Z-Transform

The Z-transform is the Discrete-Time counterpart of the Laplace Transform.

\[ \text{Laplace} \quad G(s) = \int_{-\infty}^{\infty} g(t)e^{-st}dt \]
\[ Z \quad G(z) = \sum_{n=-\infty}^{\infty} g[n]z^{-n} \]

It is

- Used in Digital Signal Processing
- Used to Define Frequency Response of Discrete-Time System.
- Used to Solve Difference Equations – use algebraic methods as we did for differential equations with Laplace Transforms; it is easier to solve the transformed equations since they are algebraic.

We will see that

1. Lines on the s-plane map to circles on the z-plane.

2. Role of \( j\omega \)-axis is replaced by unit circle, so

   (a) The DT Fourier Transform exists for a signal if the ROC includes the unit circle.

   (b) A stable system must have an ROC that contains the unit circle.

   (c) A causal and stable system must have poles inside the unit circle.
Aside: You can relate the Z transform and Laplace transform directly when you are dealing with sampled signals. First recall the definition of Laplace transform:

Take a CT signal \( g(t) \) and sample it:

\[
 g_s(t) = g(t) \sum_{n=-\infty}^{\infty} \delta(t-nT) = \sum_{n=-\infty}^{\infty} g(nT)\delta(t-nT)
\]

The Laplace transform of the sampled signal is

\[
 \mathcal{L}[g_s(t)] = \int_{-\infty}^{\infty} \left[ \sum_{n=-\infty}^{\infty} g(nT)\delta(t-nT) \right] e^{-st} dt
\]

\[
 = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} g(nT)\delta(t-nT)e^{-st} dt
\]

\[
 = \sum_{n=-\infty}^{\infty} g(nT)e^{-snT}.
\]

Let \( g[n] = g(nT) \) be the discrete representation and \( z = e^{sT} \), then

\[
 G(z) = \sum_{n=-\infty}^{\infty} g[n]z^{-n}
\]

\[
 G(z)|_{z=e^{sT}} = \sum_{n=-\infty}^{\infty} g[n]e^{-sTn}
\]

\[
 = \sum_{n=-\infty}^{\infty} g(nT)e^{-snT}
\]

\[
 = \mathcal{L}[g_s(t)]
\]

Thus, the Z transform with \( z = e^{sT} \) is the same as the Laplace transform of a sampled signal! Of course, if the signal is already discrete, the notion of sampling is unnecessary for understanding and using the Z transform.
Definitions of Z-Transforms

\[ G(z) = \sum_{n=-\infty}^{\infty} g[n] z^{-n} \]

is the bilateral (2-sided) Z-transform. Its inverse Z-transform is defined as:

\[ Z^{-1}[H(z)] = h[n] = \frac{1}{2\pi j} \oint H(z) z^{n-1} dz \]

which is a counterclockwise contour integral along a closed path in the \( z \)-plane. We will see how to take inverse Z-transforms using tables and partial fraction expansion.

Some textbooks work with the unilateral Z-transform:

\[ H_u(z) = \sum_{n=0}^{\infty} h[n] z^{-n}. \]

IMPORTANT: We do not use this at all, but make sure you know the difference in case you come across it later. In this course, our focus is on the Bilateral Z-Transform, to which we simply refer as Z-transform:

\[ G(z) = \sum_{n=-\infty}^{\infty} g[n] z^{-n}, \]

defined for 2-sided, anticausal, and causal signals (i.e. all signals).

Note that, whereas for Laplace Transform we considered where the integral converges, here we consider where the sum converges.
One must consider the Region of Convergence (ROC) of the Z-transform, because left-sided and right-sided time functions will have the same Z-transform and only the ROC will distinguish between the two possible time functions. Remember:

\[ \sum_{i=0}^{\infty} a^i = \frac{1}{1 - a}, \quad |a| < 1 \]

You’ll use this a lot!

**Ex.** Find the Z transforms of

\[ x_1[n] = a^n u[n] \quad \text{and} \quad x_2[n] = -(a^n)u[-n-1] \]

and plot the ROCs and pole/zero diagrams.

