Solutions to Exercise Set #5

1. For the (7, 4) Hamming code, calculate the probability of decoding to a wrong codeword if the code is used for a binary symmetric channel with bit error probability $p$. Evaluate the formula you obtain for the following values of $p$: $p = 0.2, 0.1, 0.05, 0.01, 0.005,$ and $0.001$.

Solution: From class, we know that the (7, 4) Hamming code can correct 1 error. Hence the probability of decoding to the correct codeword $P_c$ is

$$P_c = (1 - p)^7 + 7p(1 - p)^6$$

where $p$ is the bit error probability. Then the probability of decoding to the wrong codeword $P_w$ is

$$P_w = 1 - P_c = 1 - (1 - p)^7 - 7p(1 - p)^6$$

By evaluating the formula for the given $p$ values, we get the following:

<table>
<thead>
<tr>
<th>$p$</th>
<th>0.2</th>
<th>0.1</th>
<th>0.05</th>
<th>0.01</th>
<th>0.005</th>
<th>0.001</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_w$</td>
<td>0.42</td>
<td>0.15</td>
<td>0.044</td>
<td>0.002</td>
<td>0.00052</td>
<td>2.1 × 10^{-6}</td>
</tr>
</tbody>
</table>

2. Run a simulation that verifies the result found in Problem 1 for the case of $p = 0.05$. Do the following a sufficient number of times to get a good estimate of the decoded word error rate. Compare your simulation results with the analytic expression found in Problem 1.

(a) Generate 4 random binary digits.
(b) Encode them by calculating the 3 check digits.
(c) Put the code word through a BSC with bit error probability $p$.
(d) Decode the received 7-bit word.
(e) Compare the decoded word with the transmitted word and count word errors.

Solution: The solution depends on the particular random digits you generated so there is no universal solution.
3. Repeat Problem 2 but now only count the errors in the information positions of the code words to obtain an estimate of the decoded bit error probability. Is the bit error probability larger or smaller than the word error probability?

Solution: The decoded bit error probability for the information positions \( P_b \) cannot be larger than the word error probability \( P_w \). In fact,
\[
\frac{1}{4} P_w \leq P_b \leq P_w.
\]
This is because the upper bound of \( P_b \) can be given as the worst case where all information bits are flipped when there is a word error. The lower bound is when there is only one bit error when a word error occurs.

4. Suppose the Hamming code is used to detect errors.

(a) For \( i = 0, 1, 2, \ldots, 7 \), find the number of error patterns containing exactly \( i \) errors that the decoder will fail to detect?

(b) If the code is used for a binary symmetric channel with bit error probability \( p \), find an analytic expression for the probability of an undetected word error as a function of \( p \).

(c) Verify the result found in part (b) by simulation for \( p = 0.2 \).

Solution: Let \( \tilde{E} \) be an error pattern. The decoder will not be able to detect error patterns that match non-zero codewords, since
\[
H(\tilde{E} + \tilde{C}) = \tilde{0}
\]
for any codeword \( \tilde{C} \). By inspecting the list of codewords for the Hamming \((7, 4)\) code, we find 7 codewords with \( W_H = 3 \), 7 codewords with \( W_H = 4 \), and 1 codeword with \( W_H = 7 \). Hence
\[
P(\text{undetected error}) = 7p^3(1 - p)^4 + 7p^4(1 - p)^3 + p^7.
\]

5. Suppose the Hamming code is used to correct erasures.

(a) For \( i = 0, 1, 2, \ldots, 7 \), find the number of erasure patterns containing exactly \( i \) erasures that the decoder will fail to correct?

(b) If the code is used for a binary erasure channel with bit erasure probability \( p \), find an analytic expression for the probability of failing to correct the erasures.

(c) Verify the result found in part (b) by simulation for \( p = 0.2 \).

