\leq j}\sigma(P_{\xi_k},Q_{\xi_k})\sum_{S\in U_P^{\pi^*}(\xi_j)} P^{\xi_j}(S|\xi_j) \qquad (48)
\end{aligned}
$$

or

$$J_P^{\pi^*}(\{\xi_j\}) - J_Q^{\tilde{\pi}}(\{\xi_j\}) \leq \overbrace{\left(\frac{R_{\max}}{\sum_{S\in U_P^{\pi^*}(\xi_j)} P^{\xi_j}(S|\xi_j)} + \alpha(\xi_{j-1})\right)}^{\alpha(\xi_j)} \max_{k\leq j} \sigma(P_{\xi_k}, Q_{\xi_k}) \tag{49}$$

The proof of the induction step in Case 2 is now complete. Hence the assertion of the lemma is true.

■

Similarly we have Lemma 4.

Lemma 16: Assume that $\{\eta_1, \eta_2, \cdots, \eta_N\}$ is the nodes' strict priority ordering under policy $\tilde{\pi}$, i.e. $\eta_j >_{\tilde{\pi}} \eta_{j+1}$. We have

$$|J_P^{\tilde{\pi}}(\{\eta_j\}) - J_Q^{\tilde{\pi}}(\{\eta_j\})| \leq \gamma(\eta_j) \max_{k\leq j} \sigma(P_{\eta_k}, Q_{\eta_k}), \tag{50}$$

where $\gamma(\eta_j)$ is increasing in j and satisfies the recursion

$$\gamma(\eta_1) = 0, \qquad \gamma(\eta_i) = \begin{cases} \frac{R_{\max}}{\sum_{S\in U_P^{\tilde{\pi}}(\eta_i)} P^{\eta_i}(S|\eta_i)} + \gamma(\eta_{i-1}) & \text{if } \tilde{\pi}(\eta_i) = \eta_i \\ \gamma(\eta_{i-1}) & \text{if } \tilde{\pi}(\eta_i) = r_{\eta_i} \end{cases}$$

Proof. We, again, prove this lemma by induction on the index i of the strict ordering $\{\eta_1, \eta_2, \ldots, \eta_N\}$.

Basis of Induction: Since $\tilde{\pi}$ is an optimal routing policy under broadcast model Q, $\tilde{\pi}(\eta_1) = r_1$ and $J_Q^{\tilde{\pi}}(\{\eta_1\}) = J_P^{\tilde{\pi}}(\{\eta_1\}) = R_{\max}$ (see Section 3.3.2 in [2]). Hence we have $|J_P^{\tilde{\pi}}(\{\eta_1\}) - J_Q^{\tilde{\pi}}(\{\eta_1\})| = 0$, and equation (50) is satisfied with $\gamma(\eta_j) = 0$.

Induction Step: We assume that

$$|J_P^{\tilde{\pi}}(\{\eta_{j-1}\}) - J_Q^{\tilde{\pi}}(\{\eta_{j-1}\})| \leq \gamma(\eta_{j-1}) \max_{k\leq j-1} \sigma(P_{\eta_k}, Q_{\eta_k}), \tag{51}$$

We need to show that

$$|J_P^{\tilde{\pi}}(\{\eta_j\}) - J_Q^{\tilde{\pi}}(\{\eta_j\})| \leq \gamma(\eta_j) \max_{k\leq j} \sigma(P_{\eta_k}, Q_{\eta_k}), \tag{52}$$

where

$$\gamma(\eta_j) = \frac{R_{\max}}{\sum_{S\in U_P^{\tilde{\pi}}(\eta_j)} P^{\eta_j}(S|\eta_j)} + \gamma(\eta_{j-1}). \tag{53}$$

Case 1: If $\tilde{\pi}(\eta_j) = r_{\eta_j}$, then $J_P^{\tilde{\pi}}(\{\eta_j\}) = J_Q^{\tilde{\pi}}(\eta_j) = R_{\eta_j}$. Hence,

$$|J_P^{\tilde{\pi}}(\{\eta_j\}) - J_Q^{\tilde{\pi}}(\{\eta_j\})| = 0. \tag{54}$$

On the other hand, the right hand side of equation (50) is always a positive number. In other words,

$$|J_P^{\tilde{\pi}}(\{\eta_j\}) - J_Q^{\tilde{\pi}}(\{\eta_j\})| \leq \gamma(\eta_j) \max_{k\leq j} \sigma(P_{\eta_k}, Q_{\eta_k}) \tag{55}$$

This completes the induction step in Case 1.