We see that we must specify the ROC for the bilateral Z-transform to be unique.
Definitions and Regions of Convergence

- \( x[n] \) is right-sided if \( x[n] = 0, n < n_0 \)
- \( x[n] \) is left-sided if \( x[n] = 0, n > n_0 \)

We can write

\[
X(z) = \ldots + x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \ldots
\]

1. We’ve seen that **right-sided** signals have an ROC of the form \( |z| > r_{\text{max}} \), i.e., it converges outside the largest magnitude pole.

(Infinite Egg White)
Examine for right-sided \( x[n] \)

\[
X(z) = \sum_{n=n_0}^{\infty} x[n]z^{-n}
\]

\[
X(z) = \sum_{n=n_0}^{\infty} x[n] \left( \frac{1}{z} \right)^n
\]

As \( n \to \infty \), need \((1/z)^n \to 0\) for sum to converge.

This will happen for values of \( z \) outside rather than inside the pole, i.e. \( |z| > r_{\text{max}} \).

What about \( z = \infty \) ?

If \( x[n] \) is not causal but is still right-sided, e.g. \( x[n] = u[n + 1] \), then

\[
\hat{x}(z) = \sum_{n=-1}^{\infty} z^{-n} = z + \sum_{n=0}^{\infty} z^{-n} \quad \to \quad \text{ROC} = \{ |z| < \infty \}
\]

Will not converge at \( z = \infty \), and we won’t include it in the ROC.

Thus we can tell if a system is causal from the ROC of the Z-transform of its impulse response.

\[
|z| > r_{\text{max}} \quad \Rightarrow \quad \text{CAUSAL}
\]

\[
\infty > |z| > r_{\text{max}} \quad \Rightarrow \quad \text{right-sided but not causal}
\]
2. **Left-sided** signals have ROC of form $|z| < r_{\text{min}}$, i.e., it converges INSIDE circle $|z| < r_{\text{min}}$ (EGG YOLK).

Examine for left-sided $x[n]$

$$X(z) = \sum_{n=-\infty}^{n_0} x[n]z^{-n}$$

As $n \to -\infty$, need $(1/z)^n \to 0$ or $z^\infty \to 0$

This happens for values of $z$ inside rather than outside the poles.

What about $z = 0$ ?

If $x[n]$ is left-sided but not strictly anticausal

$(x[n] = 0$ for $n > n_0 > 0$ but $x[n_0] \neq 0)$

e.g. $x[n] = u[-n + 1]$, then

$$X(z) = \sum_{n=-\infty}^{1} z^{-n} = z^{-1} + \sum_{n=0}^{\infty} z^n$$

$$0 < |z| < 1$$

does not converge at $z = 0$ so don’t include $z = 0$ in the ROC.

3. **2-sided** signals have ROC of the form

$$r_1 < |z| < r_2$$ (BAGEL OR DONUT)

4. **Finite Duration** $x[n]$ has ROC of entire z-plane except possibly $z = 0$ or $z = \infty$

$$\delta[n - 1] \leftrightarrow z^{-1}, |z| > 0$$

$$\delta[n + 1] \leftrightarrow z, |z| < \infty$$

**FACT:** An ROC must contain the unit circle for stability – this holds for causal, anticausal, and two-sided signals.
Find the Z-Transform of $x[n] = a^n$ for $|a| < 1$.

$$X(z) = \sum_{n=-\infty}^{\infty} a^n \frac{z^{-n}}{z^{-n}} = \sum_{n=-\infty}^{\infty} a^n \frac{1}{z^n} = \sum_{k=1}^{\infty} a^k \frac{1}{z^k} + \sum_{n=0}^{\infty} (a/2)^n \frac{az}{1-az} + \frac{1}{1-a/2}$$

$$\text{ROC} = \{ |a| < 1, |a/2| < 1 \}$$
Ex. Find the Z-Transform of

\[ x[n] = 3^n u[-n - 1] + 4^n u[-n - 1]. \]

\[
\begin{align*}
\text{Roc: } & \{ |z| < 3 \} \cap \{ |z| < 4 \} = \{ |z| < 3 \} \\
3^n u[-n-1] \leftrightarrow & \frac{z}{z-3} \\
4^n u[-n-1] \leftrightarrow & \frac{-z}{z-4} \\
\end{align*}
\]

Ex. Find the Z-transform of \( \frac{1}{2} \delta[n - 1] + 3 \delta[n + 1] \).
What is its ROC?