Solution: Since the \((7, 4)\) Hamming code has \( d_{\text{min}} = 3 \), it can correct all erasure patterns with up to 2 erasures. The decoder cannot fill in any pattern with 4 or more erasures since \( H \) only has 3 rows. Hence we cannot find 4 or more linearly independent
columns in $H$. However, the decoder can correct some patterns with 3 erasures, but not all of them. We are able to fill in patterns that occur in positions that correspond to linearly independent columns in $H$. If we select 3 columns from $H$, there are

$$\binom{7}{2} = 7$$

combinations that are linearly dependent. The reason we divide by $\binom{3}{2}$ is to remove redundant sets of linearly dependent columns.

For 4, 5, 6, and 7 erasures, there are $\binom{7}{4} = 35$, $\binom{7}{5} = 21$, $\binom{7}{6} = 7$, and $\binom{7}{7} = 1$ patterns respectively. Hence the analytic expression for the probability of uncorrectable erasures is

$$P(\text{uncorrectable erasures}) = \sum_{i=3}^{7} e_i p^i (1-p)^{7-i}$$

where $e_i$ is the number of patterns with $i$ erasures that cannot be corrected calculated above, i.e., $e_3 = 7$, $e_4 = 35$, $e_5 = 21$, $e_6 = 7$, $e_7 = 1$.

6. For the (23, 12) Golay code given in the notes, calculate the probability of decoding to the wrong codeword if the code is used to correct errors caused by a binary symmetric channel with bit error probability $p$. Evaluate the formula you obtain for the following values of $p$ = 0.2, 0.1, 0.05, 0.01, 0.005, and 0.001.

**Solution:** From the lecture, we know that the Hamming (23, 12) Golay code has $d_{\text{min}} = 7$. Hence it can correct all error patterns of $\lfloor \frac{d_{\text{min}} - 1}{2} \rfloor$ errors or less. The probability of decoding to a correct codeword $P_c$ can be given as

$$P_c = \sum_{i=0}^{3} \binom{23}{i} p^i (1-p)^{23-i}$$

and the probability of decoding to a wrong codeword $P_w$ is

$$P_w = 1 - P_c.$$

By evaluating the formula for the given $p$ values, we get the following:

<table>
<thead>
<tr>
<th>$p$</th>
<th>0.2</th>
<th>0.1</th>
<th>0.05</th>
<th>0.01</th>
<th>0.005</th>
<th>0.001</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_w$</td>
<td>0.704</td>
<td>0.193</td>
<td>0.026</td>
<td>0.002</td>
<td>$5.1 \times 10^{-6}$</td>
<td>$8.7 \times 10^{-9}$</td>
</tr>
</tbody>
</table>
7. Consider the \((15,7)\) parity check code with parity check matrix given below:

\[
H = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1
\end{bmatrix}
\]

(a) Without changing the order of the digits put the parity check matrix in a form with a 8-by-8 identity matrix on the right.

(b) Give the generator matrix of the code with a 7 by 7 unit matrix on the left.

(c) What is the minimum Hamming distance of the code?

(d) Show the form of an encoder using a table look-up. What is the size of the table?

(e) Show a decoder for the code using two tables. What are the size of the tables?

**Solution:** Let \(H = [A|B]\) where \(A\) is a \(8 \times 7\) matrix and \(B\) is a \(8 \times 8\) matrix. Then

\[
H' = B^{-1}[A|B] = [B^{-1}A|I_{8 \times 8}],
\]

where \(B^{-1}\) is the modulo 2 inverse of \(B\). Using Matlab, we can find that \(B^{-1}A\) is

\[
\begin{bmatrix}
1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

Then the generator matrix is \(G = [I_{7 \times 7}|(B^{-1}A)^T]\). By observing the codeword with the minimum Hamming weight, we can find that \(d_{\text{min}} = 5\). If a table look-up was used for encoding, the size of the table would be \(2^7 \times 8\) where each row corresponds to each message sequence of length 7 and the each column corresponds to each bit of the parity sequence of length 8. The size of a decoding table would be \(2^8 \times 15\) where each row corresponds to each syndrome of length 8 and the columns correspond to the error patterns \(\bar{e}\). We would decode a received codeword \(\bar{r}\) by adding it to \(\bar{e}\).