Expected Makespan Minimization on Identical Machines in Two Interconnected Queues

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1 Introduction-Problem Formulation

We investigate the following makespan minimization problem in two interconnected queues with arrivals to the first queue.

Problem (P) Consider a system of two interconnected queues as in Fig. 1 (page 2). Initially, there are N_i jobs in the queue i (Q_i), $i = 1, 2$. New jobs arrive in the first queue according to a Poisson process with the parameter λ. There are m identical machines available in parallel, so that every machine can process any job in Q_1 and Q_2. The job processing times in Q_1 and Q_2 are exponentially distributed random variables with rates μ_1 and μ_2, respectively. After a job completes service in Q_1, it either joins Q_2 with probability p or leaves the system with probability $1-p$. When a job initially in Q_1 joins Q_2, it creates k jobs in Q_2. Whenever a job completes service in Q_2 it leaves the system. Preemption is allowed. The objective is to determine a server allocation policy that minimizes the expected value of the first time the system is empty (i. e. expected makespan).

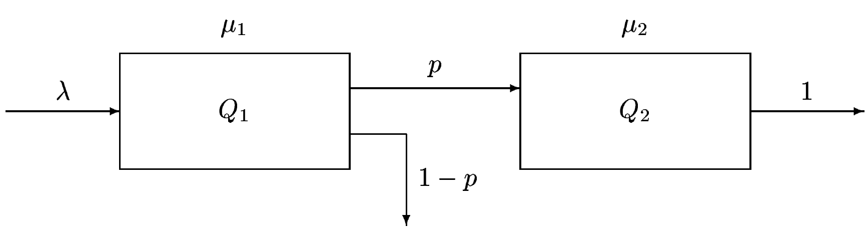

Figure 1: Two interconnected queues in Problem (**P**)

The above problem is motivated by detection and classification issues in Automated Target Recognition (ATR) systems and by quality control issues in wafer fabrication.

The ability to detect and classify objects in a timely fashion is a critical factor in determining the efficiency of ATR systems. One approach to target detection and classification in ATR systems is the following: The area under surveillance is divided into sub-patches. Initially, each sub-patch is separately processed using certain sequential detection and classification rules. One of the following situations can arise after the sub-patch processing is completed: (i) it is determined that there is no target in the sub-patch; in this situation the sub-patch is discarded and is not processed any further; (ii) it is determined that there are targets of potential interest in the sub-patch; in this situation the sub-patch is divided into k ($k \geq 2$) smaller sub-patches and a separate processor is allocated to each of the newly created sub-patches. This process is repeated until all potential targets are detected and classified. Under the assumption that there is a fixed number resources (e. g. antennas and their associated data processing systems) and a fixed set of sub-patch processing algorithms (e. g. algorithms used for detection and classification of each stage of the above process), the objective is to determine resource allocation strategies to minimize the expected length of surveillance time. This is equivalent to minimizing the expected makespan, that is, the expected value of the first time that the system is first empty. Problem (P) captures essential features of this problem as follows. The number of tasks initially present in Q_1 corresponds to the number of subpatches originally available for processing. Initially we have $N_2 = 0$. Each task that completes processing in Q_1 and leaves the system corresponds to a subpatch the processing of which is completed and no target is found. Each task that completes processing in Q_1 and then joins Q_2 and creates k new tasks in Q_2 corresponds to a subpatch that is further subdivided into k new smaller subpatches. There are no external new arrivals, hence $\lambda = 0$.

Problem (P) arises also in semiconductor manufacturing systems, specifically, in quality control of wafer fabrication. A two-stage inspection is used to speed up the quality control process in wafer fabrication. The first stage sorts out, through a preliminary test, wafers that are defected and discards them. The remaining wafers go through a second inspection; by the end of that inspection they are either declared perfect (and they can be used in logic chips, memory chips, etc.) or discarded otherwise. The above two stage inspection is can be described by the system of Fig. 1 as follows: New wafers that arrive in the system for inspection join Q_1. The preliminary stage of inspection can be modeled by the processing of a wafer in Q_1. The wafers that are found imperfect after the first inspection (i.e. after their processing in Q_1) are discarded, that is, they leave the system. The wafers that that pass the first inspection join Q_2 and go through the second inspection. The quality control process is concluded after the processing in Q_2 (i./ e./ the wafers leave the system of Fig. 1 after their second test is completed). A reasonable objective for this problem is to minimize the length of busy cycles. In our model this can be translated to minimizing the first time that the system is first empty. By the memoryless property of the arrival process, every time the system gets empty the system generates statistically similar process. As a result minimizing the expected makespan, as the first time that the system is first empty, is a reasonable goal.

So far, makespan minimization problems have been investigated either for systems of serial machines, or on parallel machines (see [16], [1], [14], [15], [10], [9], [2], and the references therein). Problem (P) is distinctly different from the makespan minimization problems on serial machines, known as shop scheduling (see [10] and [9]). Traditionally, makespan minimization problems on identical parallel machines are formulated as follows: There are jobs N to be processed by m identical machines ($m < N$). These machines are available in parallel, so that every machine can process any job. The jobs' processing times are in general random, and the objective is to find a strategy that minimizes the expected makespan . Several variations of the above problem with different assumption on the jobs' processing times have been investigated (see, for example, [16], [1], [14], [15], and [9]). The main result states that in the above problem, it is optimal to serve the jobs with the longest expected processing time (LEPT) first. The problems on makespan minimization on parallel machines investigated so far are fundamentally different from problem (P) for the following reason. In the traditional formulation, the number of jobs in the system is monotonically decreasing, in the time intervals between successive arrivals. An increase in number of jobs in the system occurs only at the arrival of the new jobs; these times are independent of the service completion times. In problem (P), with $k > 1$ we have an increase in the number of the jobs to be processed at the completion times if the interconnection occurs. Thus, the time instants where the number of jobs in the system increases are highly correlated with job completion times in Q_1. The correlation between job arrivals to Q_2 and job completions in Q_1 gives rise to problems that are conceptually different from those previously studied even when $k = 1$. In the case $k = 1$, it might appear that one can convert problem (P) into a traditional makespan minimization problem on parallel machines where the jobs' processing times are described by a random variable T_c which with a probability $1-p$ is exponential with parameter μ_1, and with probability p is equal to the sum of two independent exponential random variables with parameters μ_1 and μ_2. Such a transformation does not result in a problem equivalent to problem (P) because it discards some of the information available in problem (P). Specifically, such a transformation ignores the information about the occurrence of a job completion in Q_1 and its interconnection to Q_2. More precisely if T_i denotes the service time in Q_i, $i=1,2$ and I denotes the indicator function for the interconnection, then,

$$\sigma(T_c) \subset \sigma(T_1) \bigvee \sigma(T_2) \bigvee \sigma(I)$$

with strict inclusion, and where $\sigma(T_1) \bigvee \sigma(T_2) \bigvee \sigma(I)$ is the minimal σ-field with respect to which T_1, T_2, and I are all measurable. Note that this loss of information implies that an optimal solution to the transformed problem is not guaranteed to be an optimal solution to problem (P).

From the above discussion it is evident that problem (P) is distinctly different from traditional makespan minimization problems in [16], [1], [14], [15], [10], [9], and [2]. It is also different form [12] because it includes new arrivals.

Problem (P) can be viewed as scheduling of a random task graph where the tasks' processing times are identically distributed at each "generation" (cf. [4]-[5]) but are statistically different across different generations. In this aspect Problem (P) has features similar to those of [4]-[8], but is distinctly different from the problems investigated in [4]-[8] for the following reasons: In [4] the tasks' processing times are deterministic. In [6]-[8] the graphs are deterministic. In [5], which is an unpublished document, the processing times at all generations of the graph are identically distributed. In our opinion, the exchange arguments leading to the results of [4]-[5] depend heavily on the assumption that the the processing times are either deterministic or identically distributed across "generations", and can not be extended in a straightforward manner to the situation where the processing times are not statistically identical across generations. For the special case when the processing times are statistically identical, i.e. $\mu_1 = \mu_2$, the result of this paper is the the same as that of [5].

The main contribution of this paper is the analysis of problem (P). For problem (P) it is impossible, as demonstrated by the example in Section 3,to determine a server allocation policy that is optimal under *any* set of parameters, μ_1, μ_2, and p. Hence, we consider a specific policy g^* that gives priority to Q_1, and establish a condition on μ_1, μ_2, and p sufficient to guarantee that g^* is optimal. Interestingly, this conditions guarantees that policy g^* coincides with LEPT. Furthermore, when $k=1$, we prove that if the above condition is not satisfied, then it is always optimal to give priority to Q_2. And this again coincides with LEPT for case when $k=1$.

The paper is organized as follows: The main result on the optimality of policy g^* and its proof are stated in Section 2. In Section 2.2 we introduce the functional equation of this problem and transform and reduce this equation into a form that can be analyzed, using iteration method. In Section 2.3 an induction proof of the optimality of g^* under the specified sufficient condition is given. In Section 3 We establish a very simple example to study the necessity and sufficiency of condition 2; in this section we discuss the special case $k=1$ and prove that for $k=1$ if the condition in Section 2 is not satisfied the policy that gives priority to Q_2 is optimal.

2 Analysis of Problem (P)

2.1 The Main Result

In this section we provide the main result of the paper. We establish a condition on the processing times in Q_1 and Q_2, and the interconnection process (p), sufficient to guarantee the optimality of policy g^*, i.e. the policy that gives priority to Q_1. In contrast to the condition presented in [], our condition is independent of the number m of machines and the number k of jobs created in Q_2 after an interconnection. Note that the nature of the optimality is interesting only when the expected makespan is finite. In order to assure the finiteness of makespan we assume that $\frac{1}{\mu_1} + \frac{k}{\mu_2} < \frac{m}{\lambda}$, or equivalently $\frac{m\mu_1\mu_2}{\lambda(\mu_2+k\mu_1)} > 1$. Under this condition, we can show that even a suboptimal policy like FIFO have a finite expected makespan. The argument is similar to Whitt's proof in [17].

The following theorem gives the main results of the paper.

Theorem 1 *Assume $\frac{m\mu_1\mu_2}{\lambda(\mu_2+k\mu_1)} > 1$. Policy g^*, that is, the policy that gives priority to Q_1, is optimal for Problem (P) under the following condition:*

$$p \geq \frac{\mu_1 - \mu_2}{\mu_1}. \tag{1}$$

We proceed to establish the main result of the paper as follows: First, via some preliminary analysis, we show that Problem (P) can be reduced to a corresponding n-stage problem. By induction, we establish the optimality of policy g^* for the corresponding n-stage problem under the conditions of Theorem 1. The philosophy of our approach is quite similar to that of [2].

We note that the condition of Theorem 1 is automatically satisfied, if $\mu_1 \leq \mu_2$ or $p = 1$.

2.2 Reduction of Problem (P) to an n-stage problem

We can formulate problem (P) as a Markov decision problem. The state space in our model consists of the ordered pairs of the form (x_1, x_2), where x_i is the number of available jobs in the Q_i. The action space $A_{(x_1,x_2)}$ at state (x_1, x_2) consists of the ordered pairs of the form $\big(g_1(x_1,x_2), g_2(x_1,x_2)\big)$ where $g_i(x_1,x_2)$ are the possible allocation of servers to the jobs in Q_i. Without loss of generality we can restrict our attention to the Markov policies denoted by the vector $g = \big(g_1(x_1,x_2), g_2(x_1,x_2)\big)$ for which

$$g_1(x_1,x_2) + g_2(x_1,x_2) = \min(x_1 + x_2, m).$$

Given the state (x_1,x_2) and the action $\big(g_1(x_1,x_2), g_2(x_1,x_2)\big)$, the time until the next transition is an exponential random variable with parameter $g_1(x_1,x_2)\mu_1 + g_2(x_1,x_2)\mu_2 + \lambda$. As a result, the next state can be one of the following states with the corresponding probabilities.

$$\begin{array}{rll}
(x_1+1, x_2) & \text{with probability} & \lambda/\big(g_1(x_1,x_2)\mu_1 + g_2(x_1,x_2)\mu_2 + \lambda\big), \\
(x_1-1, x_2+1) & \text{with probability} & p\mu_1 g_1(x_1,x_2)/\big(g_1(x_1,x_2)\mu_1 + g_2(x_1,x_2)\mu_2 + \lambda\big), \\
(x_1-1, x_2) & \text{with probability} & (1-p)\mu_1 g_1(x_1,x_2)/\big(g_1(x_1,x_2)\mu_1 + g_2(x_1,x_2)\mu_2 + \lambda\big), \\
(x_1, x_2-1) & \text{with probability} & \mu_2 g_2(x_1,x_2)/\big(g_1(x_1,x_2)\mu_1 + g_2(x_1,x_2)\mu_2 + \lambda\big).
\end{array}$$

Denote by $V(x_1,x_2)$ the minimum expected makespan. Then V satisfies the dynamic programming equation

$$V(x_1,x_2) \quad = \quad \min_g \Big\{ \Big(1 + \mu_1 g_1(x_1,x_2)[pV(x_1-1, x_2+k) + (1-p)V(x_1-1,x_2)] +$$

$$
\begin{aligned}
&\mu_2 g_2(x_1,x_2)V(x_1,x_2-1)+\\
&\lambda V(x_1+1,x_2)\Big)/\Big(\mu_1 g_1(x_1,x_2)+\mu_2 g_2(x_1,x_2)+\lambda\Big)\Big\},
\end{aligned}
\tag{2}
$$

with $V(0,0)=0$. Eq. 2 ensures that for every $g=(g_1(x_1,x_2),g_2(x_1,x_2))$,

$$
\begin{aligned}
0 \quad \leq \quad & 1+\mu_1 g_1(x_1,x_2)[pV(x_1-1,x_2+k)+(1-p)V(x_1-1,x_2)]+\\
& \mu_2 g_2(x_1,x_2)V(x_1,x_2-1)-\Big(\mu_1 g_1(x_1,x_2)+\mu_2 g_2(x_1,x_2)+\lambda\Big)V(x_1,x_2)+\\
& \lambda V(x_1+1,x_2)
\end{aligned}
$$

and is equivalent to

$$
\begin{aligned}
V(x_1,x_2) \quad = \quad & \min_g\Big\{\frac{1}{R}+\frac{\mu_1 g_1(x_1,x_2)}{R}[pV(x_1-1,x_2+k)+(1-p)V(x_1-1,x_2)]+\\
& \frac{\mu_2 g_2(x_1,x_2)}{R}V(x_1,x_2-1)+\frac{\lambda}{R}V(x_1+1,x_2)+\\
& \Big(1-\frac{\mu_1 g_1(x_1,x_2)+\mu_2 g_2(x_1,x_2)+\lambda}{R}\Big)V(x_1,x_2)\Big\}
\end{aligned}
\tag{3}
$$

where R is any fixed positive number. If we pick number R such that

$$
R \geq \lambda + m\max(\mu_1,\mu_2),
\tag{4}
$$

we can interpret Eq. (3) as the functional equation for a discrete-time Markov decision process whose transitions occur at fixed rate R, i.e. the time unit between the transitions are $\frac{1}{R}$ which is independent of the policy. This independence between transitions and the policy facilitates our analysis and allows us to use successive approximation, a well-known results in dynamic programming, to analyze problem (P).