\[
H(z) = \frac{1}{2} \frac{-1}{z^2} + 3 \frac{z}{z}
\]

\[
\text{Roc} = \{ \sigma < |z| < \} 
\]
Ex. Find the Z-transform of

\[ h[n] = (0.5)^n u[n - 1] + 3^n u[-n - 1]. \]

If \( h \) is the impulse response of a system, would the system be BIBO stable?
Ex. Find the Z-transform of $x[n] = r^n \sin(bn)u[n]$ using Euler’s rule.

\[
\sin(bn) = \frac{e^{jbn} - e^{-jbn}}{2j}
\]

\[
x[n] = \frac{1}{2j} \left( (r e^{jb})^n - (r e^{-jb})^n \right) u[n]
\]

\[
X(z) = \frac{1}{2j} \left( \frac{z}{z - re^{jb}} - \frac{z}{z - re^{-jb}} \right)
\]

\[
|z| > r
\]

\[
= \frac{rz \sin(bn)}{z^2 - 2rz \cos b + r^2}
\]
Insights from the Pole-Zero Plot and ROC

Things that you can tell about a signal from its pole-zero plot (and ROC):

- When the ROC includes the unit circle, then the signal is absolutely summable. (If the signal is an impulse response ⇒ the system is stable.)

- A pole on the positive real axis corresponds to a simple decaying or growing function (of form $a^n$ for a pole at $z = a$).

- Poles off the positive real axis correspond to an oscillating signal where the frequency of oscillation is the angle from the positive real axis. (Poles on the negative real axis have an angle of $\pi$, so the frequency of oscillation is $\pi$, as in $(-1)^n$.) When the poles are...
  - on the unit circle ⇒ sinusoidal functions with constant amplitude
  - not on the unit circle ⇒ sinusoidal functions with a decaying (or growing) envelope (rate of decay/growth depends on the distance from the pole to the origin).

- Poles and zeroes must come in complex conjugate pairs for the signal to be real (consequence of the Z-transform property: $x[n]^* \leftrightarrow X^*(z^*)$).
Z-Transform Properties

The properties of Z-transform are inherited from properties of Laplace transform. The book has a longer list; here, we discuss the most important ones:

- **Linearity:**
  \[ ax[n] + by[n] \longleftrightarrow aX(z) + bY(z) \]
  where the new ROC \( R' \supset R_x \cap R_y \).

- **Time shift:**
  \[ y[n] = x[n - n_0] \longleftrightarrow z^{-n_0}X(z) \]
  where the new ROC is the same as \( R_x \) with the possible addition or deletion of the origin or infinity.

- **Convolution:**
  \[ y[n] = x[n] * h[n] \longleftrightarrow X(z)H(z) \]
  where the new ROC \( R_y \supset R_x \cap R_h \).

Linearity and the time shift property will be useful for LCCDE systems, and the convolution property lets us avoid discrete-time convolutions. We’ll use these properties a lot.
Convolution in Time

\[ y[n] = x[n] \ast h[n] \leftrightarrow \sum_{n=-\infty}^{\infty} \left[ \sum_{k=-\infty}^{\infty} x[k] h[n-k] \right] z^{-n} \]

because we have a Z-transform. Switching the order of the summations (OK except for pathological cases), we get:

\[ = \sum_{k=-\infty}^{\infty} x[k] \sum_{n=-\infty}^{\infty} h[n-k] z^{-n} \]

Now, let \( m = (n-k) \) and we get:

\[ = \sum_{k=-\infty}^{\infty} x[k] \sum_{m=-\infty}^{\infty} h[m] z^{-(m+k)} \]

\[ = \sum_{k=-\infty}^{\infty} x[k] z^{-k} \sum_{m=-\infty}^{\infty} h[m] z^{-m} = X(z)H(z) \]

The new ROC will depend on both the poles in \( X(z) \) and \( H(z) \), giving \( R_x \cap R_h \) since the ROC cannot include poles. However, if one transform has a zero that cancels a pole of the other then the ROC can be bigger, hence \( R'_y \supset R_x \cap R_h \).
LTI System Applications

Transfer Functions  The Z-transform properties are particularly useful when you have an LTI system described by an LCCDE.

\[
\sum_{k=0}^{N} a_k y[n - k] = \sum_{k=0}^{M} b_k x[n - k]
\]

\[
\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)
\]

\[
H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}
\]

An alternative way to write this is:

\[
H(z) = \frac{Y(z)}{X(z)} = \frac{p_0 \prod_{k=0}^{M} (z - \zeta_k)}{d_0 \prod_{k=0}^{N} (z - \lambda_k)}
\]

We can use this to determine outputs of LTI systems by multiplying the Z-transform with the input with \( H(z) \) to get the \( Z \)-transform of the output. Then we can recover the time domain output using the Inverse Z-transform.

