# Digital Communications III (ECE 154C)
# Introduction to Coding and Information Theory

Tara Javidi
These lecture notes were originally developed by late Prof. J. K. Wolf.
UC San Diego

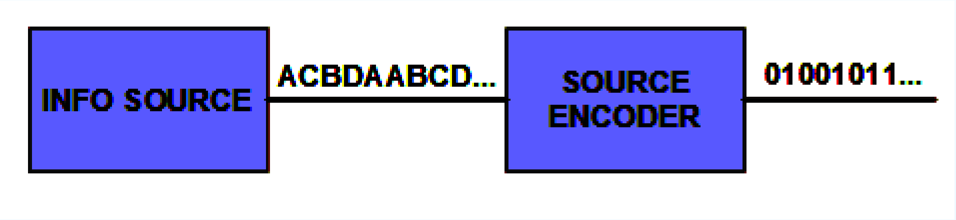
Spring 2014

# Source Coding: Lossless Compression

# Source Coding: A Simple Example

Back to our simple example of a source:

$$\mathbb{P}[A] = \frac{1}{2}, \mathbb{P}[B] = \frac{1}{4}, \mathbb{P}[C] = \frac{1}{8}, \mathbb{P}[D] = \frac{1}{8}$$




## Assumptions

1. One must be able to uniquely recover the source sequence from the binary sequence
2. One knows the start of the binary sequence at the receiver
3. One would like to minimize the average number of binary digits per source letter

# Source Coding: A Simple Example

Back to our simple example of a source:

$$\mathbb{P}[A] = \frac{1}{2}, \mathbb{P}[B] = \frac{1}{4}, \mathbb{P}[C] = \frac{1}{8}, \mathbb{P}[D] = \frac{1}{8}$$

1. $A \to 00$
   $B \to 01$ $\qquad ABAC \to 00010010 \to ABAC$
   $C \to 10$
   $D \to 11$ $\qquad \overline{L} = 2$

## Assumptions

1. One must be able to uniquely recover the source sequence from the binary sequence
2. One knows the start of the binary sequence at the receiver
3. One would like to minimize the average number of binary digits per source letter

# Source Coding: A Simple Example

Back to our simple example of a source:

$$\mathbb{P}[A] = \frac{1}{2}, \mathbb{P}[B] = \frac{1}{4}, \mathbb{P}[C] = \frac{1}{8}, \mathbb{P}[D] = \frac{1}{8}$$

1. $A \to 0$ $\quad AABD \to 00110 \to CBBA$
   $B \to 1$ $\quad (\to CBD)$
   $C \to 10$ $\quad (\to AABD)$
   $D \to 11$ $\quad \overline{L} = \frac{5}{4}$
   This code is useless. Why?

## Assumptions

1. One must be able to uniquely recover the source sequence from the binary sequence
2. One knows the start of the binary sequence at the receiver
3. One would like to minimize the average number of binary digits per source letter

# Source Coding: A Simple Example

Back to our simple example of a source:

$$\mathbb{P}[A] = \frac{1}{2}, \mathbb{P}[B] = \frac{1}{4}, \mathbb{P}[C] = \frac{1}{8}, \mathbb{P}[D] = \frac{1}{8}$$

1. $A \to 0$ $\qquad ABACD \to 0100110111 \to ABACD$
   $B \to 10$
   $C \to 110$
   $D \to 111$ $\qquad \overline{L} = \frac{7}{4}$
   Minimum length code satisfying Assumptions 1 and 2!

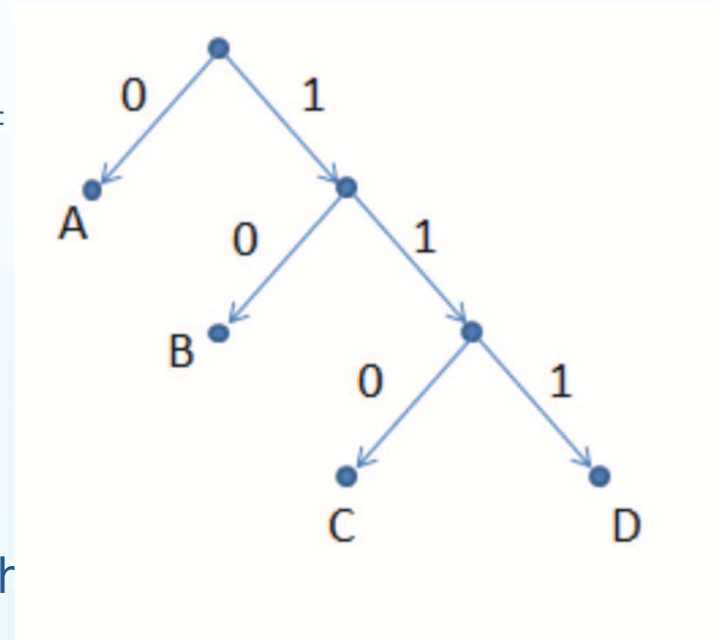
## Assumptions

1. One must be able to uniquely recover the source sequence from the binary sequence
2. One knows the start of the binary sequence at the receiver
3. One would like to minimize the average number of binary digits per source letter

# Source Coding: A Simple Example

Back to our simple example of a source:

$$\mathbb{P}[A] = \quad \ldots \quad = \frac{1}{8}$$




1. $A \to 0$ ... $ABACD$
   $B \to 10$
   $C \to 110$
   $D \to 111$
   Minimum length ... and 2!