Define for any $x_1,x_2 \geq 0$, Markov policy g, and function $f:\overline{N}^2 \mapsto \overline{R^+}$, the local cost function $c^g(x_1,x_2,f)$ by

$$
\begin{aligned}
c^g(x_1,x_2,f) \quad = \quad & \frac{1}{R}+\frac{\mu_1 g_1(x_1,x_2)}{R}[pf(x_1-1,x_2+k)+(1-p)f(x_1-1,x_2)]+\\
& \frac{\mu_2 g_2(x_1,x_2)}{R}f(x_1,x_2-1)+\frac{\lambda}{R}f(x_1+1,x_2)+\\
& \Big(1-\frac{\mu_1 g_1(x_1,x_2)+\mu_2 g_2(x_1,x_2)+\lambda}{R}\Big)f(x_1,x_2)
\end{aligned}
\tag{5}
$$

Thus, because of Eq. (3) the minimum expected makespan is the solution to the equation:

$$
V(x_1,x_2)=\min_g c^g(x_1,x_2,V)
\tag{6}
$$

Construct the sequence of $\{V_n(x_1,x_2):n=0,1,2,\ldots\}$ as follows

$$
\begin{aligned}
V_0(x_1,x_2) \quad &= \quad 0, \text{ for all} x_1,x_2 && (7)\\
V_n(0,0) \quad &= \quad 0, \text{ for all } n\geq 0 && (8)\\
V_n(x_1,x_2) \quad &= \quad \min_g c^g(x_1,x_2,V_{n-1}), \text{ for all } n\geq 1. && (9)
\end{aligned}
$$

Lemma 1 *Assume policy $\hat{g} = (\hat{g}_1(x_1, x_2), \hat{g}_2(x_1, x_2))$ achieves the minimum in Eq. (9) for all n. Then $\hat{g}$ is an optimal allocation policy for problem (P).*

Proof. Define by $\hat{g}(n)$ the policy that satisfies

$$V_n(x_1, x_2) = \min_g c^g(x_1, x_2, V_{n-1}) = c^{\hat{g}}(x_1, x_2, V_{n-1}),$$

where c^g is defined in Eq. 5 and $V_{n-1}(x_1, x_2)$ is known for all $x_1 \geq 0$, $x_1 \geq 0$. It is known ([2], [13], [18]) that if $\hat{g}(n) = \hat{g}$ for all $n \geq 1$, then $\hat{g}$ satisfies

$$V(x_1, x_2) = \min_g c^g(x_1, x_2, V) = c^{\hat{g}}(x_1, x_2, V),$$

where $V(x_1, x_2)$ is the minimum expected makespan under the initial condition (x_1, x_2). Furthermore, for all (x_1, x_2)

$$V(x_1, x_2) = \lim_{n \to \infty} V_n(x_1, x_2).$$

□

The argument used in the proof of Lemma 1 is a special case of a general relationship between negative dynamic programming and successive approximation first introduced by Strauch [13]. Note that since our action space is finite, lemmas 3.1 and 9.1 in [13] provide the argument sufficient for establishing the Lemma 1.

Notice that the set of Eqs. (7)-(9) are called the optimality equations for n-stage problem. In the next section we study this problem and its interpretation, and prove that if $p \geq \frac{\mu_1 - \mu_2}{\mu_1}$, policy g^* is optimal for every $n \geq 1$.

2.3 Solution to the n-stage problem

Consider the sequence constructed by Eqs. (7)-(9). This sequence has the following interpretation. Consider the following finite horizon problem with n transitions defined as horizon. Assume that in the queueing system in Figure 1, transitions occur at a fixed rate R, which is independent of the servicing policy. With probability $g_i(x_1, x_2)\frac{\mu_i}{R}$ the transition occurs in Q_i, with probability $\frac{\lambda}{R}$ there is an arrival, and with probability $1 - \frac{\mu_1 g_1(x_1,x_2) + \mu_2 g_2(x_1,x_2) + \lambda}{R}$ there is a transition to the same state. Define the stopping time τ_n as the minimum of the makespan and horizon $\frac{n}{R}$. The objective is to minimize the expected value of τ_n. It is easy to see that the optimality equation for this system coincides with Eqs. (7)-(9). This formulation is called the n-stage problem. This justifies why in Section 2.2 Eqs. (7)-(9) are called the optimality equations for the n-stage problem. Naturally, if a policy achieves the minimum in Eq. (9) for a specific number n, the policy is called optimal at stage n. In fact, Lemma 1 shows the relationship between problem (P) and the n-stage problem. In Lemma 1 we showed that solving problem (P) can be reduced to solving the n-stage problem. Hence, proving theorem 1 is equivalent to proving the following theorem

Theorem 2 *If $\frac{m\mu_1\mu_2}{\lambda(\mu_2 + k\mu_1)} > 1$ and $p \geq \frac{\mu_1 - \mu_2}{\mu_1}$. then the policy that gives priority to Q_1 is optimal at every stage n.*

Proof. The proof of this theorem requires a lengthy argument. For this reason to clarify the ideas we proceed as follows: First we present an outline of the proof where the main ideas are outlined. Then, we present all the details of the proof.

2.3.1 Outline of the Proof of Theorem 2

We prove Theorem 2 by induction on n, number of stages. For that matter we define

Condition 1 $\frac{m\mu_1\mu_2}{\lambda(\mu_2+k\mu_1)} > 1$.

Condition 2 $p \geq \frac{\mu_1-\mu_2}{\mu_1}$.

$$\begin{aligned}
V_n^1(x_1,x_2) &= \mu_1\left[pV_n(x_1-1,x_2+k)+(1-p)V_n(x_1-1,x_2)-V_n(x_1,x_2)\right] && (10)\\
V_n^2(x_1,x_2) &= \mu_2\left[V_n(x_1,x_2-1)-V_n(x_1,x_2)\right] && (11)\\
\mathcal{D}_n(x_1,x_2) &= V_n^1(x_1,x_2)-V_n^2(x_1,x_2) && (12)\\
\mathcal{F}_n(x_1,x_2) &= pV_n(x_1-1,x_2+k-1)+(1-p)V_n(x_1-1,x_2)-V_n(x_1,x_2-1) && (13)\\
\mathcal{G}_n(x_1,x_2) &= V_n(x_1-1,x_2)-V_n(x_1,x_2-1) && (14)
\end{aligned}$$

Using the above expressions, we state the induction hypotheses for stage n as

$(\mathcal{H}0)_n$	policy g^*, that gives priority to Q_1, is optimal at stage n
$(\mathcal{H}1)_n$	$\mathcal{D}_n(x_1,x_2) \leq 0$, for every $x_1 \geq 0,\ x_2 \geq 0$
$(\mathcal{H}2)_n$	$\mathcal{F}_n(x_1,x_2) \leq 0$, for every $x_1 \geq 0,\ x_2 \geq 0$
$(\mathcal{H}3)_n$	$\mathcal{G}_n(x_1,x_2) \leq 0$, for every $x_1 \geq 0,\ x_2 \geq 0$

The induction then proceeds as follows. First we verify the validity of $(\mathcal{H}0)_1$-$(\mathcal{H}3)_1$; this establishes the basis of the induction. For the induction step we assume that $(\mathcal{H}0)_n$-$(\mathcal{H}3)_n$ are valid, and prove that under this assumption and Condition 2 $(\mathcal{H}0)_{n+1}$-$(\mathcal{H}3)_{n+1}$ are true.

We note that $(\mathcal{H}0)$ is sufficient to assert the validity of Theorem 2. However, to prove inductively that that $(\mathcal{H}0)$ is true we need $(\mathcal{H}1)$-$(\mathcal{H}3)$. Specifically, $(\mathcal{H}0)_n$-$(\mathcal{H}3)_n$ are used to inductively establish Theorem 2 via the following lemmas:

Lemma 2:	$(\mathcal{H}0)_n$ and $(\mathcal{H}1)_n \Longrightarrow (\mathcal{H}0)_{n+1}$.
Lemma 3:	$(\mathcal{H}0)_{n+1}$ and $(\mathcal{H}3)_n \Longrightarrow (\mathcal{H}3)_{n+1}$.
Lemma 4:	$(\mathcal{H}0)_{n+1}$, $(\mathcal{H}3)_n$, and $(\mathcal{H}2)_n \Longrightarrow (\mathcal{H}2)_{n+1}$.
Lemma 5:	$(\mathcal{H}0)_{n+1}$, $(\mathcal{H}2)_n$, and $(\mathcal{H}1)_n \Longrightarrow (\mathcal{H}1)_{n+1}$.

Condition 1 is sufficient to guarantee that the minimum expected makespan is finite.

2.3.2 Proof of Theorem 2

we proceed in two steps: First we establish the basis for the induction. Then, we complete the proof by establishing the induction step.

Basis for induction:

From Eqs. (5), (7), and (9) we have $V_1(x_1,x_2) = \min_g(\frac{1}{R})$. In this case, policy g^* performs as well as any other policy at the first stage, so $(\mathcal{H}0)_1$ is valid. As a result, $V_1(x_1,x_2) = \frac{1}{R}$ for every $x_1 \geq 0,\ x_2 \geq 0$; hence $\mathcal{D}_1(x_1,x_2) = \mathcal{F}_1(x_1,x_2) = \mathcal{G}_1(x_1,x_2) = 0$, which proves the validity of $(\mathcal{H}1)_1$-$(\mathcal{H}3)_1$. The basis for the induction argument is now complete.

Induction Step:

As stated in Section 2.3.1, we assume that $(\mathcal{H}0)_n$-$(\mathcal{H}3)_n$ are valid and proceed to show that $(\mathcal{H}0)_{n+1}$-$(\mathcal{H}3)_{n+1}$ are also true via the proof of Lemmas 2-5. For the proof of Lemmas 2-5 we first establish the following facts,

Fact. 1 *For every $n \geq 1, x_1 \geq 0, x_2 \geq 0$, the following relations hold.*

$$V_n(x_1+1, x_2) \geq V_n(x_1, x_2), \tag{15}$$
$$V_n(x_1, x_2+1) \geq V_n(x_1, x_2). \tag{16}$$

Proof. As it was discussed before $V_n(x_1, x_2)$ can be defined as the minimum of the expected value of the stopping time τ_n in the n-stage problem. For the n-stage problem, under any specific policy, and along every sample path of service completions, job interconnection, and job arrivals, the stopping time corresponding to the initial state (x_1, x_2) is no longer than than the stopping time corresponding to the initial state (x_1+1, x_2) or to the initial state (x_1, x_2+1). Therefore, the above inequalities hold.

□

Fact. 2 *For every $n \geq 1, x_1 \geq 0, x_2 \geq 0$, the following relations hold.*

$$V_n^1(x_1, x_2) \leq 0, \tag{17}$$
$$V_n^2(x_1, x_2) \leq 0. \tag{18}$$

Proof. In the n-stage problem, consider the effect of giving a small amount of processing time to a single job. Specifically, let $\delta\overline{V_n^1(x_1, x_2)}$ denote (to first order in δ) the amount by which the expected value of τ_n would change from $V_n(x_1, x_2)$ if we were to give a job in Q_1 an extra δ amount of preprocessing time. Then,

$$\begin{aligned}
\delta\overline{V_n^1(x_1, x_2)} &= \mu_1\delta\Big(pV_n(x_1-1, x_2+k) + (1-p)V_n(x_1-1, x_2)\Big) + \\
&\quad \lambda\delta V_n(x_1+1, x_2) + (1-\mu_1\delta-\lambda\delta)V_n(x_1, x_2) - V_n(x_1, x_2) \\
&= \mu_1\delta\Big(pV_n(x_1-1, x_2+k) + (1-p)V_n(x_1-1, x_2) - V_n(x_1, x_2)\Big) + \\
&\quad \lambda\delta\Big(V_n(x_1+1, x_2) - V_n(x_1, x_2)\Big) \\
&\geq \delta V_n^1(x_1, x_2)
\end{aligned}$$

where the inequality holds because of Eq. (15).

On the other hand, since the processing times are exponential we have

$$\delta\overline{V_n^1(x_1, x_2)} \leq 0, \text{ for all } x_1, x_2.$$

which results in 17.

Eq. (18) can be proved in a similar way.

□

Note that the proof of this fact gives us a very intuitive interpretation of the functions $V_n^1(x_1, x_2)$, $V_n^2(x_1, x_2)$, and $\mathcal{D}_n(x_1, x_2)$ for the n-stage problem. From the above discussion it is clear that $V_n^i(x-1, x_2)$ can be viewed as the change in the expected value of τ_n, in the absence of arrivals, as a result of giving a preprocessing to a job in Q_i. $\mathcal{D}_n(x_1, x_2)$ can be interpreted as the cost advantage of preprocessing a job in Q_2 over a job in Q_1, in the n-stage problem. Note that $\mathcal{D}_n(x_1, x_2)$ remains the same whether or not arrivals are included in the problem formulation. $(\mathcal{H}1)_n$ implies that it is better to give the preprocessing to Q_1, which can be shown to be equivalent to the optimality of policy g^*.