Note that finding inverse needs paying special attention to ROC.
Given a difference equation,

\[ y[n] - .3y[n - 1] = x[n] \]

find the Z-transform of the equation and then find the response \( Y(z) \) of the system to an input \( x[n] = (.6)^nu[n] \).

\[
H(z) = \frac{1}{(1 - .3z^{-1})} = \frac{z}{z - .3} \quad |z| > .3
\]

\[
X(z) = \frac{z}{z - (.6)} \quad |z| > .6
\]

\[
Y(z) = \frac{z^2}{(z - .3)(z - .6)} \quad \leftarrow \frac{Z^{-1}}{2(.6)^nu[n] - (.1)^nu[n]}
\]

\[
\frac{z^2}{(z - .3)(z - .6)} = \frac{A2 + B}{z - .3} + \frac{Cz + D}{z - .6}
\]

| A + C = 1 \quad (B + D - .3A - .6C) = 0 \quad B + 2D = 0
| \begin{align*}
A + C &= 1 \\
B + 2D &= 0 \\
A + 2 &= 0 \\
C &= -1
\end{align*} \quad \Rightarrow \quad A = 2, C = -1

What if you wanted to find the response in the time domain?

⇒ We can use **Partial Fraction Expansion** to invert the Z-transform.
As we saw for Laplace Transforms,

\[ Y(z) = \frac{N(z)}{D(z)} = \sum_{k=1}^{N} \frac{r_k z}{z - p_k} \]

where

\[ p_k = \text{pole} \quad r_k = \text{residue} \]

Then use tables to invert the Z-transform, e.g.

\[ a^n u[n] \leftrightarrow \frac{z}{z - a} \]

Back to our previous example ...

At home \{ cheek \}

\[ P_1 = .3 \quad , \quad P_2 = .6 \]

\[ r_1 = -1 \quad \quad r_2 = 2 \]
Ex. Find Inverse Z-Transform of

\[ X(z) = \frac{2z^2 - 5z}{(z - 2)(z - 3)}, \quad |z| > 3 \]

Expand:

\[ x(z)^{-1} = \frac{X(z)}{z} = \frac{2z - 5}{(z - 2)(z - 3)} \]

\[ = \frac{A}{z - 2} + \frac{B}{z - 3} \]

\[ A = \frac{2 \cdot 2 - 5}{2 - 3} \bigg|_{z=2} = \frac{-1}{-1} = 1 \]

\[ B = \frac{2 \cdot 3 - 5}{3 - 2} \bigg|_{z=3} = 1 \]

\[ X(z) = \frac{z}{z - 2} + \frac{z}{z - 3}, \quad |z| > 3 \]

\[ 2^n u[n] + 3^n u[n] \]

\[ \text{If } |z| < 2 \Rightarrow \sum_{n} (2)^n u[-n-1] \]

\[ - (3)^n u[-n-1] \]
Ex. Given \( h[n] = a^n u[n] \) (\(|a| < 1\) and \( x[n] = u[n] \), find \( y[n] = x[n] \ast h[n] \).

What if \( x[n] = u[n - 2] \)?
Ex. Find the output $y[n]$ to an input $x[n] = u[n]$ and an LTI system with impulse response

$$h[n] = -3^n u[-n - 1].$$

$\text{Roc}(x) = \{|z| > 1\}$

$\text{Roc}(h) = \{|z| < 3\}$

$\text{Roc}(y) = \{|z| < 3\}$
Another method to invert Z-transforms is the **Power Series Expansion**. Using

$$\delta[n - k] \longleftrightarrow z^{-k}$$

$$X(z) = \sum_{k=0}^{\infty} x[k] z^{-k} = x[0] + x[1] z^{-1} + x[2] z^{-2} + \cdots$$

$$x[n] = \sum_{k=0}^{\infty} x[k] \delta[n - k] = x[0] \delta[n] + x[1] \delta[n - 1] + x[2] \delta[n - 2] + \cdots$$

So if you can expand $X(z)$ like this as a series in $z^{-1}$, you can pick off $x[n]$ as the coefficients of the series.

