Delay Optimal Transmission Policy in a Wireless Multi-access Channel

Navid Ehsan and Tara Javidi
Department of Electrical and Computer Engineering
University of California, San Diego
La Jolla, CA 92093
Email: navid@soe.ucsd.edu , tara@ece.ucsd.edu

Abstract

In this paper we consider the problem of optimal rate allocation in a (potentially asymmetric) multi-access channel. The rate feasibility region of such a network is well-studied. In other words, it is known that for the case of saturated sources any point on the boundary maximizes the throughput. We consider this problem with unsaturated sources, i.e. jobs arrive at sources at random times and the source has the possibility of being empty. In such a setting, all stable rate allocation policies result in a throughput matched with the average arrival rate. Hence, we are interested in rate allocation policies that minimize expected delay in the system. In this paper, we show that a policy of threshold type is optimal in minimizing the average queueing delay. We study the average delay criterion as the limit of an infinite horizon discounted cost function when the discount factor approaches 1.

I. Introduction

In this paper, we consider optimal rate assignment for a general class of multi user system where the achievable rate region (denoted by $\mathcal{R}$) is time invariant and is given by a set of inequalities corresponding to polymatroid structure. At any time instant we can operate at any point in the set $\mathcal{R}$. We study the qualitative structure of the optimal policy for choosing this point as a function of queue lengths to minimize the average job size in the queues.

Consider set of users denoted by $\mathcal{N}$ sharing a common channel and let r_i be the service rate for queue i. The allocated rates must satisfy the following constraints. For any $\mathcal{S} \subset \mathcal{N}$ we must have,

$$\sum_{i \in \mathcal{S}} r_i \leq k_{\mathcal{S}} \ , \tag{1}$$

where $k_{\mathcal{S}}$ is a constant that depends on the subset $\mathcal{S}$.

This multi access scenario is a general form for many specific models as illustrated by the examples below .

Example 1: Additive Gaussian Multi-Access Channel with Slow Fading- The information theoretic capacity of this channel can be derived as follows [1].

$$\sum_{i \in \mathcal{S}} r_i \leq W \log_2 (1 + \frac{\sum_{i \in \mathcal{S}} P_i}{N}) \ , \ \forall \ \mathcal{S} \subset \mathcal{N} \ , \tag{2}$$

where W is the channel bandwidth, P_i is the received power from user i and N is the Gaussian noise power.

Example 2: MIMO Multi-access channel with No CSI at Transmitter-

Consider a Multiple Input Multiple Output (MIMO) channel where the input-output relationship is given by

$$\mathbf{y}[m] = \sum_{k=1}^{2} H_k[m] \mathbf{x}_k[m] + \mathbf{w}[m] \ ,$$

where vector $\mathbf{y}[m]$ ($\mathbf{x}_k[m]$) represents the signal received at the MIMO receiver (transmitted signal of user k), matrix H_k represents the channel matrix for user k, and $\mathbf{w}[m]$ represents the Gaussian noise term.

The information theoretic capacity of this channel for a two user scenario $\mathcal{N} = \{1, 2\}$ can be derived as follows [2].

$$\begin{aligned} r_i &\leq E\left[W \log_2(1 + ||\mathbf{H}_i P_i||^2)\right] \\ r_1 + r_2 &\leq E\left[W \log_2 \det(\mathbf{I} + \frac{\mathbf{H}\mathbf{K}_\mathbf{x}\mathbf{H}^*}{N})\right], \end{aligned}$$

where W is the channel bandwidth, P_i is the vector of transmit power from user i, $\mathbf{K}_\mathbf{x}$ is the covariance matrix, and N is the Gaussian noise power.

In this paper, we study the optimal rate assignment for such multi-user systems where the rate region is defined with the set of inequalities in (1) for a two user scenario.

In this paper, we are not concerned with the computation and derivation of capacity/rate region as they have been widely studied, e.g. see [1], [3], [4]. Instead we are interested in the delay optimality of various rate allocation policies when bits arrive stochastically and whose average rate of arrivals lie inside the capacity region. The stability of a multiple access channel has been studied in [5], [6]. When the arrival rates lie in the capacity region, these papers introduce policies that guarantee stability of the queues. However these policies do not necessarily minimize the delay.