Fact. 3 *Under Condition 2, i.e.* $p \geq \frac{\mu_1-\mu_2}{\mu_1}$,

$$V_n^1(x_1, x_2-1) - V_n^2(x_1-1, x_2) \geq \mu_1 \mathcal{G}_n(x_1, x_2) \tag{19}$$

Proof.

$$\begin{aligned}
&V_n^1(x_1, x_2-1) - V_n^2(x_1-1, x_2) \\
&\quad = \mu_1[pV_n(x_1-1, x_2+k-1) + (1-p)V_n(x_1-1, x_2-1) - V_n(x_1, x_2-1)] - \\
&\quad\quad \mu_2[V_n(x_1-1, x_2-1) - V_n(x_1-1, x_2)] \\
&\quad \geq \mu_1[pV_n(x_1-1, x_2+k-1) + (1-p)V_n(x_1-1, x_2-1) - V_n(x_1, x_2-1)] - \\
&\quad\quad \mu_1(1-p)[V_n(x_1-1, x_2-1) - V_n(x_1-1, x_2)] \\
&\quad = \mu_1\Big\{p[V_n(x_1-1, x_2+k-1) - V_n(x_1-1, x_2)] + [V_n(x_1-1, x_2) - V_n(x_1, x_2-1)\Big\} \\
&\quad \geq \mu_1 \mathcal{G}_n(x_1, x_2)
\end{aligned}$$

□

The first inequality is a result of condition 2 and the second inequality follows from Eq. (16) and $k \geq 1$. Intuitively, $\mathcal{G}_n(x_1, x_2)$ can be interpreted as follows: Consider the n-stage problem where a new job is added to the system, and one has the option of assigning it to either Q_1 or Q_2. Then $\mathcal{G}_n(x_1, x_2)$ gives the cost advantage of assigning the extra job to Q_1 instead of Q_2. $(\mathcal{H}3)_n$ implies that at every stage n, it is better to assign the extra job to Q_1.

Fact. 4 *Under Condition 2, i.e.* $p \geq \frac{\mu_1-\mu_2}{\mu_1}$,

$$\begin{aligned}
&V_n^1(x_1, x_2-1) - pV_n^2(x_1-1, x_2+k-1) - (1-p)V_n^2(x_1-1, x_2) \\
&\quad \geq \mu_1\Big\{-(1-p)\mathcal{G}_n(x_1, x_2-1) - p\mathcal{G}_n(x_1, x_2) - (1-p)\mathcal{F}_n(x_1, x_2-1) + \\
&\quad\quad\quad (1-p)\mathcal{F}_n(x_1, x_2)\Big\}.
\end{aligned} \tag{20}$$

Proof.

$$\begin{aligned}
&V_n^1(x_1, x_2-1) - pV_n^2(x_1-1, x_2+k-1) - (1-p)V_n^2(x_1-1, x_2) \\
&\quad = \mu_1\Big[pV_n(x_1-1, x_2+k-1) + (1-p)V_n(x_1-1, x_2-1) - V(x_1-1, x_2)\Big)] - \\
&\quad\quad \mu_2\Big[pV_n(x_1-1, x_2+k-2) - pV_n(x_1-1, x_2+k-1) + \\
&\quad\quad (1-p)V_n(x_1-1, x_2-1) - (1-p)V_n(x_1-1, x_2)\Big] \\
&\quad \geq \mu_1\Big[pV_n(x_1-1, x_2+k-1) + (1-p)V_n(x_1-1, x_2-1) - V(x_1-1, x_2)\Big)] - \\
&\quad\quad \mu_1(1-p)\Big[pV_n(x_1-1, x_2+k-2) - pV_n(x_1-1, x_2+k-1) + \\
&\quad\quad (1-p)V_n(x_1-1, x_2-1) - (1-p)V_n(x_1-1, x_2)\Big] \\
&\quad = \mu_1\Big\{p[V_n(x_1-1, x_2+k-1) - V_n(x_1, x_2-1)] + (1-p)V_n(x_1-1, x_2-1) - \\
&\quad\quad (1-p)[V_n(x_1, x_2-1) - pV_n(x_1-1, x_2+k-1) - (1-p)V_n(x_1-1, x_2)] +
\end{aligned}$$

$$
\begin{aligned}
& (1-p)[-pV_n(x_1-1,x_2+k-2)-(1-p)V_n(x_1-1,x_2-1)]\Big\} \\
\geq\ & \mu_1\Big\{-p\mathcal{G}_n(x_1,x_2)+(1-p)V_n(x_1-1,x_2-1)+ \\
& (1-p)\mathcal{F}_n(x_1,x_2)-(1-p)[\mathcal{F}_n(x_1,x_2-1)+V_n(x_1,x_2-2)]\Big\} \\
=\ & \mu_1\Big\{-p\mathcal{G}_n(x_1,x_2)+(1-p)[V_n(x_1-1,x_2-1)-V_n(x_1,x_2-2)]+ \\
& (1-p)\mathcal{F}_n(x_1,x_2)-(1-p)\mathcal{F}_n(x_1,x_2-1)\Big\} \\
=\ & \mu_1\Big\{-p\mathcal{G}_n(x_1,x_2)-(1-p)\mathcal{G}_n(x_1,x_2-1)+ \\
& (1-p)\mathcal{F}_n(x_1,x_2)-(1-p)\mathcal{F}_n(x_1,x_2-1)\Big\}
\end{aligned}
$$

□

The first inequality is a result of condition 2 and the second inequality follows from Eq. (16) where $k \geq 1$, and Eq. (14).

Based on facts 1-4 we prove Lemmas 2-5 that are needed for the completion of the induction step‘.

Lemma 2 *Assume Conditions 1 and 2 hold. If $\mathcal{D}_n(x_1,x_2) \leq 0$ for all $x_1 \geq 0$, $x_2 \geq 0$, then g^* is optimal at stage $n+1$.*

Proof. By the definition of g^*, which gives priority to Q_1, we know that for any arbitrary policy g,

$g_1(x_1,x_2) \leq g_1^*(x_1,x_2)$ for all $x_1,\ x_2$.

Without loss of generality we have restricted out attention to policies for which

$g_1(x_1,x_2)+g_2(x_1,x_2)=g_1^*(x_1,x_2)+g_2^*(x_1,x_2)=m.$

If $x_1+x_2 \leq m$, g^*, by its definition, utilizes x_1+x_2 servers, therefore, it results in an optimal action. When $x_1+x_2 > m$, we prove that any policy g that $g_1(x_1,x_2) < g_1^*(x_1,x_2) = \min(m,x_1)$, can be imposed at stage $n+1$ by reallocating one server from a job in Q_2 to a job Q_1. Reception of the same argument shows that g^* is optimal at stage $n+1$. Therefore, to compute the proof, we compare, at stage $n+1$, the local cost functions due to policy g^I and g, where

$g_1^I(x_1,x_2)=g_1(x_1,x_2)+1 \quad$ and $\quad g_2^I(x_1,x_2)=g_2(x_1,x_2)-1.$

We have,

$$
\begin{aligned}
& c^{g^I}(x_1,x_2,V_n)-c^g(x_1,x_2,V_n) \\
&\quad = \frac{\mu_1}{R}\Big(pV_n(x_1-1,x_2+k)+(1-p)V_n(x_1-1,x_2)\Big)- \\
&\qquad \frac{\mu_2}{R}V_n(x_1,x_2-1)-\frac{\mu_1}{R}V_n(x_1,x_2)+\frac{\mu_2}{R}V_n(x_1,x_2) \\
&\quad = \frac{\mu_1}{R}\Big(pV_n(x_1-1,x_2+k)+(1-p)V_n(x_1-1,x_2)-V_n(x_1,x_2)\Big)- \\
&\qquad \frac{\mu_2}{R}\Big(V_n(x_1,x_2-1)-Vnn\Big) \\
&\quad = \frac{1}{R}\mathcal{D}_n(x_1,x_2) \leq 0.
\end{aligned} \tag{21}
$$

The inequality in Eq. (21) together with argument proceeding it completes the proof of Lemma 2.

$\square$

Lemma 3 *Assume that conditions 1 and 2 hold. If g^* is optimal at stage $n+1$ and $\mathcal{G}_n(x_1, x_2) \leq 0$ for all x_1, x_2, then $\mathcal{G}_{n+1}(x_1, x_2) \ \leq \ 0$.*

Proof.
In Appendix A, we show that
(**i**) If $x_1 > m$,

$$
\begin{aligned}
&\mathcal{G}_{n+1}(x_1, x_2) \\
&\quad = \frac{\mu_1 m}{R}[p\mathcal{G}_n(x_1-1, x_2+k) + (1-p)\mathcal{G}_n(x_1-1, x_2)] + \\
&\qquad \frac{\lambda}{R}\mathcal{G}_n(x_1+1, x_2) + \Big(1 - \frac{\mu_1 m + \lambda}{R}\Big)\mathcal{G}_n(x_1, x_2) \qquad (22)
\end{aligned}
$$

(**ii**) If $x_1 \leq m < x_1 + x_2$,

$$
\begin{aligned}
&\mathcal{G}_{n+1}(x_1, x_2) \\
&\quad \leq \frac{\mu_1(x_1-1)}{R}[p\mathcal{G}_n(x_1-1, x_2+k) + (1-p)\mathcal{G}_n(x_1-1, x_2)] + \\
&\qquad \frac{\mu_2(m-x_1)}{R}V_n(x_1, x_2-1) + \frac{\lambda}{R}\mathcal{G}_n(x_1+1, x_2) + \\
&\qquad \Big(1 - \frac{\mu_1 x_1 + \mu_2(m-x_1) + \lambda}{R}\Big)\mathcal{G}_n(x_1, x_2) \qquad (23)
\end{aligned}
$$

and
(**iii**) If $x_1 + x_2 \leq m$,

$$
\begin{aligned}
&\mathcal{G}_{n+1}(x_1, x_2) \\
&\quad \leq \frac{\mu_1(x_1-1)}{R}[p\mathcal{G}_n(x_1-1, x_2+k) + (1-p)\mathcal{G}_n(x_1-1, x_2)] + \\
&\qquad \frac{\mu_2(x_2-1)}{R}\mathcal{G}_n(x_1, x_2-1) + \frac{\lambda}{R}\mathcal{G}_n(x_1+1, x_2) + \\
&\qquad \Big(1 - \frac{\mu_1 x_1 + \mu_2(x_2-1) + \lambda}{R}\Big)\mathcal{G}_n(x_1, x_2) \qquad (24)
\end{aligned}
$$

Note that using Eq. (4) and the induction hypothesis, namely, $\mathcal{G}_n(x_1, x_2) \leq 0$ for all x_1, x_2, we can see that the left-hand-sides in Eqs. (22)-(24) are non-positive, i.e.

$$\mathcal{G}_{n+1}(x_1, x_2) \leq 0.$$

$\square$

Lemma 4 *Assume that conditions 1 and 2 hold. If g^* is optimal at stage $n+1$, $\mathcal{G}_n(x_1, x_2) \leq 0$, and $\mathcal{F}_n(x_1, x_2) \leq 0$ for all x_1, x_2, then $\mathcal{F}_{n+1}(x_1, x_2) \leq 0$.*

Proof.
In Appendix A, we show that
(i) If $x_1 > m$,

$$\begin{aligned}
&\mathcal{F}_{n+1}(x_1, x_2) \\
&\quad = \frac{\mu_1 m}{R}[p\mathcal{F}_n(x_1 - 1, x_2 + k) + (1-p)\mathcal{F}_n(x_1 - 1, x_2)] + \\
&\quad\quad \frac{\lambda}{R}\mathcal{F}_n(x_1 + 1, x_2) + \Big(1 - \frac{\mu_1 m + \lambda}{R}\Big)\mathcal{F}_n(x_1, x_2) \qquad (25)
\end{aligned}$$

(ii) If $x_1 \le m < x_1 + x_2$,

$$\begin{aligned}
&\mathcal{F}_{n+1}(x_1, x_2) \\
&\quad \le \frac{\mu_1(x_1 - 1)}{R}[p\mathcal{F}_n(x_1 - 1, x_2 + k) + (1-p)\mathcal{F}_n(x_1 - 1, x_2)] + \\
&\quad\quad \frac{\mu_2(m - x_1)}{R}\mathcal{F}_n(x_1, x_2 - 1) + \frac{\lambda}{R}\mathcal{F}_n(x_1 + 1, x_2) + \\
&\quad\quad \Big(1 - \frac{\mu_1(x_1 - 1) + \mu_2(m - x_1) + \lambda}{R}\Big)\mathcal{F}_n(x_1, x_2) + \\
&\quad\quad \frac{\mu_1}{R}\Big\{(1-p)\mathcal{G}_n(x_1, x_2 - 1) + p\mathcal{G}_n(x_1, x_2) + (1-p)\mathcal{F}_n(x_1, x_2 - 1)\Big\} \qquad (26)
\end{aligned}$$

(iii) If $x_1 + x_2 \le m$,

$$\begin{aligned}
&\mathcal{F}_{n+1}(x_1, x_2) \\
&\quad \le \frac{\mu_1(x_1 - 1)}{R}[p\mathcal{F}_n(x_1 - 1, x_2 + k) + (1-p)\mathcal{F}_n(x_1 - 1, x_2)] + \\
&\quad\quad \frac{\mu_2(x_2 - 1)}{R}\mathcal{F}_n(x_1, x_2 - 1) + \frac{\lambda}{R}\mathcal{F}_n(x_1 + 1, x_2) + \\
&\quad\quad \Big(1 - \frac{\mu_1(x_1 - 1) + \mu_2(x_2 - 1) + \lambda}{R}\Big)\mathcal{F}_n(x_1, x_2) + \\
&\quad\quad \frac{\mu_1}{R}\Big\{(1-p)\mathcal{G}_n(x_1, x_2 - 1) + p\mathcal{G}_n(x_1, x_2) + (1-p)\mathcal{F}_n(x_1, x_2 - 1)\Big\} \qquad (27)
\end{aligned}$$