**Ex.** Find the Inverse Z-Transform of

$$H(z) = 1 + 2z^{-1} + 3z^{-2}$$

**Note:** If $H(z)$ has only finitely many $z^m$, $m \in \mathbb{Z}$ terms, we can conclude that $h[n]$ has only a finite set of non-zero elements. Such a scenario, then, corresponds to a FIR.
Ex. Find the inverse Z-transform of

\[ X(z) = \frac{8z - 19}{z^2 - 5z + 6} = (z - 2)(z - 3) \]

\[ |z| > 3 \]
Ex. Find the inverse Z-transform of

\[ H(z) = \frac{2z^2 - \frac{5}{2}z}{z^2 - \frac{5}{2}z + 1}, \]

\[ \frac{1}{2} < |z| < 2. \]

Would a system having this Z-transform be BIBO stable?
Find the inverse Z-transform of

\[ W(z) = \frac{z^{-4}}{z^2 - 2z - 3}, \quad |z| > 3 \]
**Frequency Response**

If we evaluate $H(z)$ at $z = e^{j\omega}$, i.e., on the unit circle (as long as unit circle is in the ROC), then we get the Fourier transform $H(e^{j\omega})$ (or as we will see the DTFT $H(\Omega)$) which we call the frequency response. Recall that $H(e^{j\omega})$ is periodic in $\omega$ with period $2\pi$.

Recall that we saw the magnitude of $H(e^{j\omega})$

$$|H(e^{j\omega})| = \sqrt{H(e^{j\omega})H^*(e^{j\omega})} = \sqrt{H(e^{j\omega})H(e^{-j\omega})} = \sqrt{|H(z)| |H(z^{-1})|}$$

Find the phase of $H(e^{j\omega})$

$$\angle H(e^{j\omega}) = \frac{1}{2j} \ln \left[ \frac{H(z)}{H(z^{-1})} \right]_{z = e^{j\omega}}$$
Now we can calculate the frequency response for a stable rational transfer function with real-coefficients:

\[ |H(e^{j\omega})|^2 = \left| \frac{p_0}{d_0} \right|^2 \prod_{k=1}^{M} \left| \frac{(e^{j\omega} - \zeta_k)(e^{j\omega} - \zeta_k^*)}{(e^{j\omega} - \lambda_k)(e^{j\omega} - \lambda_k^*)} \right| \]

and

\[ \angle H(e^{j\omega}) = \angle \frac{p_0}{d_0} + \omega (N - M) + \sum_{k=1}^{M} \angle (e^{j\omega} - \zeta_k) + \sum_{k=1}^{M} \angle (e^{j\omega} - \lambda_k) \]

Geometric Interpretation:
Stability

As we saw earlier, for BIBO stability of a causal LTI system, all roots of the system characteristic equation lie within the unit circle in the $z$-plane.

This is equivalent to stating that all poles of the transfer function $H(z)$ must lie within the unit circle on the $z$-plane. We point out that $H(z)$ does not converge at its poles.

Because causal systems have Regions of Convergence that lie outside the largest magnitude pole, an equivalent condition for BIBO stability is that the ROC must contain the unit circle.

Ex. Find the Z-Transform of the unit step $u[n]$. Would an LTI system with $u[n]$ as its system function be BIBO stable?

\[
\text{ROC}(u[n]) = \{ |z| > 1 \}
\]

Ex. Find the Z-transform of $x[n] = (.9)^nu[n]$. Would an LTI system with $x[n]$ as its system function be BIBO stable?

\[
\text{ROC}(0.9^n u[n]) = \{ |z| > 0.9 \}
\]
Invertibility

\[ h[n] * h_i[n] = \delta[n] \Rightarrow H(z)H_i(z) = 1 \]

Ex. Find the inverse system \( h_i[n] \) of \( h[n] = a^n u[n] \).
Check your results by taking the convolution of \( h[n] \) with \( h_i[n] \).

\[ H_i(z) = \frac{1}{H(z)} \]

Exclude \( z \)'s which \( A(z) = 0 \).
Ex. Find the inverse system of $h[n]$ where

$$H(z) = \frac{z - a}{z - b}.$$ 

For BIBO stability of both systems (assuming they are both causal), where must all poles and zeros of $H(z)$ lie?