Yeh, et. al, in [7], considered a time varying but symmetric version of our problem and studied the optimal policy for minimizing the average *bit delay*. It was shown that the policy that serves the longest queue with the highest rate (in case of our Example 1, this translate to choosing oder of cancellation) minimizes the average bit delay. This work is closest to our work in that their special case of time-invariant channel scenario coincides with our special case when the arrival processes as well as capacity region is symmetric. In this paper we assume a time-invariant rate region (this can be the model of a slow fading channel, ergodic capacity region, or the case of no channel state information at the transmitter) and consider the problem of minimizing the average job delay in presence of asymmetric arrival processes and channel conditions.

The remainder of this paper is organized as follows. In the next section, we introduce the equivalent queueing model and formulate the problem of delay optimal scheduling as a Markov decision problem with an average cost criterion. In Section III, we introduce a finite horizon discounted variant of the problem, whose solution is then used to construct a solution to the infinite horizon version of the problem (both for a discounted as well as an average case) in Section IV. Finally, in Section V, we summarize and conclude the paper.

II. Problem Formulation and Assumptions

In the case of a two user multi-access system (1) can be written as,

$$\begin{aligned} r_i &\leq k_i\ ,\ i \in \{1,2\} \\ r_1 + r_2 &\leq k_3 \end{aligned} \tag{3}$$

Consider two queues with Poisson arrivals with rates λ_1 and λ_2. Packets have a length exponentially distributed with mean L. Each queue has a dedicated server which serves the packets with constant rate C_i. In addition to the private servers, there is a server with rate C that can be allocated to either queue at any time.

Let $\mu_i = \frac{C_i}{L}$ and $\mu = \frac{C}{L}$. This model corresponds to the capacity region described in (3) as follows by putting,

$$\begin{aligned} \mu_1 &= k_3 - k_2\ , \\ \mu_2 &= k_3 - k_1\ , \\ \mu &= k_1 + k_2 - k_3\ . \end{aligned} \tag{4}$$

Note that in order to avoid triviality we have assumed $k_3 \geq k_1, k_2$, and $k_1 + k_2 \geq k_3$. Note that by time sharing on the shared server, all the points on the boundary of the capacity region described by (3) can be achieved. Although all those points are throughput optimal at any given instant of time, they are not necessarily optimal in terms of minimizing the average job delay. It is clear that restricting attention to policies which do not allow for rate splitting (i.e. simultaneous use of the shared server) does not incur any loss of optimality.

The goal in this paper is to allocate the common server in order to minimize the average delay in the system. By Little's theorem we have $D = (\lambda_1 + \lambda_2)(E(x_1) + E(x_2))$ where D is the average delay, x_i is the number of packets in queue i and $E(.)$ is the expectation. Therefore in order to minimize the delay it suffices to minimize the average number of packets in the system. In this paper we consider a slightly more generalized version of this problem, i.e. the goal is to minimize,

$$J_{av} = \lim_{T \to \infty} \frac{1}{T} \int_{t=0}^{\infty} c(\mathbf{x}(t))dt \ ,$$

where $c(\mathbf{x})$ is a general function for now (later we will restrict $c(\mathbf{x})$ to belong to a specific class of functions). Note that for $c(\mathbf{x}) = x_1 + x_2$, this is minimizing the average number of packets which corresponds to minimizing the average delay. Note that due to exponential assumption on job sizes, the above problem can also be used to minimize the average bit delay by considering the total number of bits in each queue. In general, though, such goals do not coincide.

We study this problem as the limit of an infinite horizon discounted cost dynamic programming problem as the discount factor goes to one [8]. The infinite horizon discounted cost problem can be realized as the limit of the finite horizon discounted cost problem as will be discussed in Section IV. In the next section we study the finite horizon optimization problem.

For the notation in this paper, we will use bold face letters to represent vectors and normal-size letters to represent scalars or random variables. For example $\mathbf{x}(t) = (x_1, x_2)$ is the buffer occupancy of user 1 and 2 at time t. We use $\mathbf{e}_i$ to denote the i-th unit vector.

III. Finite horizon discounted cost problem

Due to the memoryless property of the exponential distribution, the Markovian state of the system at each time can be represented by the number of packets in each queue $\mathbf{x}(t)$. The optimal policy only changes the server allocation when the state changes, i.e. at instance of new arrivals or departures. An event (potential departure or arrival) occurs in the system with rate $\Lambda = \lambda_1 + \lambda_2 + \mu_1 + \mu_2 + \mu$.