Note that again the right-hand-sides in Eqs. (25)-(27) are non-positive, because of Eq. (4) and the assumption that $\mathcal{F}_n(x_1, x_2) \le 0$ for all x_1, x_2, i.e.

$$\mathcal{F}_{n+1}(x_1, x_2) \le 0.$$

$\square$

Lemma 5 *Assume that conditions 1 and 2 hold. If g^* is optimal at stage $n+1$, $\mathcal{F}_n(x_1, x_2) \le 0$, and $\mathcal{D}_n(x_1, x_2) \le 0$ for all x_1, x_2, then $\mathcal{D}_{n+1}(x_1, x_2) \le 0$.*

Proof. In Appendix A, we show that
(i) If $x_1 > m$,

$$
\begin{aligned}
&\mathcal{D}_{n+1}(x_1, x_2) \\
&\quad = \frac{\mu_1 m}{R}[p\mathcal{D}_n(x_1-1, x_2+k) + (1-p)\mathcal{D}_n(x_1-1, x_2)] + \\
&\quad\quad \frac{\lambda}{R}\mathcal{D}_n(x_1+1, x_2) + \Big(1 - \frac{\mu_1 m + \lambda}{R}\Big)\mathcal{D}_n(x_1, x_2)
\end{aligned} \tag{28}
$$

(ii) If $x_1 \leq m < x_1 + x_2$,

$$
\begin{aligned}
&\mathcal{D}_{n+1}(x_1, x_2) \\
&\quad = \frac{\mu_1(x_1-1)}{R}[p\mathcal{D}_n(x_1-1, x_2+k) + (1-p)\mathcal{D}_n(x_1-1, x_2)] + \\
&\quad\quad \frac{\mu_2(m-x_1)}{R}\mathcal{D}_n(x_1, x_2-1) + \frac{\lambda}{R}\mathcal{D}_n(x_1+1, x_2) + \\
&\quad\quad \Big(1 - \frac{\mu_1 x_1 + \mu_2(m-x_1) + \lambda}{R}\Big)\mathcal{D}_n(x_1, x_2)
\end{aligned} \tag{29}
$$

(iii) If $x_1 + x_2 \leq m$,

$$
\begin{aligned}
&\mathcal{D}_{n+1}(x_1, x_2) \\
&\quad \leq \frac{\mu_1(x_1-1)}{R}[p\mathcal{D}_n(x_1-1, x_2+k) + (1-p)\mathcal{D}_n(x_1-1, x_2)] + \\
&\quad\quad \frac{\mu_2(x_2-1)}{R}\mathcal{D}_n(x_1, x_2-1) + \frac{\lambda}{R}\mathcal{D}_n(x_1+1, x_2) + \\
&\quad\quad \Big(1 - \frac{\mu_1 x_1 + \mu_2 x_2 + \lambda}{R}\Big)\mathcal{D}_n(x_1, x_2) + \frac{\mu_1\mu_2}{R}\mathcal{F}_n(x_1, x_2)
\end{aligned} \tag{30}
$$

Because of Eq. (4) and the induction hypothesis on $\mathcal{D}_n(x_1, x_2)$ and $\mathcal{F}_n(x_1, x_2)$, namely, for all x_1, x_2 $\mathcal{D}_n(x_1, x_2) \leq 0$, $\mathcal{F}_n(x_1, x_2) \leq 0$, we know that the right-hand-side of the Eqs. (28)-(30) are non-positive. i.e. for all x_1, x_2,

$$\mathcal{D}_{n+1}(x_1, x_2) \leq 0.$$

□

The induction step is now complete as $(\mathcal{H}0)_{n+1}$-$(\mathcal{H}3)_{n+1}$ are true because of Lemmas 2,5,4, and 3, respectively. This concludes the proof of Theorem 2.

□

3 Discussion

In this section we first present an example that illustrates the role of the condition expressed by Eq. (1) in Theorem 1. The example shows that Eq. (1) is sufficient but not necessary to ensure the optimality of policy g^* that gives priority to Queue 1. Consequently, the result of Theorem 1 does not provide any information about the optimal allocation strategy for Problem (P) when Eq. (1) is not satisfied. We identify an instance where it is impossible to determine an optimal allocation strategy for Problem (P) when Eq. (1) is not satisfied. We discuss this instance after Example 3.1.

3.1 Example

Consider the system in Figure 1. Let the initial state be $(x_1, x_2) = (2, 2)$. Suppose there are three machine available, i.e. $m = 3$, and each interconnected job from Q_1 to Q_2 creates two jobs, i.e. $k = 2$. Let $\mu_1 = 2\mu_2$ and $\lambda = 0$. Hence the requirement expressed by Eq. (1) is

$$p \geq \frac{1}{2}. \tag{31}$$

Now as discussed before we only need to restrict attention to non-idling allocation strategies. Thus, when the state (x_1, x_2) is such that $x_1 + x_2 \leq m$ the allocation decision is trivial. For our example, there are only three states encountered for which the allocation is not trivial: when the system is at the initial state $(2, 2)$ and if the system enters states $(1, 4)$ or $(1, 3)$. We consider two non-idling priority allocation strategies: Policy g^* that gives priority to Q_1 and policy g that gives priority to Q_2. These two policies differ from each other in their allocation of servers if the system is in states $(2, 2)$, $(1, 4)$, and $(1, 3)$ (for other states both policies act in the same form).

Define the functions V^{g^*} and V^g as the expected makespan under policy g^* and policy g, respectively. At each state (x_1, x_2) the function V is defined as the minimum expected makespan. Then,

$$\begin{aligned}
V^{g^*}(0,1) &= V^g(0,1) = \frac{1}{\mu_2} = \frac{2}{\mu_1} \\
V^{g^*}(0,2) &= V^g(0,2) = \frac{3}{2\mu_2} = \frac{3}{\mu_1} \\
V^{g^*}(0,3) &= V^g(0,3) = \frac{11}{6\mu_2} = \frac{11}{3\mu_1} \\
V^{g^*}(0,4) &= V^g(0,4) = \frac{13}{6\mu_2} = \frac{13}{3\mu_1} \\
V^{g^*}(0,5) &= V^g(0,5) = \frac{15}{6\mu_2} = \frac{15}{3\mu_1} \\
V^{g^*}(1,0) &= V^g(1,0) = \frac{1}{\mu_1} + pV(0,2) = \frac{3p+1}{\mu_1} \\
V^{g^*}(1,1) &= V^g(1,1) = \frac{1}{\mu_1+\mu_2} + \frac{\mu_1}{\mu_1+\mu_2}[pV(0,3) + (1-p)V(0,1)] + \frac{\mu_2}{\mu_1+\mu_2}V(1,0) \\
&= \frac{19p+21}{9\mu_1} \\
V^{g^*}(1,2) &= V^g(1,2) = \frac{1}{\mu_1+2\mu_2} + \frac{\mu_1}{\mu_1+2\mu_2}[pV(0,4) + (1-p)V(0,2)] + \frac{2\mu_2}{\mu_1+2\mu_2}V(1,1) \\
&= \frac{31p+57}{18\mu_1}
\end{aligned}$$

At the states $(1, 3)$, $(1, 4)$ and $(2, 2)$ the two strategies g^* and g allocate servers differently. Thus we have

$$\begin{aligned}
V^{g^*}(1,3) &= \frac{1}{\mu_1+2\mu_2} + \frac{\mu_1}{\mu_1+2\mu_2}[pV(0,5) + (1-p)V(0,3)] + \frac{2\mu_2}{\mu_1+2\mu_2}V(1,2) \\
&= \frac{55p+141}{36\mu_1}.
\end{aligned} \tag{32}$$

$$
\begin{aligned}
V^{g}(1,3) &= \frac{1}{3\mu 2} + V(1,2) = \frac{31p+69}{36\mu_2} && (33)\\
V^{g^*}(1,4) &= \frac{1}{\mu_1+2\mu_2} + \frac{\mu_1}{\mu_1+2\mu_2}[pV(0,6)+(1-p)V(0,4)] + \frac{2\mu_2}{\mu_1+2\mu_2}V(1,3) && (34)\\
V^{g}(1,4) &= \frac{1}{3\mu 2} + V(1,3) = \frac{31p+69}{36\mu_2} && (35)\\
V(2,1) &= \frac{1}{2\mu_1+\mu_2} + \frac{2\mu_1}{2\mu_1+\mu_2}[pV(1,3)+(1-p)V(1,1)] + \frac{\mu_2}{2\mu_1+\mu_2}V(1,2). && (36)\\
V^{g^*}(2,2) &= \frac{1}{2\mu_1+\mu_2} + \frac{2\mu_1}{2\mu_1+\mu_2}[pV(1,4)+(1-p)V(1,2)] + \frac{\mu_2}{2\mu_1+\mu_2}V(2,1) && (37)\\
V^{g}(2,2) &= \frac{1}{\mu_1+2\mu_2} + \frac{\mu_1}{\mu_1+2\mu_2}[pV(1,4)+(1-p)V(1,2)] + \frac{2\mu_2}{\mu_1+2\mu_2}V(2,1) && (38)
\end{aligned}
$$

where as mentioned before at each state (x_1, x_2), $V(x_1, x_2)$ is defined as

$$V(x_1, x_2) = \min\{V^{g^*}(x_1, x_2), V^{g}(x_1, x_2)\}.$$

The optimal policy in this case can be easily determined by policy iteration. In order to do so, we first look at the expected makespan under strategies g and g^* if the system is at either states $(1,3)$ or $(1,4)$. comparing the expected makespan associated with each of these strategies, we identify the least expected makespan for different values of p, i.e.

$$V(1,3) = \begin{cases} V^{g^*}(1,3) & \text{if } p \geq 0.4285 \\ V^{g}(1,3) & \text{otherwise} \end{cases} \tag{39}$$

and

$$V(1,4) = \begin{cases} V^{g^*}(1,4) & \text{if } p \geq 0.4285 \\ V^{g}(1,4) & \text{otherwise} \end{cases} \tag{40}$$

Now we first assume $p \geq 0.4285$, using Eqs. (39) and (40) it can be seen that

$$V(2,2) = \quad V^{g^*}(2,2) \quad \text{if } p \geq 0.4285 \ . \tag{41}$$

Then we assume $p \leq 0.4285$, using Eqs. (39) and (40) we have

$$V(2,2) = \begin{cases} V^{g}(2,2) & \text{if } p \leq 0.4267 \\ V^{g^*}(2,2) & \text{if } 0.4267 \leq p \leq 0.4267 \end{cases} \tag{42}$$

In summary, we can identify three regions for value of parameter p as follows:

1. $p \in [0, 0.4267]$: In this region, it is always optimal to follow policy g
2. $p \in [0.4267, 0.4285]$: In this region, the optimal policy depends on the state; at states $(1,3)$ and $(1,4)$ it is optimal to follow g, while at $(2,2)$ the optimal allocation coincides with policy g^*
3. $p \in [0.4285, 1]$: In this region, policy g^* is optimal

Comparing (31) with the above result, we conclude that : (i) If the requirement expressed by Eq. (1) is not satisfied the policy g^* is not, in general, optimal. (ii) The requirement expressed by Eq. (1) is only sufficient but not necessary to guarantee the optimality of policy g^*. (iii) The optimal policy need not be a priority policy, and an optimal allocation can be dependent on the state of the system.

3.2 Special Case, $k = 1$

As mentioned at the beginning of Section 3, if Eq. (1) is not satisfied, the result of this paper provides no conclusive evidence about the form of an optimal allocation policy. However, it is possible to determine an optimal allocation strategy for Problem (P) when Eq. (1) is not true and $k = 1$. Such a strategy is desired in the following theorem

Theorem 3 *Assume* $\frac{m\mu_1\mu_2}{\lambda(\mu_2+k\mu_1)} > 1$ *and* $p \leq \frac{\mu_1-\mu_2}{\mu_1}$. *Under these conditions, Policy* $\tilde{g}$ *that gives priority to* Q_2, *is an optimal policy for Problem (P) when* $k = 1$.

Theorem 3 can be proved by arguments similar to those used to prove Theorem 1. These arguments are presented in Appendix B.

Theorem 1 and 3 shows that it is possible to determine an optimal allocation policy for any combination of μ_1, μ_2, and p when each job interconnecting from Q_1 to Q_2 creates exactly one job in Q_2.

4 Conclusion

The main result of the paper provides a condition sufficient to guarantee the optimality of the policy that gives priority to Q_1 for Problem (P). In this case where every task interconnecting from Q_1 to Q_2 creates exactly one job in Q_2, i.e. $k = 1$, we have determined an optimal allocation policy for any combination of values of μ_1, μ_2, and p.

Determining an optimal allocation policy for any combination of values of μ_1, μ_2, and p when $k > 1$ is a problem worthy of investigation. Another situation similar to Problem (P) that is practically significant arises when the number of processors required to process a job at a certain queue depends on the queue. Determining an optimal allocation strategy for such a class of problems appears to be a challenging task. The determination of an optimal allocation strategy that stochastically minimizes makespan for each of the above problems is also an interesting problem worthy of investigation.