Case 2: If $\tilde{\pi}(\eta_j) = \eta_j$, then

$$
\begin{aligned}
&|J_P^{\tilde{\pi}}(\{\eta_j\}) - J_Q^{\tilde{\pi}}(\{\eta_j\})| \\
&= |PJ_P^{\tilde{\pi}}(\{\eta_j\},\eta_j) - QJ_Q^{\tilde{\pi}}(\{\eta_j\},\eta_j)| \\
&= |PJ_P^{\tilde{\pi}}(\{\eta_j\},\eta_j) - PJ_Q^{\tilde{\pi}}(\{\eta_j\},\eta_j) + PJ_Q^{\tilde{\pi}}(\{\eta_j\},\eta_j) - QJ_Q^{\tilde{\pi}}(\{\eta_j\},\eta_j)| \\
&\leq |PJ_Q^{\tilde{\pi}}(\{\eta_j\},\eta_j) - QJ_Q^{\tilde{\pi}}(\{\eta_j\},\eta_j)| + |PJ_P^{\tilde{\pi}}(\{\eta_j\},\eta_j) - PJ_Q^{\tilde{\pi}}(\{\eta_j\},\eta_j)| \\
&\leq (\sup_S J_Q^{\tilde{\pi}}(S) - \inf_S J_Q^{\tilde{\pi}}(S))\sigma(P_{\eta_j},Q_{\eta_j}) + |PJ_P^{\tilde{\pi}}(\{\eta_j\},\eta_j) - PJ_Q^{\tilde{\pi}}(\{\eta_j\},\eta_j)| \\
&\leq R_{\max}\sigma(P_{\eta_j},Q_{\eta_j}) + |J_P^{\tilde{\pi}}(\{\eta_j\}) - J_Q^{\tilde{\pi}}(\{\eta_j\})| \sum_{S\in L_P^{\pi^*}(\eta_j)} P^{\eta_j}(S|\eta_j) \\
&+ \sum_{S\in U_P^{\tilde{\pi}}(\eta_j)} P^{\eta_j}(S|\eta_j)|J_P^{\tilde{\pi}}(\{\tilde{\pi}(S)\}) - J_Q^{\tilde{\pi}}(\{\tilde{\pi}(S)\})| \\
&\leq R_{\max}\sigma(P_{\eta_j},Q_{\eta_j}) + |J_P^{\tilde{\pi}}(\{\eta_j\}) - J_Q^{\tilde{\pi}}(\{\eta_j\})| \sum_{S\in L_P^{\pi^*}(\eta_j)} P^{\eta_j}(S|\eta_j) \\
&+ \sum_{S\in U_P^{\tilde{\pi}}(\eta_j)} P^{\eta_j}(S|\eta_j)\gamma(\eta_{j-1}) \max_{k\leq j-1} \sigma(P_{\eta_k},Q_{\eta_k}) \\
&= R_{\max}\sigma(P_{\eta_j},Q_{\eta_j}) + |J_P^{\tilde{\pi}}(\{\eta_j\}) - J_Q^{\tilde{\pi}}(\{\eta_j\})| \sum_{S\in L_P^{\pi^*}(\eta_j)} P^{\eta_j}(S|\eta_j) \\
&+ \gamma(\eta_{j-1}) \max_{k\leq j-1} \sigma(P_{\eta_k},Q_{\eta_k}) \sum_{S\in U_P^{\tilde{\pi}}(\eta_j)} P^{\eta_j}(S|\eta_j) \qquad (56)
\end{aligned}
$$

where the second inequality holds because of Fact 7, the third equality again holds since $\tilde{\pi}$ is an optimal policy in the context of Q and the fact that at each node the expected reward of an optimal policy always belongs to the interval $[0, R_{\max}]$, and the fourth inequality follows from the induction hypothesis, the fact that for all S such that $\eta_j \in S$ and $\tilde{\pi}(S) \neq \eta_j$ we have $\tilde{\pi}(S) \in \{\eta_1, \eta_2, \dots, \eta_{j-1}\}$, and the increasing property of $\gamma(\xi_l)$ in l.

An argument similar to that following equation (46) in Lemma 3 results in the following inequality

$$
|J_P^{\tilde{\pi}}(\{\eta_j\}) - J_Q^{\tilde{\pi}}(\{\eta_j\})| \leq \overbrace{\left(\frac{R_{\max}}{\sum_{S\in U_P^{\tilde{\pi}}(\eta_j)} P^{\eta_j}(S|\eta_j)} + \gamma(\eta_{j-1}) \right)}^{\gamma(\eta_j)} \max_{k\leq j} \sigma(P_{\eta_k},Q_{\eta_k}) \qquad (57)
$$

and this completes the proof of induction step in Case 2. Hence, the assertion of the lemma is true.

■