Consider the following cost function:

$$J_{discounted} = \int_{t=0}^{T_K} e^{-\alpha t} c\mathbf{x}(t)dt, \tag{5}$$

where T_K is the time when K^{th} event occurs. Let $\mathbf{x}_k$ be the queue length right after the k^{th} event. Using standard uniformization techniques [9], this problem can be reformulated as a discrete time optimization problem with the following cost:

$$J_{discrete} = \sum_{k=0}^{K-1} \beta^k c(\mathbf{x}_k) \tag{6}$$

where $\beta = \frac{\alpha}{\alpha + \Lambda}$.

Define the k step value function as follows:

$$V_k(\mathbf{x}) = \min_{\pi} E[\sum_{k'=0}^{k-1} \beta^{k'} c(\mathbf{x}_k) | \mathbf{x}_0 = \mathbf{x}] \ , \tag{7}$$

where π belongs to the set of Markov policies. This value function can be found recursively as follows [8]:

$$\begin{aligned} V_0(\mathbf{x}) &= 0 \\ V_k(\mathbf{x}) &= c(\mathbf{x}) + \beta\Lambda^{-1}\{\sum_{i=1}^{2}(\lambda_i V_{k-1}(\mathbf{x}+\mathbf{e}_i) + \mu_i V_{k-1}([\mathbf{x}-\mathbf{e}_i]^+)) + \mu \min_i\{V_{k-1}([\mathbf{x}-\mathbf{e}_i]^+)\}\} \end{aligned} \tag{8}$$

In subsequent discussion we set $V_k(x_1, x_2) = V_k(\max(0, x_1), \max(0, x_2))$ to simplify the notation (i.e. we let $V_k(.)$ to be defined on negative values as well). Note that in order to solve (8) we only need to consider $\mathbf{x} \geq 0$.

Definition 1: A function $f : \mathbb{Z}_+^2 \rightarrow \mathbb{R}$ belongs to the set $\mathcal{F}$ if $f(\mathbf{x})$ satisfies the following conditions:

C.1 (monotonicity or non-decreasing condition)

$f(\mathbf{x}) \leq f(\mathbf{x}+\mathbf{e}_i), \quad i \in \{1, 2\};$

C.2 (supermodularity condition)

$f(\mathbf{x}+\mathbf{e}_1) + f(\mathbf{x}+\mathbf{e}_2) \leq f(\mathbf{x}) + f(\mathbf{x}+\mathbf{e}_1+\mathbf{e}_2);$

C.3.a (superconvexity condition)

$f(\mathbf{x}+\mathbf{e}_1) + f(\mathbf{x}+\mathbf{e}_1+\mathbf{e}_2) \leq f(\mathbf{x}+\mathbf{e}_2) + f(\mathbf{x}+2\mathbf{e}_1);$

C.3.b (superconvexity condition)

$f(\mathbf{x}+\mathbf{e}_2) + f(\mathbf{x}+\mathbf{e}_1+\mathbf{e}_2) \leq f(\mathbf{x}+\mathbf{e}_1) + f(\mathbf{x}+2\mathbf{e}_2).$

Here the terminologies follow that used in [10]. Note that these are rather benign conditions, and they specify a very large class of cost functions of practical interest. For example all functions of the form $c_1 x_1^{n_1} + c_2 x_2^{n_2}$ (for any $c_1, c_2, n_1, n_2 > 0$) satisfy these conditions. An example of a function that does not satisfy the above conditions is $f(x_1, x_2) = x_1 x_2$ which fails conditions **C.3.a** and **C.3.b**. Also note that conditions **C.2** and **C.3.a** result in the convexity of f in x_1. Similarly, **C.2** and **C.3.b** imply the convexity of f in x_2.

We are only interested in cost functions $c(\mathbf{x})$ that satisfy these conditions. For the rest of the paper we assume that $c(\mathbf{x}) \in \mathcal{F}$.

Theorem 1: The optimal value function $V_k(.)$, belongs to $\mathcal{F}$ for all $k \geq 0$. Furthermore the optimal policy at each step is of threshold type.