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Appendices

A Proof of Eqs. (22)-(30)

To prove Eqs. (22)-(30), we incorpate the assumption that g^* is optimal at stage $n+1$, which translates to

$$V_{n+1}(x_1, x_2) = c^{g^*}(x_1, x_2, V_n). \tag{43}$$

A.1 Proof of Equation (22)

If $x_1 > m$,

$$\begin{array}{ll} g_1^*(x_1-1, x_2) = m & g_2^*(x_1-1, x_2) = 0 \\ g_1^*(x_1, x_2-1) = m & g_2^*(x_1, x_2-1) = 0. \end{array} \tag{44}$$

So we have

$$\begin{aligned}
&\mathcal{G}_{n+1}(x_1, x_2) \\
&\quad = V_{n+1}(x_1-1, x_2) - V_{n+1}(x_1, x_2-1) \\
&\quad = \frac{\mu_1 m}{R}[pV_n(x_1-2, x_2+k) + (1-p)V_n(x_1-2, x_2)] + \\
&\qquad \frac{\lambda}{R}V_n(x_1, x_2) + \left(1 - \frac{\mu_1 m + \lambda}{R}\right)V_n(x_1-1, x_2) - \\
&\qquad \left\{\frac{\mu_1 m}{R}[pV_n(x_1-1, x_2+k-1) + (1-p)V_n(x_1-1, x_2-1)] + \right. \\
&\qquad \left. \frac{\lambda}{R}V_n(x_1+1, x_2-1) + \left(1 - \frac{\mu_1 m + \lambda}{R}\right)V_n(x_1, x_2-1)\right\} \\
&\quad = \frac{\mu_1 m}{R}[p\mathcal{G}_n(x_1-1, x_2+k) + (1-p)\mathcal{G}_n(x_1-1, x_2)] + \\
&\qquad \frac{\lambda}{R}\mathcal{G}_n(x_1+1, x_2) + \left(1 - \frac{\mu_1 m + \lambda}{R}\right)\mathcal{G}_n(x_1, x_2)
\end{aligned}$$

The first and third equalities follow from Eq. (14). The second equality follows from Eqs. (44), (43), and (5).

□

A.2 Proof of Equation (23)

If $x_1 \leq m < x_1 + x_2$,

$$\begin{array}{ll} g_1^*(x_1-1, x_2) = x_1 - 1 & g_2^*(x_1-1, x_2) = m - x_1 + 1 \\ g_1^*(x_1, x_2-1) = x_1 & g_2^*(x_1, x_2-1) = m - x_1. \end{array} \tag{45}$$

Then

$$
\begin{aligned}
&\mathcal{G}_{n+1}(x_1, x_2) \\
&\quad = V_{n+1}(x_1-1, x_2) - V_{n+1}(x_1, x_2-1) \\
&\quad = \frac{\mu_1(x_1-1)}{R}[pV_n(x_1-2, x_2+k) + (1-p)V_n(x_1-2, x_2)] + \\
&\qquad \frac{\mu_2(m-x_1+1)}{R}V_n(x_1-1, x_2-1) + \frac{\lambda}{R}V_n(x_1, x_2) + \\
&\qquad \Big(1 - \frac{\mu_1(x_1-1)+\mu_2(m-x_1+1)+\lambda}{R}\Big)V_n(x_1-1, x_2) - \\
&\qquad \Big\{\frac{\mu_1 x_1}{R}[pV_n(x_1-1, x_2+k-1) + (1-p)V_n(x_1-1, x_2-1)] + \\
&\qquad \frac{\mu_2(m-x_1)}{R}V_n(x_1, x_2-2) + \frac{\lambda}{R}V_n(x_1, x_2) + \\
&\qquad \Big(1 - \frac{\mu_1 x_1+\mu_2(m-x_1)+\lambda}{R}\Big)V_n(x_1, x_2-1)\Big\} \\
&\quad = \frac{\mu_1(x_1-1)}{R}[p\mathcal{G}_n(x_1-1, x_2+k) + (1-p)\mathcal{G}_n(x_1-1, x_2)] + \\
&\qquad \frac{\mu_2(m-x_1)}{R}V_n(x_1, x_2-1) + \frac{\lambda}{R}\mathcal{G}_n(x_1+1, x_2) + \\
&\qquad \Big(1 - \frac{\mu_1(x_1-1)+\mu_2(m-x_1)+\lambda}{R}\Big)\mathcal{G}_n(x_1, x_2) - \\
&\qquad \frac{\mu_1}{R}[pV_n(x_1-1, x_2+k-1) + (1-p)V_n(x_1-1, x_2-1) - V_n(x_1, x_2-1)] + \\
&\qquad \frac{\mu_2}{R}[V_n(x_1-1, x_2-1) - V_n(x_1-1, x_2)] \\
&\quad = \frac{\mu_1 x_1 - 1}{R}[p\mathcal{G}_n(x_1-1, x_2+k) + (1-p)\mathcal{G}_n(x_1-1, x_2)] + \\
&\qquad \frac{\mu_2(m-x_1)}{R}V_n(x_1, x_2-1) + \frac{\lambda}{R}\mathcal{G}_n(x_1+1, x_2) + \\
&\qquad \Big(1 - \frac{\mu_1(x_1-1)+\mu_2(m-x_1)+\lambda}{R}\Big)\mathcal{G}_n(x_1, x_2) - \\
&\qquad \frac{1}{R}V_n^1(x_1, x_2-1) + \frac{1}{R}V_n^2(x_1-1, x_2) \\
&\quad \leq \frac{\mu_1(x_1-1)}{R}[p\mathcal{G}_n(x_1-1, x_2+k) + (1-p)\mathcal{G}_n(x_1-1, x_2)] + \\
&\qquad \frac{\mu_2(m-x_1)}{R}V_n(x_1, x_2-1) + \frac{\lambda}{R}\mathcal{G}_n(x_1+1, x_2) + \\
&\qquad \Big(1 - \frac{\mu_1(x_1-1)+\mu_2(m-x_1)+\lambda}{R}\Big)\mathcal{G}_n(x_1, x_2) - \\
&\qquad \frac{\mu_1}{R}\mathcal{G}_n(x_1, x_2) \\
&\quad = \frac{\mu_1(x_1-1)}{R}[p\mathcal{G}_n(x_1-1, x_2+k) + (1-p)\mathcal{G}_n(x_1-1, x_2)] +
\end{aligned}
$$

$$\frac{\mu_2(m-x_1)}{R}V_n(x_1,x_2-1)+\frac{\lambda}{R}\mathcal{G}_n(x_1+1,x_2)+$$
$$\Big(1-\frac{\mu_1 x_1+\mu_2(m-x_1)+\lambda}{R}\Big)\mathcal{G}_n(x_1,x_2)$$

The first and third equalities follow from the definition of $\mathcal{G}_{n+1}(x_1,x_2)$ and $\mathcal{G}_n(x_1,x_2)$. The second equality is a result of Eqs. (45), (43), and (5). The fourth equality follows from the definitions (10) and (11). The inequality holds becasue of Fact 3 (Eq. (19)).

□

A.3 Proof of Equation (24)

If $x_1+x_2\le m$,

$$\begin{aligned} g_1^*(x_1-1,x_2) &= x_1-1 \qquad & g_2^*(x_1-1,x_2) &= x_2 \\ g_1^*(x_1,x_2-1) &= x_1 \qquad & g_2^*(x_1,x_2-1) &= x_2-1. \end{aligned} \tag{46}$$

Then

$$\begin{aligned}
&\mathcal{G}_{n+1}(x_1,x_2)\\
&= V_{n+1}(x_1-1,x_2)-V_{n+1}(x_1,x_2-1)\\
&= \frac{\mu_1(x_1-1)}{R}[pV_n(x_1-2,x_2+k)+(1-p)V_n(x_1-2,x_2)]+\\
&\quad \frac{\mu_2 x_2}{R}V_n(x_1-1,x_2-1)+\frac{\lambda}{R}V_n(x_1,x_2)+\\
&\quad \Big(1-\frac{\mu_1(x_1-1)+\mu_2x_2+\lambda}{R}\Big)V_n(x_1-1,x_2)-\\
&\quad \Big\{\frac{\mu_1x_1}{R}[pV_n(x_1-1,x_2+k-1)+(1-p)V_n(x_1-1,x_2-1)]+\\
&\quad \frac{\mu_2(x_2-1)}{R}V_n(x_1,x_2-2)+\frac{\lambda}{R}V_n(x_1,x_2)+\\
&\quad \Big(1-\frac{\mu_1x_1+\mu_2(x_2-1)+\lambda}{R}\Big)V_n(x_1,x_2-1)\Big\}\\
&= \frac{\mu_1(x_1-1)}{R}[p\mathcal{G}_n(x_1-1,x_2+k)+(1-p)\mathcal{G}_n(x_1-1,x_2)]+\\
&\quad \frac{\mu_2(x_2-1)}{R}\mathcal{G}_n(x_1,x_2-1)+\frac{\lambda}{R}\mathcal{G}_n(x_1+1,x_2)+\\
&\quad \Big(1-\frac{\mu_1(x_1-1)+\mu_2(x_2-1)+\lambda}{R}\Big)\mathcal{G}_n(x_1,x_2)-\\
&\quad \frac{\mu_1}{R}[pV_n(x_1-1,x_2+k-1)+(1-p)V_n(x_1-1,x_2-1)-V_n(x_1,x_2-1)]+\\
&\quad \frac{\mu_2}{R}[V_n(x_1-1,x_2-1)-V_n(x_1-1,x_2)]\\
&= \frac{\mu_1(x_1-1)}{R}[p\mathcal{G}_n(x_1-1,x_2+k)+(1-p)\mathcal{G}_n(x_1-1,x_2)]+\\
&\quad \frac{\mu_2(x_2-1)}{R}\mathcal{G}_n(x_1,x_2-1)+\frac{\lambda}{R}\mathcal{G}_n(x_1+1,x_2)+
\end{aligned}$$

$$
\begin{aligned}
& \Big(1 - \frac{\mu_1(x_1-1)+\mu_2(x_2-1)+\lambda}{R}\Big)\mathcal{G}_n(x_1,x_2) - \\
& \frac{1}{R}V_n^1(x_1,x_2-1) + \frac{1}{R}V_n^2(x_1-1,x_2) \\
\leq\ & \frac{\mu_1(x_1-1)}{R}[p\mathcal{G}_n(x_1-1,x_2+k)+(1-p)\mathcal{G}_n(x_1-1,x_2)] + \\
& \frac{\mu_2(x_2-1)}{R}\mathcal{G}_n(x_1,x_2-1) + \frac{\lambda}{R}\mathcal{G}_n(x_1+1,x_2) + \\
& \Big(1 - \frac{\mu_1(x_1-1)+\mu_2(x_2-1)+\lambda}{R}\Big)\mathcal{G}_n(x_1,x_2) - \\
& \frac{\mu_1}{R}\mathcal{G}_n(x_1,x_2) \\
=\ & \frac{\mu_1(x_1-1)}{R}[p\mathcal{G}_n(x_1-1,x_2+k)+(1-p)\mathcal{G}_n(x_1-1,x_2)] + \\
& \frac{\mu_2(x_2-1)}{R}\mathcal{G}_n(x_1,x_2-1) + \frac{\lambda}{R}\mathcal{G}_n(x_1+1,x_2) + \\
& \Big(1 - \frac{\mu_1 x_1+\mu_2(x_2-1)+\lambda}{R}\Big)\mathcal{G}_n(x_1,x_2)
\end{aligned}
$$

Again, the first and third equalities follow from the definition of $\mathcal{G}_{n+1}(x_1,x_2)$ and $\mathcal{G}_n(x_1,x_2)$. The second equality is a result of Eqs. (46), (43), and (5). The fourth equality follows from the definitions (10) and (11). The inequality holds becasue of Fact 3 (Eq. (19)).

□

A.4 Proof of Equation (25)

If $x_1 > m$,

$$
\begin{aligned}
g_1^*(x_1-1,x_2) &= m & g_2^*(x_1-1,x_2) &= 0 \\
g_1^*(x_1-1,x_2+k-1) &= m & g_2^*(x_1-1,x_2+k-1) &= 0 \\
g_1^*(x_1,x_2-1) &= m & g_2^*(x_1,x_2-1) &= 0.
\end{aligned}
\tag{47}
$$

Using Eqs. (43), (47), (13) and the induction hypothesis on $\mathcal{F}_n(x_1,x_2)$ we obtain

$$
\begin{aligned}
& \mathcal{F}_{n+1}(x_1,x_2) \\
&\quad = pV_{n+1}(x_1-1,x_2+k-1)+(1-p)V_{n+1}(x_1-1,x_2) - V_{n+1}(x_1,x_2-1) \\
&\quad = p\Big\{\frac{\mu_1 m}{R}[pV_n(x_1-2,x_2+2k-1)+(1-p)V_n(x_1-2,x_2+k-1)] + \\
&\qquad \frac{\lambda}{R}V_n(x_1,x_2+k-1) + \Big(1-\frac{\mu_1 m+\lambda}{R}\Big)V_n(x_1-1,x_2+k-1)\Big\} + \\
&\qquad (1-p)\Big\{\frac{\mu_1 m}{R}[pV_n(x_1-2,x_2+k)+(1-p)V_n(x_1-2,x_2)] + \\
&\qquad \frac{\lambda}{R}V_n(x_1,x_2) + \Big(1-\frac{\mu_1 m+\lambda}{R}\Big)V_n(x_1-1,x_2)\Big\} - \\
&\qquad \Big\{\frac{\mu_1 m}{R}[pV_n(x_1-1,x_2+k-1)+(1-p)V_n(x_1-1,x_2-1)] +
\end{aligned}
$$

$$\frac{\lambda}{R}V_n(x_1+1,x_2-1)+\Big(1-\frac{\mu_1 m+\lambda}{R}\Big)V_n(x_1,x_2-1)\Big\}$$
$$= \frac{\mu_1 m}{R}[p\mathcal{F}_n(x_1-1,x_2+k)+(1-p)\mathcal{F}_n(x_1-1,x_2)]+$$
$$\frac{\lambda}{R}\mathcal{F}_n(x_1+1,x_2)+\Big(1-\frac{\mu_1 m+\lambda}{R}\Big)\mathcal{F}_n(x_1,x_2)$$

The first and third equalities follow from Eq. (13). The second equality follows from Eqs. (47), (43), and (5).

□

A.5 Proof of Equation (26)

If $x_1 \le m < x_1+x_2$,

$$\begin{aligned}
g_1^*(x_1-1,x_2) &= x_1-1 & g_2^*(x_1-1,x_2) &= m-x_1+1\\
g_1^*(x_1-1,x_2+k-1) &= x_1-1 & g_2^*(x_1-1,x_2+k-1) &= m-x_1+1\\
g_1^*(x_1,x_2-1) &= x_1 & g_2^*(x_1,x_2-1) &= m-x_1.
\end{aligned} \tag{48}$$

Then

$$\begin{aligned}
&\mathcal{F}_{n+1}(x_1,x_2)\\
&= pV_{n+1}(x_1-1,x_2+k-1)+(1-p)V_{n+1}(x_1-1,x_2)-V_{n+1}(x_1,x_2-1)\\
&= p\Big\{\frac{\mu_1(x_1-1)}{R}[pV_n(x_1-2,x_2+k)+(1-p)V_n(x_1-2,x_2)]+\\
&\quad \frac{\mu_2(m-x_1+1)}{R}V_n(x_1-1,x_2-1)+\frac{\lambda}{R}V_n(x_1,x_2)+\\
&\quad \Big(1-\frac{\mu_1(x_1-1)+\mu_2(m-x_1+1)+\lambda}{R}\Big)V_n(x_1-1,x_2)\Big\}+\\
&\quad (1-p)\Big\{\frac{\mu_1(x_1-1)}{R}[pV_n(x_1-2,x_2+k)+(1-p)V_n(x_1-2,x_2)]+\\
&\quad \frac{\mu_2(m-x_1+1)}{R}V_n(x_1-1,x_2-1)+\frac{\lambda}{R}V_n(x_1,x_2)+\\
&\quad \Big(1-\frac{\mu_1(x_1-1)+\mu_2(m-x_1+1)+\lambda}{R}\Big)V_n(x_1-1,x_2)\Big\}-\\
&\quad \Big\{\frac{\mu_1 x_1}{R}[pV_n(x_1-1,x_2+k-1)+(1-p)V_n(x_1-1,x_2-1)]+\\
&\quad \frac{\mu_2(m-x_1)}{R}V_n(x_1,x_2-2)+\frac{\lambda}{R}V_n(x_1,x_2)+\\
&\quad \Big(1-\frac{\mu_1 x_1+\mu_2(m-x_1)+\lambda}{R}\Big)V_n(x_1,x_2-1)\Big\}\\
&= \frac{\mu_1(x_1-1)}{R}[p\mathcal{F}_n(x_1-1,x_2+k)+(1-p)\mathcal{F}_n(x_1-1,x_2)]+\\
&\quad \frac{\mu_2(m-x_1)}{R}\mathcal{F}_n(x_1,x_2-1)+\frac{\lambda}{R}\mathcal{F}_n(x_1+1,x_2)+
\end{aligned}$$

$$
\begin{aligned}
& \Big(1 - \frac{\mu_1(x_1-1)+\mu_2(m-x_1)+\lambda}{R}\Big)\mathcal{F}_n(x_1,x_2) - \\
& \frac{\mu_1}{R}[p\mathcal{F}_n(x_1-1,x_2+k-1)+(1-p)V_n(x_1-1,x_2-1)-V_n(x_1,x_2-1)] + \\
& \frac{\mu_2}{R}p[V_n(x_1-1,x_2+k-2)-V_n(x_1-1,x_2+k-1)] + \\
& \frac{\mu_2}{R}(1-p)[V_n(x_1-1,x_2-1)-V_n(x_1-1,x_2)] \\
= \; & \frac{\mu_1(x_1-1)}{R}[p\mathcal{F}_n(x_1-1,x_2+k)+(1-p)\mathcal{F}_n(x_1-1,x_2)] + \\
& \frac{\mu_2(m-x_1)}{R}\mathcal{F}_n(x_1,x_2-1)+\frac{\lambda}{R}\mathcal{F}_n(x_1+1,x_2) + \\
& \Big(1 - \frac{\mu_1(x_1-1)+\mu_2(m-x_1)+\lambda}{R}\Big)\mathcal{F}_n(x_1,x_2) - \\
& \frac{1}{R}V_n^1(x_1,x_2-1)+\frac{p}{R}V_n^2(x_1-1,x_2+k-1)+\frac{1-p}{R}V_n^2(x_1-1,x_2) \\
\leq \; & \frac{\mu_1(x_1-1)}{R}[p\mathcal{F}_n(x_1-1,x_2+k)+(1-p)\mathcal{F}_n(x_1-1,x_2)] + \\
& \frac{\mu_2(m-x_1)}{R}\mathcal{F}_n(x_1,x_2-1)+\frac{\lambda}{R}\mathcal{F}_n(x_1+1,x_2) + \\
& \Big(1 - \frac{\mu_1(x_1-1)+\mu_2(m-x_1)+\lambda}{R}\Big)\mathcal{F}_n(x_1,x_2) + \\
& \frac{\mu_1}{R}\Big\{(1-p)\mathcal{G}_n(x_1,x_2-1)+p\mathcal{G}_n(x_1,x_2)+(1-p)\mathcal{F}_n(x_1,x_2-1) - \\
& (1-p)\mathcal{F}_n(x_1,x_2)\Big\} \\
= \; & \frac{\mu_1(x_1-1)}{R}[p\mathcal{F}_n(x_1-1,x_2+k)+(1-p)\mathcal{F}_n(x_1-1,x_2)] + \\
& \frac{\mu_2(m-x_1)}{R}\mathcal{F}_n(x_1,x_2-1)+\frac{\lambda}{R}\mathcal{F}_n(x_1+1,x_2) + \\
& \Big(1 - \frac{\mu_1(x_1-1)+\mu_2(m-x_1)+\lambda}{R}\Big)\mathcal{F}_n(x_1,x_2) - (1-p)\frac{\mu_1}{R}\mathcal{F}_n(x_1,x_2) + \\
& \frac{\mu_1}{R}\Big\{(1-p)\mathcal{G}_n(x_1,x_2-1)+p\mathcal{G}_n(x_1,x_2)+(1-p)\mathcal{F}_n(x_1,x_2-1)\Big\} \\
= \; & \frac{\mu_1(x_1-1)}{R}[p\mathcal{F}_n(x_1-1,x_2+k)+(1-p)\mathcal{F}_n(x_1-1,x_2)] + \\
& \frac{\mu_2(m-x_1)}{R}\mathcal{F}_n(x_1,x_2-1)+\frac{\lambda}{R}\mathcal{F}_n(x_1+1,x_2) + \\
& \Big(1 - \frac{\mu_1(x_1-1)+\mu_2(m-x_1)+\lambda}{R}\Big)\mathcal{F}_n(x_1,x_2) + \\
& \frac{\mu_1}{R}\Big\{(1-p)\mathcal{G}_n(x_1,x_2-1)+p\mathcal{G}_n(x_1,x_2)+(1-p)\mathcal{F}_n(x_1,x_2-1)\Big\} \qquad (49)
\end{aligned}
$$

The first and third equalities follow from Eq. (13). The second equality follows from Eqs. (48), (43), and (5). The fourth equality follows from the definitions (10) and (11). The last two equalitites are simple grouping the terms. The inequality in Eq. (49) is a subsequent of Eq. (20) in fact 4.

A.6 Proof of Equation (27)

If $x_1 + x_2 \leq m$,

$$\begin{aligned}
g_1^*(x_1-1,x_2) &= x_1 - 1 & g_2^*(x_1-1,x_2) &= x_2 \\
g_1^*(x_1-1,x_2+k-1) &= x_1 - 1 & g_2^*(x_1-1,x_2+k-1) &= x_2 + N \\
g_1^*(x_1,x_2-1) &= x_1 & g_2^*(x_1,x_2-1) &= x_2 - 1.
\end{aligned} \tag{50}$$

where

$$N = \min(k-1, m-x_1-x_2+1). \tag{51}$$

Thus

$$\begin{aligned}
&\mathcal{F}_{n+1}(x_1,x_2) \\
&\quad= pV_{n+1}(x_1-1,x_2+k-1) + (1-p)V_{n+1}(x_1-1,x_2) - V_{n+1}(x_1,x_2-1) \\
&\quad= p\Big\{\frac{\mu_1(x_1-1)}{R}[pV_n(x_1-2,x_2+k) + (1-p)V_n(x_1-2,x_2)] + \\
&\qquad \frac{\mu_2(x_2+n)}{R}V_n(x_1-1,x_2-1) + \frac{\lambda}{R}V_n(x_1,x_2) + \\
&\qquad \Big(1 - \frac{\mu_1(x_1-1)+\mu_2(x_2+n)+\lambda}{R}\Big)V_n(x_1-1,x_2)\Big\} + \\
&\qquad (1-p)\Big\{\frac{\mu_1(x_1-1)}{R}[pV_n(x_1-2,x_2+k) + (1-p)V_n(x_1-2,x_2)] + \\
&\qquad \frac{\mu_2(x_2)}{R}V_n(x_1-1,x_2-1) + \frac{\lambda}{R}V_n(x_1,x_2) + \\
&\qquad \Big(1 - \frac{\mu_1(x_1-1)+\mu_2 x_2+\lambda}{R}\Big)V_n(x_1-1,x_2)\Big\} - \\
&\qquad \Big\{\frac{\mu_1 x_1}{R}[pV_n(x_1-1,x_2+k-1) + (1-p)V_n(x_1-1,x_2-1)] + \\
&\qquad \frac{\mu_2(x_2-1)}{R}V_n(x_1,x_2-2) + \frac{\lambda}{R}V_n(x_1,x_2) + \\
&\qquad \Big(1 - \frac{\mu_1 x_1+\mu_2(x_2-1)+\lambda}{R}\Big)V_n(x_1,x_2-1)\Big\} \\
&\quad= \frac{\mu_1(x_1-1)}{R}[p\mathcal{F}_n(x_1-1,x_2+k) + (1-p)\mathcal{F}_n(x_1-1,x_2)] + \\
&\qquad \frac{\mu_2(x_2-1)}{R}\mathcal{F}_n(x_1,x_2-1) + \frac{\lambda}{R}\mathcal{F}_n(x_1+1,x_2) + \\
&\qquad \Big(1 - \frac{\mu_1(x_1-1)+\mu_2(x_2-1)+\lambda}{R}\Big)\mathcal{F}_n(x_1,x_2) - \\
&\qquad \frac{\mu_1}{R}[pV_n(x_1-1,x_2+k-1) + (1-p)V_n(x_1-1,x_2-1) - V_n(x_1,x_2-1)] + \\
&\qquad \frac{\mu_2}{R}p[V_n(x_1-1,x_2+k-2) - V_n(x_1-1,x_2+k-1)] + \\
&\qquad \frac{\mu_2}{R}(1-p)[V_n(x_1-1,x_2-1) - V_n(x_1-1,x_2)] + \\
&\qquad N\frac{\mu_2}{R}[V_n(x_1-1,x_2+k-2) - V_n(x_1-1,x_2+k-1)]
\end{aligned}$$

$$
\begin{aligned}
= \; & \frac{\mu_1(x_1-1)}{R}[p\mathcal{F}_n(x_1-1,x_2+k)+(1-p)\mathcal{F}_n(x_1-1,x_2)]+ \\
& \frac{\mu_2(x_2-1)}{R}\mathcal{F}_n(x_1,x_2-1)+\frac{\lambda}{R}\mathcal{F}_n(x_1+1,x_2)+ \\
& \Big(1-\frac{\mu_1(x_1-1)+\mu_2(x_2-1)+\lambda}{R}\Big)\mathcal{F}_n(x_1,x_2)- \\
& \frac{1}{R}V_n^1(x_1,x_2-1)+\frac{p}{R}V_n^2(x_1-1,x_2+k-1)+\frac{1-p}{R}V_n^2(x_1-1,x_2) \\
& N\frac{\mu_2}{R}V_n^2(x_1-1,x_2+k-1) \\
\leq \; & \frac{\mu_1(x_1-1)}{R}[p\mathcal{F}_n(x_1-1,x_2+k)+(1-p)\mathcal{F}_n(x_1-1,x_2)]+ \\
& \frac{\mu_2(x_2-1)}{R}\mathcal{F}_n(x_1,x_2-1)+\frac{\lambda}{R}\mathcal{F}_n(x_1+1,x_2)+ \\
& \Big(1-\frac{\mu_1(x_1-1)+\mu_2(x_2-1)+\lambda}{R}\Big)\mathcal{F}_n(x_1,x_2)- \\
& \frac{1}{R}V_n^1(x_1,x_2-1)+\frac{p}{R}V_n^2(x_1-1,x_2+k-1)+\frac{1-p}{R}V_n^2(x_1-1,x_2) \\
\leq \; & \frac{\mu_1(x_1-1)}{R}[p\mathcal{F}_n(x_1-1,x_2+k)+(1-p)\mathcal{F}_n(x_1-1,x_2)]+ \\
& \frac{\mu_2(x_2-1)}{R}\mathcal{F}_n(x_1,x_2-1)+\frac{\lambda}{R}\mathcal{F}_n(x_1+1,x_2)+ \\
& \Big(1-\frac{\mu_1(x_1-1)+\mu_2(x_2-1)+\lambda}{R}\Big)\mathcal{F}_n(x_1,x_2)+ \\
& \frac{\mu_1}{R}\Big\{(1-p)\mathcal{G}_n(x_1,x_2-1)+p\mathcal{G}_n(x_1,x_2)+(1-p)\mathcal{F}_n(x_1,x_2-1)- \\
& (1-p)\mathcal{F}_n(x_1,x_2)\Big\} \\
= \; & \frac{\mu_1(x_1-1)}{R}[p\mathcal{F}_n(x_1-1,x_2+k)+(1-p)\mathcal{F}_n(x_1-1,x_2)]+ \\
& \frac{\mu_2(x_2-1)}{R}\mathcal{F}_n(x_1,x_2-1)+\frac{\lambda}{R}\mathcal{F}_n(x_1+1,x_2)+ \\
& \Big(1-\frac{\mu_1(x_1-1)+\mu_2(x_2-1)+\lambda}{R}\Big)\mathcal{F}_n(x_1,x_2) \\
& \frac{\mu_1}{R}\Big\{(1-p)\mathcal{G}_n(x_1,x_2-1)+p\mathcal{G}_n(x_1,x_2)+(1-p)\mathcal{F}_n(x_1,x_2-1)\Big\}
\end{aligned} \tag{52}
$$

The first and third equalities follow from Eq. (13). The second equality follows from Eqs. (50), (43), and (5). The fourth equality follows from the definitions (10) and (11). The first inequality in Eq. (52) holds because of $N \geq 0$ (N is given by (51)), and the second inequality is a subsequent of Eq. (20) in fact 4.