Proof - Since $V_0(\mathbf{x}) = 0$, we have that $V_0(\mathbf{x}) \in \mathcal{F}$. On the other hand we have $c(.) \in \mathcal{F}$. Therefore using induction (similar to the method used in [11], [12]) it can be easily shown that $V_k(\mathbf{x}) \in \mathcal{F}$ for all $k \geq 0$.

The optimality of a threshold policy is a direct result of superconvexity property. By the first part of this theorem, $V_{k-1} \in \mathcal{F}$ for all $0 < k \leq K$. Thus by property **C.3.a** we have $V_{k-1}(\mathbf{x}+\mathbf{e}_1) + V_{k-1}(\mathbf{x}+\mathbf{e}_1+\mathbf{e}_2) \leq V_{k-1}(\mathbf{x}+2\mathbf{e}_1) + V_{k-1}(\mathbf{x}+\mathbf{e}_2)$.

By replacing $\mathbf{x}$ with $\mathbf{x}-\mathbf{e}_1-\mathbf{e}_2$ we have $V_{k-1}(\mathbf{x}-\mathbf{e}_2) + V_{k-1}(\mathbf{x}) \leq V_{k-1}(\mathbf{x}+\mathbf{e}_1-\mathbf{e}_2) + V_{k-1}(\mathbf{x}-\mathbf{e}_1)$. Rearranging, we get

$$V_{k-1}(\mathbf{x}-\mathbf{e}_2) - V_{k-1}(\mathbf{x}-\mathbf{e}_1) \leq V_{k-1}(\mathbf{x}+\mathbf{e}_1-\mathbf{e}_2) - V_{k-1}(\mathbf{x}) \ .$$

The last inequality suggests that if the left hand side is non-negative, then the right hand side is also non-negative. Therefore if the optimal decision is to allocate the shared server to the first queue when the state is $\mathbf{x}$ for some $\mathbf{x}$, then it is optimal to allocate the server to the first queue when the state is $\mathbf{x}+\mathbf{e}_1$. Similarly using $\mathbf{C.3.b}$ we can show that if the optimal decision is to allocate the shared server to the second queue when the state is $\mathbf{x}$, then it is optimal to allocate the server to the second queue when the state is $\mathbf{x}+\mathbf{e}_2$. We can then define a threshold as follows.

$$g_k(x_1) = \min\{x_2 | V_{k-1}(\mathbf{x}-\mathbf{e}_2) \leq V_{k-1}(\mathbf{x}-\mathbf{e}_1)\}, \tag{9}$$

and $g_k(x_1) = \infty$ when the above set is empty. If for time horizon k we have $x_2 \geq g_k(x_1)$ then the optimal policy is to assign the shared server at time t to queue 2, otherwise to queue 1 (if the set is empty then the threshold is infinity), proving the optimality of a threshold policy. ■

While Theorem 1 shows that the optimal scheduler is of the threshold type, it is worth pointing out that it is in general difficult to obtain the quantitative value of the threshold. The threshold is given by Equation (9), where the current cost-to-go function needs to be calculated. This can be computationally expensive.

IV. Infinite horizon

In this section we let K (and as a result T_K) to go to infinity. For the infinite horizon case, we can define two different cost functions, the discounted cost function and the average cost function. We will study these two extensions separately.

A. Discounted cost

We define the infinite horizon discrete time discounted cost function as follows.

$$J^{\infty}_{discrete} = \lim_{K\to\infty} \sum_{k=0}^{K-1} \beta^k c(\mathbf{x}_k) \tag{10}$$

The value function can be defined as:

$$V_\infty(\mathbf{x}) = \min_\pi E[\lim_{k\to\infty} \sum_{k'=0}^{k-1} \beta^{k'} c(\mathbf{x}_{k'})|\mathbf{x}_0 = \mathbf{x}] \tag{11}$$

In this case under the additional constraint that $c(\mathbf{x}) \geq 0$ for all $\mathbf{x} \geq 0$, we have the following result.

Lemma 1: If $c(\mathbf{x}) \geq 0$ for all $\mathbf{x} \geq 0$ then we have $V_\infty(\mathbf{x}) = \lim_{k\to\infty} V_k(\mathbf{x})$.

A proof of this lemma is provided in [13], Chapter 5.4. For the rest of the paper we will assume that $c(\mathbf{x}) \geq 0$.

Theorem 2: We have $V_\infty(.) \in \mathcal{F}$. Furthermore, the optimal policy for the infinite horizon optimization is of threshold type.

Proof - Note that the set $\mathcal{F}$ is closed under limit operation. Since $V_k \in \mathcal{F}$ for all k, therefore by Lemma 1, we have $V_\infty \in \mathcal{F}$. Therefore by superconvexity of $V_\infty(.)$ the optimality of a threshold policy is concluded. ■

Note that in the infinite horizon scenario the policy is stationary whereas in the finite horizon case the threshold can change as the time index changes.

B. Average cost

Define the average cost as follows.

$$J_{av} = \lim_{K\to\infty} \frac{1}{K} \sum_{k=0}^{K-1} c(\mathbf{x}_k) \tag{12}$$

We consider this cost as the limit of the discounted cost problem as $\beta \to 1^-$. We define the value function the same as in the previous section, however instead of the subscript ∞ we use the subscript β in order to emphasize the dependency on β:

$$V_\beta(\mathbf{x}) = \min_\pi E[\lim_{k\to\infty} \sum_{k'=0}^{k-1} \beta^{k'} c(\mathbf{x}_{k'})|\mathbf{x}_0 = \mathbf{x}] \tag{13}$$

By Lemma 1 the function $V_\beta(\mathbf{x})$ satisfies the following recursion:

$$V_\beta(\mathbf{x}) = c(\mathbf{x}) + \beta\Lambda^{-1}\{\sum_{i=1}^{2}(\lambda_i V_\beta(\mathbf{x}+\mathbf{e}_i) + \mu_i V_\beta([\mathbf{x}-\mathbf{e}_i]^+)) + \mu \min_i\{V_\beta([\mathbf{x}-\mathbf{e}_i]^+)\}\} \tag{14}$$

Define the operator T as follows:

$$Tf(\mathbf{x}) = \sum_{i=1}^{2}(\lambda_i f(\mathbf{x}+\mathbf{e}_i) + \mu_i f([\mathbf{x}-\mathbf{e}_i]^+)) + \mu \min_i \{f([\mathbf{x}-\mathbf{e}_i]^+)\} \tag{15}$$

Then Equation (14) can be written as:

$$V_\beta(\mathbf{x}) = c_(\mathbf{x}) + \beta\Lambda^{-1}TV_\beta(\mathbf{x}) \tag{16}$$

Consider the following assumption which is required to guarantee the existence of a policy for which both queues are stable:

Assumption 1: We have $\lambda_1 + \lambda_2 < \mu_1 + \mu_2 + \mu$.

Theorem 3: Suppose $c(\mathbf{x}) \geq 0$ for all $\mathbf{x} \geq \mathbf{0}$ and that Assumption 1 holds. Then,

(a) There exists a finite constant J^* that satisfies the following inequality:

$$J^* + h(\mathbf{x}) \geq \bar{c}(\mathbf{x}) + Th(\mathbf{x}) \ . \tag{17}$$

(b) Let π^* be a policy that minimizes the right hand side of (17). Then π^* is the optimal average cost policy.

(c) J^* is the optimum average cost.

In order to prove theorem 3 we need the following lemmas.

Lemma 2: Starting from any state $\mathbf{x}$, there exists a policy $\pi_\mathbf{x}$ so that under this policy the system will go to zero state with finite cost. We denote this finite cost by $U(\mathbf{x})$.

This is a direct result of Assumption 1 and the fact that under this assumption there exists a policy under which the system is stable.

Lemma 3: $V_\beta(\mathbf{x})$ is non-decreasing in $\mathbf{x}$. Moreover, under Assumption 1 we have

$$V_\beta(\mathbf{x}) - V_\beta(\mathbf{0}) \leq U(\mathbf{x}) \ . \tag{18}$$

Proof - In Theorem 2 we have shown that $V_\beta(\mathbf{x})$ is non-decreasing. To show that (18) holds, consider the policy π^* that follows policy $\pi_\mathbf{x}$ until the first time state $\mathbf{0}$ is reached and then follows the optimal policy. Therefore we have

$$V_\beta(\mathbf{x}) \leq V_\beta^{\pi^*}(\mathbf{x}) = U(\mathbf{x}) + V_\beta(\mathbf{0}),$$

thus proving the lemma. ∎

Lemma 4: Suppose $c(\mathbf{x}) \geq 0$ for all $\mathbf{x} \geq \mathbf{0}$. Then under Assumption 1 the quantity $(1-\beta)W_\beta(\mathbf{0})$ is bounded for $\beta \in (0,1)$.