□

A.7 Proof of Equation (28)

If $x_1 > m$,

$$
\begin{array}{ll}
g_1^*(x_1-1,x_2)=m & g_2^*(x_1-1,x_2)=0 \\
g_1^*(x_1-1,x_2+k)=m & g_2^*(x_1-1,x_2+k)=0 \\
g_1^*(x_1,x_2-1)=m & g_2^*(x_1,x_2-1)=0 \\
g_1^*(x_1,x_2)=m & g_2^*(x_1,x_2)=0.
\end{array} \tag{53}
$$

Then,

$$
\begin{aligned}
&V^1_{n+1}(x_1,x_2)\\
&\quad= \mu_1\Big(pV_{n+1}(x_1-1,x_2+k)+(1-p)V_{n+1}(x_1-1,x_2)-V_{n+1}(x_1,x_2)\Big)\\
&\quad= \mu_1\Big(p\Big\{\frac{\mu_1 m}{R}[pV_n(x_1-2,x_2+2k)+(1-p)V_n(x_1-2,x_2+k)]+\\
&\qquad \frac{\lambda}{R}V_n(x_1,x_2+k)+\Big(1-\frac{\mu_1 m+\lambda}{R}\Big)V_n(x_1-1,x_2+k)\Big\}+\\
&\qquad (1-p)\Big\{\frac{\mu_1 m}{R}[pV_n(x_1-2,x_2+k)+(1-p)V_n(x_1-2,x_2)]+\\
&\qquad \frac{\lambda}{R}V_n(x_1,x_2)+\Big(1-\frac{\mu_1 m+\lambda}{R}\Big)V_n(x_1-1,x_2)\Big\}-\\
&\qquad \Big\{\frac{\mu_1 m}{R}[pV_n(x_1-1,x_2+k)+(1-p)V_n(x_1-1,x_2)]+\\
&\qquad \frac{\lambda}{R}V_n(x_1+1,x_2)+\Big(1-\frac{\mu_1 m+\lambda}{R}\Big)V_n(x_1,x_2)\Big\}\Big)\\
&\quad= \frac{\mu_1 m}{R}[pV^1_n(x_1-1,x_2+k)+(1-p)V^1_n(x_1-1,x_2)]+\\
&\qquad \frac{\lambda}{R}V^1_n(x_1+1,x_2)+\Big(1-\frac{\mu_1 m+\lambda}{R}\Big)V^1_n(x_1,x_2)
\end{aligned} \tag{54}
$$

The first and third equalities follow from Eq. (10). The second equality follows from Eqs. (53), (43), and (5).

$$
\begin{aligned}
&V^2_{n+1}(x_1,x_2)\\
&\quad= \mu_2\Big(V_{n+1}(x_1,x_2-1)-V_{n+1}(x_1,x_2)\Big)\\
&\quad= \mu_2\Big(\frac{\mu_1 m}{R}[pV_n(x_1-1,x_2+k-1)+(1-p)V_n(x_1-1,x_2-1)]+\\
&\qquad \frac{\lambda}{R}V_n(x_1+1,x_2-1)+\Big(1-\frac{\mu_1 m+\lambda}{R}\Big)V_n(x_1,x_2-1)-\\
&\qquad \Big\{\frac{\mu_1 m}{R}[pV_n(x_1-1,x_2+k)+(1-p)V_n(x_1-1,x_2)]+\\
&\qquad \frac{\lambda}{R}V_n(x_1+1,x_2)+\Big(1-\frac{\mu_1 m+\lambda}{R}\Big)V_n(x_1,x_2)\Big\}\Big)\\
&\quad= \frac{\mu_1 m}{R}[pV^2_n(x_1-1,x_2+k)+(1-p)V^2_n(x_1-1,x_2)]+\\
&\qquad \frac{\lambda}{R}V^2_n(x_1+1,x_2)+\Big(1-\frac{\mu_1 m+\lambda}{R}\Big)V^2_n(x_1,x_2)
\end{aligned} \tag{55}
$$

The first and third equalities follow from Eq. (11). The second equality follows from Eqs. (53), (43), and (5).

Because of Eqs. (54), and (55), we can write

$$
\begin{aligned}
&\mathcal{D}_{n+1}(x_1, x_2) \\
&\quad= \frac{\mu_1 m}{R}[pV_n^1(x_1-1, x_2+k) + (1-p)V_n^1(x_1-1, x_2)] + \\
&\qquad \frac{\lambda}{R}V_n^1(x_1+1, x_2) + \Big(1 - \frac{\mu_1 m + \lambda}{R}\Big)V_n^1(x_1, x_2) - \\
&\qquad \Big\{\frac{\mu_1 m}{R}[pV_n^2(x_1-1, x_2+k) + (1-p)V_n^2(x_1-1, x_2)] + \\
&\qquad \frac{\lambda}{R}V_n^2(x_1+1, x_2) + \Big(1 - \frac{\mu_1 m + \lambda}{R}\Big)V_n^2(x_1, x_2)\Big\} \\
&\quad= \frac{\mu_1 m}{R}[p\mathcal{D}_n(x_1-1, x_2+k) + (1-p)\mathcal{D}_n(x_1-1, x_2)] + \\
&\qquad \frac{\lambda}{R}\mathcal{D}_n(x_1+1, x_2) + \Big(1 - \frac{\mu_1 m + \lambda}{R}\Big)\mathcal{D}_n(x_1, x_2)
\end{aligned}
$$

The first and third inequalities holds because of Eq. (12). The second equality holds because of Eqs. (54) and (55).

□

A.8 Proof of Equation (29)

If $x_1 \le m < x_1 + x_2$,

$$
\begin{aligned}
g_1^*(x_1-1, x_2) &= x_1 - 1 & g_2^*(x_1-1, x_2) &= m - x_1 + 1 \\
g_1^*(x_1-1, x_2+k) &= x_1 - 1 & g_2^*(x_1-1, x_2+k) &= m - x_1 + 1 \\
g_1^*(x_1, x_2-1) &= x_1 & g_2^*(x_1, x_2-1) &= m - x_1 \\
g_1^*(x_1, x_2) &= x_1 & g_2^*(x_1, x_2) &= m - x_1.
\end{aligned} \tag{56}
$$

Then,

$$
\begin{aligned}
&V_{n+1}^1(x_1, x_2) \\
&\quad= \mu_1\Big(pV_{n+1}(x_1-1, x_2+k) + (1-p)V_{n+1}(x_1-1, x_2) - V_{n+1}(x_1, x_2)\Big) \\
&\quad= \mu_1(p\Big\{\frac{\mu_1(x_1-1)}{R}[pV_n(x_1-2, x_2+2k) + (1-p)V_n(x_1-2, x_2+k)] + \\
&\qquad \frac{\mu_2(m-x_1+1)}{R}V_n(x_1-1, x_2+k-1) + \frac{\lambda}{R}V_n(x_1, x_2+k) + \\
&\qquad \Big(1 - \frac{\mu_1(x_1-1) + \mu_2(m-x_1+1) + \lambda}{R}\Big)V_n(x_1-1, x_2+k)\Big\} +
\end{aligned}
$$

$$
\begin{aligned}
& (1-p)\Big\{\frac{\mu_1(x_1-1)}{R}[pV_n(x_1-2,x_2+k)+(1-p)V_n(x_1-2,x_2)]+ \\
& \frac{\mu_2(m-x_1+1)}{R}V_n(x_1-1,x_2-1)\frac{\lambda}{R}V_n(x_1,x_2)+ \\
& \Big(1-\frac{\mu_1(x_1-1)+\mu_2(m-x_1+1)+\lambda}{R}\Big)V_n(x_1-1,x_2)\Big\}- \\
& \Big\{\frac{\mu_1 x_1}{R}[pV_n(x_1-1,x_2+k)+(1-p)V_n(x_1-1,x_2)]+ \\
& \frac{\mu_2(m-x_1)}{R}V_n(x_1,x_2-1)+\frac{\lambda}{R}V_n(x_1+1,x_2)+ \\
& \Big(1-\frac{\mu_1 x_1+\mu_2(m-x_1)+\lambda}{R}\Big)V_n(x_1,x_2)\Big\}\Big) \\
= \; & \frac{\mu_1(x_1-1)}{R}[pV_n^1(x_1-1,x_2+k)+(1-p)V_n^1(x_1-1,x_2)]+ \\
& \frac{\mu_2(m-x_1)}{R}V_n^1(x_1,x_2-1)+\frac{\lambda}{R}V_n^1(x_1+1,x_2)+ \\
& \Big(1-\frac{\mu_1(x_1-1)+\mu_2(m-x_1)+\lambda}{R}\Big)V_n^1(x_1,x_2)-\frac{\mu_1}{R}V_n^1(x_1,x_2)+ \\
& \frac{\mu_1}{R}[pV_n^2(x_1-1,x_2+k)+(1-p)V_n^2(x_1-1,x_2+k)] \qquad (57)
\end{aligned}
$$

The first and third equalities follow from Eq. (10). The second equality follows from Eqs. (56), (43), and (5).

$$
\begin{aligned}
& V_{n+1}^2(x_1,x_2) \\
= \; & \mu_2\Big(V_{n+1}(x_1,x_2-1)-V_{n+1}(x_1,x_2)\Big) \\
= \; & \mu_2\Big(\frac{\mu_1(x_1-1)}{R}[pV_n(x_1-1,x_2+k-1)+(1-p)V_n(x_1-1,x_2-1)]+ \\
& \frac{\mu_2(m-x_1)}{R}V_n(x_1,x_2-2)+\frac{\lambda}{R}V_n(x_1+1,x_2-1)+ \\
& \Big(1-\frac{\mu_1 x_1+\mu_2(m-x_1)+\lambda}{R}\Big)V_n(x_1,x_2-1)- \\
& \Big\{\frac{\mu_1 x_1}{R}[pV_n(x_1-1,x_2+k)+(1-p)V_n(x_1-1,x_2)]+ \\
& \frac{\mu_2(m-x_1)}{R}V_n(x_1,x_2-1)\frac{\lambda}{R}V_n(x_1+1,x_2)+ \\
& \Big(1-\frac{\mu_1 x_1+\mu_2(m-x_1)+\lambda}{R}\Big)V_n(x_1,x_2)\Big\}\Big) \\
= \; & \frac{\mu_1 x_1}{R}[pV_n^2(x_1-1,x_2+k)+(1-p)V_n^2(x_1-1,x_2)]+ \\
& \frac{\mu_2(m-x_1)}{R}V_n^2(x_1,x_2-1)+\frac{\lambda}{R}V_n^2(x_1+1,x_2)+ \\
& \Big(1-\frac{\mu_1 x_1+\mu_2(m-x_1)+\lambda}{R}\Big)V_n^2(x_1,x_2) \qquad (58)
\end{aligned}
$$

The first and third equalities follow from Eq. (11). The second equality follows from Eqs. (56), (43), and (5).

From Eqs. (57), (58), and (12) we have

$$
\begin{aligned}
&\mathcal{D}_{n+1}(x_1,x_2)\\
&= V^1_{n+1}(x_1,x_2) - V^2_{n+1}(x_1,x_2)\\
&= \frac{\mu_1(x_1-1)}{R}[pV^1_n(x_1-1,x_2+k)+(1-p)V^1_n(x_1-1,x_2)]+\\
&\quad \frac{\mu_2(m-x_1)}{R}V^1_n(x_1,x_2-1)+\frac{\lambda}{R}V^1_n(x_1+1,x_2)+\\
&\quad \Big(1-\frac{\mu_1(x_1-1)+\mu_2(m-x_1)+\lambda}{R}\Big)V^1_n(x_1,x_2)-\frac{\mu_1}{R}V^1_n(x_1,x_2)+\\
&\quad \frac{\mu_2}{R}[pV^2_n(x_1-1,x_2+k)+(1-p)V^2_n(x_1-1,x_2)]-\\
&\quad \Big\{\frac{\mu_1 x_1}{R}[pV^2_n(x_1-1,x_2+k)+(1-p)V^2_n(x_1-1,x_2)]+\\
&\quad \frac{\mu_2(m-x_1)}{R}V^2_n(x_1,x_2-1)+\frac{\lambda}{R}V^2_n(x_1+1,x_2)+\\
&\quad \Big(1-\frac{\mu_1 x_1+\mu_2(m-x_1)+\lambda}{R}V^2_n(x_1,x_2)\Big\}\Big)\\
&= \frac{\mu_1(x_1-1)}{R}[p\mathcal{D}_n(x_1-1,x_2+k)+(1-p)\mathcal{D}_n(x_1-1,x_2)]+\\
&\quad \frac{\mu_2(m-x_1)}{R}\mathcal{D}_n(x_1,x_2-1)+\frac{\lambda}{R}\mathcal{D}_n(x_1+1,x_2)+\\
&\quad \Big(1-\frac{\mu_1(x_1-1)+\mu_2(m-x_1)+\lambda}{R}\Big)\mathcal{D}_n(x_1,x_2)-\\
&\quad \frac{\mu_1}{R}V^1_n(x_1,x_2)+\frac{\mu_1}{R}V^2_n(x_1,x_2)\\
&= \frac{\mu_1(x_1-1)}{R}[p\mathcal{D}_n(x_1-1,x_2+k)+(1-p)\mathcal{D}_n(x_1-1,x_2)]+\\
&\quad \frac{\mu_2(m-x_1)}{R}\mathcal{D}_n(x_1,x_2-1)+\frac{\lambda}{R}\mathcal{D}_n(x_1+1,x_2)+\\
&\quad \Big(1-\frac{\mu_1 x_1+\mu_2(m-x_1)+\lambda}{R}\Big)\mathcal{D}_n(x_1,x_2)
\end{aligned}
$$

The first, third, and fourth equalities holds because of Eq. (12). The second equality holds because of Eqs. (57) and (58).