Proof - Note that when $c(\mathbf{x}) \geq 0$, Assumption 1 implies that $E^{\pi_\mathbf{0}}[c(\mathbf{x}_k)|\mathbf{x}_0 = \mathbf{0}] \leq U(\mathbf{0})$. This can be argued as follows. Under policy $\pi_\mathbf{0}$, state $\mathbf{0}$ is a recurrent state and thus any state at time t lies in between two consecutive occurrences of state $\mathbf{0}$. Since the expected sum of all costs in between those two occurrences is less than or equal to $U(\mathbf{0})$ and all costs are non-negative, the cost at each time step has to be less than or equal to $U(\mathbf{0})$. Thus we have

$$\begin{aligned}
(1-\beta)V_\beta(\mathbf{0}) \leq (1-\beta)V_\beta^{\pi_\mathbf{0}}(\mathbf{0}) &= (1-\beta)E^{\pi_\mathbf{0}}[\lim_{k\to\infty}\sum_{k'=0}^{k-1}\beta^{k'}c(\mathbf{x}_u)|\mathbf{x}_0 = \mathbf{0}] \\
&= (1-\beta)\lim_{k\to\infty}\sum_{k'=0}^{k-1}\beta^{k'}E^{\pi_\mathbf{0}}[c(\mathbf{x}_{k'})|\mathbf{x}_0 = \mathbf{0}] \\
&\leq (1-\beta)\lim_{k\to\infty}\sum_{k'=0}^{k-1}\beta^{k'}\cdot U(\mathbf{0}) = U(\mathbf{0}) \ ,
\end{aligned}$$

where the first inequality is due to the fact that $\pi_{\mathbf{0}}$ is not necessarily the optimal policy. The exchange of the limit and expectation is a result of the assumption that $c(\mathbf{x}) \geq 0$ (and consequently the fact that the sum inside the expectation is non-decreasing) and the last inequality holds by Assumption 1. ∎

Lemma 5: Let β_n be a sequence of real numbers such that $\beta_n \to 1^-$ as $n \to \infty$. If Assumption 1 holds, then there exists a subsequence α_n such that

$$\lim_{n\to\infty}(V_{\alpha_n}(\mathbf{x}) - V_{\alpha_n}(0)) = h(\mathbf{x}) \ ,$$

where $0 \leq h(\mathbf{x}) \leq U(\mathbf{x})$ for all $\mathbf{x} > \mathbf{0}$.

Proof - Note that $h_{\beta_n} = V_{\beta_n}(\mathbf{x}) - V_{\beta_n}(0) \leq U(\mathbf{x})$ by Lemma 3. The sequence h_{β_n} can be considered as a point in the product topology $\prod_{n=1}^{\infty}[0, U(\mathbf{x})]$ which is a compact space by Tychnoff theorem [14]. Therefore there exists a subsequence α_n for which $h_{\alpha_n}(\mathbf{x})$ converges. Let $h(\mathbf{x})$ be the limit point of $h_{\alpha_n}(\mathbf{x})$. Since $0 \leq h_{\alpha_n}(\mathbf{x}) \leq U(\mathbf{x})$ for all n we have $0 \leq h(\mathbf{x}) \leq U(\mathbf{x})$. ∎

Now define $h_\beta(\mathbf{x})$ as follows:

$$h_\beta(\mathbf{x}) = V_\beta(\mathbf{x}) - V_\beta(\underline{0}) \tag{19}$$

If $\beta_n \to 1^-$, then it is shown in Lemma 5 that under Assumption 1 one can find a subsequence α_n such that $\lim_{n\to\infty} h_{\alpha_n}(\mathbf{x})$ exists. We call this limit function $h(\mathbf{x})$. We then have all components needed for proof of Theorem 3.

Proof of Theorem 3 - Let β_n be a sequence of real numbers such that $\beta_n \to 1^-$ as $n \to \infty$. If we add $\beta_n V_{\beta_n}(\mathbf{0})$ to both sides of (14) we get:

$$(1-\beta_n)V_{\beta_n}(\mathbf{0}) + (V_{\beta_n}(\mathbf{x}) - V_{\beta_n}(\mathbf{0})) = c_(\mathbf{x}) + \beta_n \Lambda^{-1} T(V_{\beta_n}(\mathbf{x}) - V_{\beta_n}(\mathbf{0}))$$

By Lemma 5, there exists a sequence $\alpha_n \to 1^-$ such that $\lim_{n\to\infty}(V_{\alpha_n}(\mathbf{x}) - V_{\alpha_n}(0)) = h(\mathbf{x})$. By Lemma 4 the value $(1-\alpha_n)V_{\alpha_n}(\mathbf{0})$ is bounded. Therefore there exists a sequence $\gamma_n \to 1^-$ for which this value converges to a real number which we call J. Therefore by replacing γ_n. Replacing γ_n in place of β_n in the above equation, taking the limit and using Fatou's lemma we have;

$$J^* + h(\mathbf{x}) \geq \bar{c}(\mathbf{x}) + Th(\mathbf{x}) \ .$$

Now assume that policy π^* minimizes the right hand side of (17). First we show that $\bar{J}^{\pi^*} \leq J^*$. Let $\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_k$ be the (random)states that are visited during times $1, 2, \cdots, k+1$, then using (17) we have,

$$\begin{aligned}
J^* + h(\mathbf{x}_0) &\geq \bar{c}(\mathbf{x}_0) + E[h(\mathbf{x}_1)|\mathbf{x}_0], \\
J^* + h(\mathbf{x}_1) &\geq \bar{c}(\mathbf{x}_1) + E[h(\mathbf{x}_2)|\mathbf{x}_1], \\
&\cdots \\
J^* + h(\mathbf{x}_{k-1}) &\geq \bar{c}(\mathbf{x}_{k-1}) + E[h(\mathbf{x}_k)|\mathbf{x}_{k-1}]
\end{aligned}$$

Taking the expected value on both sides and adding the equations we get,

$$\frac{1}{k}\sum_{u=0}^{k-1} E[c(\mathbf{x}_u)] \leq J^* + \frac{(h(\mathbf{x}_1) - h(\mathbf{x}_k))}{k} \leq J^* + \frac{h(\mathbf{x}_0)}{k}, \tag{20}$$

where the second inequality is due to the fact that $h(\mathbf{x}_k) \geq 0$. Taking the limit from both sides of (20) as $t \to \infty$ and using the fact that $h(\mathbf{x}) \leq U(\mathbf{x})$ we have $\bar{J}^{\pi^*} \leq J^*$.

Now consider any other policy π'.

$$J^{\pi^*} \leq J^* \leq \limsup_{\beta \to 1^-}(1-\beta)V_\beta(\mathbf{x}) \leq \limsup_{\beta \to 1^-}(1-\beta)V_\beta^{\pi'}(\mathbf{x}) \leq J^{\pi'} \tag{21}$$

Therefore π is the optimal average cost policy. On the other hand if we let $\pi' = \pi^*$, then we scan see that J^* is the optimal average cost. ∎

From this theorem we also have the following corollary.

Corollary 1: We have $h(\mathbf{x}) \in \mathcal{F}$. Hence, the optimal policy is of threshold type.

Proof - Let β_n be a sequence such that $\beta_n \to 1^-$. Then by Lemma 5, there exists a subsequence α_n such that $\lim_{n\to\infty} h_{\alpha_n}(\mathbf{x})$ exists. On the other hand we have for any α_n,

$$h_{\alpha_n}(\mathbf{x}) = V_{\alpha_n}(\mathbf{x}) - V(\alpha_n)(\underline{0}) \ .$$

Since we have $V_{\alpha_n} \in \mathcal{F}$ for all α_n, taking the limit of both sides as $\alpha_n \to 1^-$ and noting that $\mathcal{F}$ is closed under the limit operation, we conclude that $h(\mathbf{x}) \in \mathcal{F}$.