□

A.9 Proof of Equation (30)

If $x_1 + x_2 \le m$,

$$g^*_1(x_1-1,x_2) = x_1 - 1 \qquad g^*_2(x_1-1,x_2) = x_2$$

$$
\begin{array}{ll}
g_1^*(x_1-1,x_2+k)=x_1-1 & g_2^*(x_1-1,x_2+k)=x_2+L \\
g_1^*(x_1,x_2-1)=x_1 & g_2^*(x_1,x_2-1)=x_2-1 \\
g_1^*(x_1,x_2)=x_1 & g_2^*(x_1,x_2)=x_2.
\end{array}
\tag{59}
$$

where

$$
L=\min(k,m-x_1-x_2+1). \tag{60}
$$

Then,

$$
\begin{aligned}
&V_{n+1}^1(x_1,x_2)\\
&= \mu_1\Big(pV_{n+1}(x_1-1,x_2+k)+(1-p)V_{n+1}(x_1-1,x_2)-V_{n+1}(x_1,x_2)\Big)\\
&= \mu_1\Big(p\Big\{\frac{\mu_1(x_1-1)}{R}[pV_n(x_1-2,x_2+2k)+(1-p)V_n(x_1-2,x_2+k)]+\\
&\quad \frac{\mu_2(x+L)}{R}V_n(x_1-1,x_2+k-1)+\frac{\lambda}{R}V_n(x_1,x_2+k)+\\
&\quad \Big(1-\frac{\mu_1(x_1-1)+\mu_2(x_2+L)+\lambda}{R}\Big)V_n(x_1-1,x_2+k)\Big\}+\\
&\quad (1-p)\Big\{\frac{\mu_1(x-1-1)}{R}[pV_n(x_1-2,x_2+k)+(1-p)V_n(x_1-2,x_2)]+\\
&\quad \frac{\mu_2x_2}{R}V_n(x_1-1,x_2-1)\frac{\lambda}{R}V_n(x_1,x_2)+\\
&\quad \Big(1-\frac{\mu_1(x_1-1)+\mu_2x_2+\lambda}{R}\Big)V_n(x_1-1,x_2)\Big\}-\\
&\quad \Big\{\frac{\mu_1x_1}{R}[pV_n(x_1-1,x_2+k)+(1-p)V_n(x_1-1,x_2)]+\\
&\quad \frac{\mu_2x_2}{R}V_n(x_1,x_2-1)+\frac{\lambda}{R}V_n(x_1+1,x_2)+\\
&\quad \Big(1-\frac{\mu_1x_1+\mu_2x_2+\lambda}{R}\Big)V_n(x_1,x_2)\Big\}\Big)\\
&= \frac{\mu_1(x_1-1)}{R}[pV_n^1(x_1-1,x_2+k)+(1-p)V_n^1(x_1-1,x_2)]+\\
&\quad \frac{\mu_2x_2}{R}V_n^1(x_1,x_2-1)+\frac{\lambda}{R}V_n^1(x_1+1,x_2)+\\
&\quad \Big(1-\frac{\mu_1(x_1-1)+\mu_2(m-x_1)+\lambda}{R}\Big)V_n^1(x_1,x_2)-\frac{\mu_1}{R}V_n^1(x_1,x_2)+\\
&\quad \frac{\mu_1L}{R}pV_n^2(x_1-1,x_2+k).
\end{aligned}
\tag{61}
$$

The first and third equalities follow from Eq. (10). The second equality follows from Eqs. (56), (43), and (5).

$$
V_{n+1}^2(x_1,x_2)
$$

$$
\begin{aligned}
&= \mu_2\Big(V_{n+1}(x_1,x_2-1)-V_{n+1}(x_1,x_2)\Big)\\
&= \mu_2\Big(\frac{\mu_1 x_1}{R}[pV_n(x_1-1,x_2+k-1)+(1-p)V_n(x_1-1,x_2-1)]+\\
&\quad \frac{\mu_2(x_2-1)}{R}V_n(x_1,x_2-2)+\frac{\lambda}{R}V_n(x_1+1,x_2-1)+\\
&\quad \Big(1-\frac{\mu_1x_1+\mu_2(x_2-1)+\lambda}{R}\Big)V_n(x_1,x_2-1)-\\
&\quad \Big\{\frac{\mu_1x_1}{R}[pV_n(x_1-1,x_2+k)+(1-p)V_n(x_1-1,x_2)]+\\
&\quad \frac{\mu_2x_2}{R}V_n(x_1,x_2-1)+\frac{\lambda}{R}V_n(x_1+1,x_2)+\\
&\quad \Big(1-\frac{\mu_1x_1+\mu_2x_2+\lambda}{R}\Big)V_n(x_1,x_2)\Big\}\Big)\\
&= \frac{\mu_1x_1}{R}[pV_n^2(x_1-1,x_2+k)+(1-p)V_n^2(x_1-1,x_2)]+\\
&\quad \frac{\mu_2(x_2-1)}{R}V_n^2(x_1,x_2-1)+\frac{\lambda}{R}V_n^2(x_1+1,x_2)+\\
&\quad \Big(1-\frac{\mu_1x_1+\mu_2(x_2-1)+\lambda}{R}\Big)V_n^2(x_1,x_2)-\frac{\mu_2}{R}V_n^2(x_1,x_2). \qquad (62)
\end{aligned}
$$

The first and third equalities follow from Eq. (11). The second equality follows from Eqs. (59), (43), and (5).

$$
\begin{aligned}
&\mathcal{D}_{n+1}(x_1,x_2)\\
&\quad= V_{n+1}^1(x_1,x_2)-V_{n+1}^2(x_1,x_2)\\
&\quad= \frac{\mu_1(x_1-1)}{R}[pV_n^1(x_1-1,x_2+k)+(1-p)V_n^1(x_1-1,x_2)]+\\
&\qquad \frac{\mu_2x_2}{R}V_n^1(x_1,x_2-1)+\frac{\lambda}{R}V_n^1(x_1+1,x_2)+\\
&\qquad \Big(1-\frac{\mu_1(x_1-1)+\mu_2(m-x_1)+\lambda}{R}\Big)V_n^1(x_1,x_2)-\frac{\mu_1}{R}V_n^1(x_1,x_2)+\\
&\qquad \frac{\mu_1L}{R}pV_n^2(x_1-1,x_2+k)-\\
&\qquad \Big\{\frac{\mu_1x_1}{R}[pV_n^2(x_1-1,x_2+k)+(1-p)V_n^2(x_1-1,x_2)]+\\
&\qquad \frac{\mu_2(x_2-1)}{R}V_n^2(x_1,x_2-1)+\frac{\lambda}{R}V_n^2(x_1+1,x_2)+\\
&\qquad \Big(1-\frac{\mu_1x_1+\mu_2(x_2-1)+\lambda}{R}\Big)V_n^2(x_1,x_2)-\frac{\mu_2}{R}V_n^2(x_1,x_2)\Big\}\\
&\quad= \frac{\mu_1(x_1-1)}{R}[p\mathcal{D}_n(x_1-1,x_2+k)+(1-p)\mathcal{D}_n(x_1-1,x_2)]+\\
&\qquad \frac{\mu_2(x_2-1)}{R}\mathcal{D}_n(x_1,x_2-1)+\frac{\lambda}{R}\mathcal{D}_n(x_1+1,x_2)+\\
&\qquad \Big(1-\frac{\mu_1(x_1-1)+\mu_2(x_2-1)+\lambda}{R}\Big)\mathcal{D}_n(x_1,x_2)-\frac{\mu_1}{R}V_n^1(x_1,x_2)+
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\mu_2}{R}[V_n^1(x_1, x_2-1) - V_n^1(x_1,x_2)] + \frac{\mu_2}{R}V_n^2(x_1,x_2) + \\
& p\frac{\mu_1 L}{R}V_n^2(x_1-1, x_2+k) + \frac{\mu_1}{R}V_n^2(x_1,x_2) - \\
& \frac{\mu_1}{R}[pV_n^2(x_1-1,x_2+k) + (1-p)V_n^2(x_1-1,x_2)] \\
= \; & \frac{\mu_1(x_1-1)}{R}[p\mathcal{D}_n(x_1-1,x_2+k) + (1-p)\mathcal{D}_n(x_1-1,x_2)] + \\
& \frac{\mu_2(x_2-1)}{R}\mathcal{D}_n(x_1,x_2-1) + \frac{\lambda}{R}\mathcal{D}_n(x_1+1,x_2) + \\
& \left(1 - \frac{\mu_1 x_1 + \mu_2 x_2 + \lambda}{R}\right)\mathcal{D}_n(x_1,x_2) + p\frac{\mu_1(L-1)}{R}V_n^2(x_1-1,x_2+k) + \\
& \frac{\mu_2}{R}V_n^1(x_1,x_2-1) - \frac{\mu_1}{R}V_n^2(x_1-1,x_2) \\
= \; & \frac{\mu_1(x_1-1)}{R}[p\mathcal{D}_n(x_1-1,x_2+k) + (1-p)\mathcal{D}_n(x_1-1,x_2)] + \\
& \frac{\mu_2(x_2-1)}{R}\mathcal{D}_n(x_1,x_2-1) + \frac{\lambda}{R}\mathcal{D}_n(x_1+1,x_2) + p\frac{\mu_1(L-1)}{R}V_n^2(x_1-1,x_2+k) + \\
& \left(1 - \frac{\mu_1 x_1 + \mu_2 x_2 + \lambda}{R}\right)\mathcal{D}_n(x_1,x_2) + \frac{\mu_1\mu_2}{R}\mathcal{F}_n(x_1,x_2) \\
\le \; & \frac{\mu_1(x_1-1)}{R}[p\mathcal{D}_n(x_1-1,x_2+k) + (1-p)\mathcal{D}_n(x_1-1,x_2)] + \\
& \frac{\mu_2(x_2-1)}{R}\mathcal{D}_n(x_1,x_2-1) + \frac{\lambda}{R}\mathcal{D}_n(x_1+1,x_2) + \\
& \left(1 - \frac{\mu_1 x_1 + \mu_2 x_2 + \lambda}{R}\right)\mathcal{D}_n(x_1,x_2) + \frac{\mu_1\mu_2}{R}\mathcal{F}_n(x_1,x_2)
\end{aligned}
$$

The first and third equalities holds because of Eq. (12). The second equality holds because of Eqs. (61) and (62). The fifth equality follows from Eq. (13) The inequality holds because of Eq. (18) and $L \geq 1$.

□

B Outline of the proof of Theorem 3

In this appendix, we provide the reader with the essence of the proof of Theorem 3. First define

Condition 3 $\quad p \leq \dfrac{\mu_1 - \mu_2}{\mu_1}.$

as the compliment of Condition 2. Theroem 3 can be proved by the same argument as in Section 2.2; that is Problem (P) can be reduced to proving the optimality of $\tilde{g}$ under Condition 3 for the n-stage problem described by Eqs. (7)-(9). To proceed with the proof, we establish an induction similar to that of Section 2.3.1. Notice that since $k=1$, $\mathcal{G}_n(x_1,x_2) = \mathcal{F}_n(x_1,x_2)$ and the induction hypotheses for stage n will be

$(\mathcal{H}'0)_n$	policy $\tilde{g}$, that gives priority to Q_2, is optimal at stage n
$(\mathcal{H}'1)_n$	$\mathcal{D}_n(x_1,x_2) \geq 0$, for every $x_1 \geq 0,\ x_2 \geq 0$
$(\mathcal{H}'2)_n$	$\mathcal{G}_n(x_1,x_2) \geq 0$, for every $x_1 \geq 0,\ x_2 \geq 0$

The basis of induction is verified through the same argument as in Section 2.3.2. Assuming that the induction hypotheses are valid at stage n, we use a number of facts and lemmas to prove the validity of $(\mathcal{H}'0)_{n+1}$-$(\mathcal{H}'2)_{n+1}$. These facts and lemmas, which are stated below, and their proofs are very similar to those of Section 2.3.2.

Fact. 5 *For every $n \geq 1, x_1 \geq 0, x_2 \geq 0$, the following relations hold.*

$$V_n(x_1+1, x_2) \geq V_n(x_1, x_2), \tag{63}$$
$$V_n(x_1, x_2+1) \geq V_n(x_1, x_2). \tag{64}$$

Fact. 6 *For every $n \geq 1, x_1 \geq 0, x_2 \geq 0$, the following relations hold.*

$$V_n^1(x_1, x_2) \leq 0, \tag{65}$$
$$V_n^2(x_1, x_2) \leq 0. \tag{66}$$

Fact. 7 *Under Condition 3, i.e. $p \leq \frac{\mu_1 - \mu_2}{\mu_1}$,*

$$V_n^1(x_1, x_2-1) - V_n^2(x_1-1, x_2) \leq \mu_1 \mathcal{G}_n(x_1, x_2) \tag{67}$$

Lemma 6 *Assume Conditions 1 and 3 hold. If $\mathcal{D}_n(x_1, x_2) \geq 0$ for all $x_1 \geq 0$, $x_2 \geq 0$, then $\tilde{g}$ is optimal at stage $n+1$.*

Lemma 7 *Assume that Conditions 1 and 3 hold. If $\tilde{g}$ is optimal at stage $n+1$ and $\mathcal{G}_n(x_1, x_2) \geq 0$ for all x_1, x_2, then $\mathcal{G}_{n+1}(x_1, x_2) \geq 0$.*

Lemma 8 *Assume that Conditions 1 and 3 hold. If $\tilde{g}$ is optimal at stage $n+1$, $\mathcal{G}_n(x_1, x_2) \geq 0$, and $\mathcal{D}_n(x_1, x_2) \geq 0$ for all x_1, x_2, then $\mathcal{D}_{n+1}(x_1, x_2) \geq 0$.*

$\square$