Note that we have,

$$Th(\mathbf{x}) = \sum_{i=1}^{2}(\lambda_i h(\mathbf{x}+\mathbf{e}_i) + \mu_i h([\mathbf{x}-\mathbf{e}_i]^+)) + \mu \min_i\{h([\mathbf{x}-\mathbf{e}_i]^+)\} \ ,$$

Therefore the optimal policy allocates the server to the queue i that minimizes $h([\mathbf{x}-\mathbf{e}_i]^+)$. Using the fact that $h(.) \in \mathcal{F}$ and using similar arguments made in the proof of Theorem 1 we can show that the optimal policy is of threshold type. The threshold is given by the following equation,

$$g(x_1) = \min\{x_2 | h(\mathbf{x}-\mathbf{e}_2) \leq h(\mathbf{x}-\mathbf{e}_1)\}.$$

Note that for the average cost criterion, the threshold is stationary and does not depend on time. ∎

V. Conclusion

In this paper we considered a general potentially asymmetric multi-user system which can be used to model many communication systems with shared resources. In such a problem, we studied the problem of optimally allocating the transmission rates from the capacity region in order to minimize an average or discounted cost function. The cost at each time instant, is a function of the number of packets in each queue and for the special case of symmetric linear function results in a delay optimal strategy. In this paper, we showed that the optimal policy in such a setting is of threshold type. Note that the key issue in finding the optimal policy is the following fact. When one queue becomes empty while the other is full, the private server allocated to the empty queue is wasted. In other words, to minimize delay, it becomes important to balance the queues, in the anticipation of such an event in future. This balancing act depends on the expected arrivals in the future, holding cost, and the asymmetry in the capacity region. This intuitive observation is exactly what our threshold structure provides.

Again we emphasize that although in this paper we proved the optimality of a threshold policy, finding these thresholds can be computationally expensive. Also extending these results to more than two users still remains an open problem.

REFERENCES

[1] D. Tse, "Optimal power allocation over parallel gaussian broadcast channels," *Proceedings of International Symposium on Information*, June 1997.

[2] D. Tse and P. Viswanath, *Fundamentals of Wireless Communication*, Cambridge University Press, 2005.

[3] D. Tse and S. Hanly, "Multi-access fading channels: Part i: Polymatroid structure, optimal resource allocation and throughput capacities," *IEEE Transactions on Information Theory*, vol. 44, no. 7, pp. 2796–2815, November 1998.

[4] L. Li and A. J. Goldsmith, "Outage capacities and optimal power allocation for fading multiple-access channels," *IEEE Transactions on Information Theory*, vol. 51, no. 4, pp. 1326–1347, April 2005.

[5] M. J. Neely, E. Modiano, and C. E. Rohrs, "Power allocation and routing in multibeam satellites with time-varying channels," *IEEE/ACM Tansactions on Networking*, vol. 11, no. 1, pp. 138–152, 2003.

[6] M. J. Neely, E. Modiano, and C. E. Rohrs, "Dynamic power allocation and routing for time-varying wireless networks," *IEEE Journal on Selected Areas in Communications, Special Issue on Wireless Ad-Hoc Networks*, vol. 23, no. 1, pp. 89–103, 2005.

[7] E. M. Yeh, "Minimum delay multiaccess communication for general packet length distributions," *Proceedings of the Allerton Conference on Communication, Control, and Computing*, pp. 1536–1545, September 2004.

[8] P. R. Kumar and P. Varaiya, *Stochastic Systems, Estimation, Identification and Adaptive Control*, Prentice Hall, 1986.

[9] C. G. Cassandras and S. Lafortune, *Introduction To Discrete Event Systems*, Kluwer Academic Publishers, 1999.

[10] G. M. Koole, "Structural results for the control of queueing systems using event-based dynamic programming," *Queueing Systems*, vol. 30, pp. 323–339, 1998.

[11] N. Ehsan and M. Liu, "Optimal server allocation in batches," *IEEE Journal on Selected Areas in Communications (JSAC), special issue on Nonlinear Optimization of Communication Systems*, vol. 24, no. 8, pp. 1614–1626, August 2006.

[12] B. Hajek, "Optimal control of two interacting service stations," *IEEE Trans. Auto. Control. AC-29*, pp. 491–499, 1984.

[13] D. P. Bertsekas, *Dynamic Programming, Deterministic And Stochastic Models*, Prentice Hall, 1987.

[14] J. R. Munkres, *Topology*, Prentice Hall, second edition, 2000.