## Assumptions

1. One must be able to uniquely recover the source sequence from the binary sequence
2. One knows the start of the binary sequence at the receiver
3. One would like to minimize the average number of binary digits per source letter

# Source Coding: Basic Definitions

## Codeword (aka Block Code)

Each source symbol is represented by some sequence of coded symbols called a code word

## Non-Singular Code

Code words are distinct

## Uniquely Decodable (U.D.) Code

Every distinct concatenation of $m$ code words is distinct for every finite $m$

## Instantaneous Code

A U.D. Code where we can decode each code word without seeing subsequent code words

# Source Coding: Basic Definitions



## Example

Back to the simple case of 4-letter DMS:

| Source Symbols | Code 1 | Code 2 | Code 3 | Code 4 |
|---|---|---|---|---|
| A | 0 | 00 | 0 | 0 |
| B | 1 | 01 | 10 | 01 |
| C | 00 | 10 | 110 | 011 |
| D | 01 | 11 | 111 | 111 |

| | | | | |
|---|---|---|---|---|
| Non-Singular | Yes | Yes | Yes | Yes |
| U.D. | No | Yes | Yes | Yes |
| Instan | | Yes | Yes | Yes |

A NECESSARY AND SUFFICIENT CONDITION for a code to be instantaneous is that no code word be a PREFIX of any other code word.

# Coding Several Source Symbol at a Time

# Example 1: 3 letter DMS

| Source Symbols | Probability | U.D. Code |
|---|---|---|
| A | .5 | 0 |
| B | .35 | 10 |
| C | .85 | 11 |

$\overline{L}_1 = 1.5$ (Bits / Symbol)

# Example 1: 3 letter DMS

| Source Symbols | Probability | U.D. Code |
|---|---|---|
| A | .5 | 0 |
| B | .35 | 10 |
| C | .85 | 11 |

$\overline{L}_1 = 1.5$ (Bits / Symbol)

Let us consider two consecutive source symbols at a time:

| 2 Symbols | Probability | U.D. Code |
|---|---|---|
| AA | .25 | 01 |
| AB | .175 | 11 |
| AC | .075 | 0010 |
| BA | .175 | 000 |
| BB | .1225 | 101 |
| BC | .0525 | 1001 |
| CA | .075 | 0011 |
| CB | .0525 | 10000 |
| CC | .0225 | 10001 |

$\overline{L}_2 = 2.9275$ (Bits/ 2 Symbols)

$\frac{\overline{L}_2}{2} = 1.46375$ (Bits / Symbol)

# Example 1: 3 letter DMS

In other words,

1. It is more efficient to build a code for 2 source symbols!
2. Is it possible to decrease the length more and more by increasing the alphabet size?

To see the answer to the above question, it is useful if we can say precisely characterize the best code. The codes given above are *Huffman Codes*. The procedure for making *Huffman Codes* will be described next.

# Minimizing average length

# Binary Huffman Codes



## Binary Huffman Codes

1. Order probabilities - Highest to Lowest
2. Add two lowest probabilities
3. Reorder probabilities
4. Break ties in any way you want

# Binary Huffman Codes



## Binary Huffman Codes

1. Order probabilities - Highest to Lowest
2. Add two lowest probabilities
3. Reorder probabilities
4. Break ties in any way you want

### Example

1. $\{.1, .2, .15, .3, .25\} \xrightarrow{order} \{.3, .25, .2, .15, .1\}$
2. $\{.3, .25, .2, \underbrace{.15, .1}_{.25}\}$
3. Get either $\{.3, \underbrace{(.15, .1)}_{.25}, .25, .2\}$ or

   $\{.3, .25, \underbrace{(.15, .1)}_{.25}, .2\}$

# Binary Huffman Codes



## Binary Huffman Codes

1. Order probabilities - Highest to Lowest
2. Add two lowest probabilities
3. Reorder probabilities
4. Break ties in any way you want
5. Assign 0 to top branch and 1 to bottom branch (or vice versa)
6. Continue until we have only one probability equal to 1
7. $\overline{L}$ = Sum of probabilities of combined nodes (i.e., the circled ones)

# Binary Huffman Codes

### Binary Huffman Codes

1. Order probabilities - Highest to Lowest
2. Add two lowest probabilities
3. Reorder probabilities
4. Break ties in any way you want
5. Assign 0 to top branch and 1 to bottom branch (or vice versa)
6. Continue until we have only one probability equal to 1
7. $\overline{L}$ = Sum of probabilities of combined nodes (i.e., the circled ones)

### Optimality of Huffman Coding

1. Binary Huffman code will have the shortest average length as compared with any U.D. Code for set of probabilities.
2. The Huffman code is not unique. Breaking ties in different ways can result in very different codes. The average length, however, will be the same for all of these codes.

# Huffman Coding: Example

## Example Continued

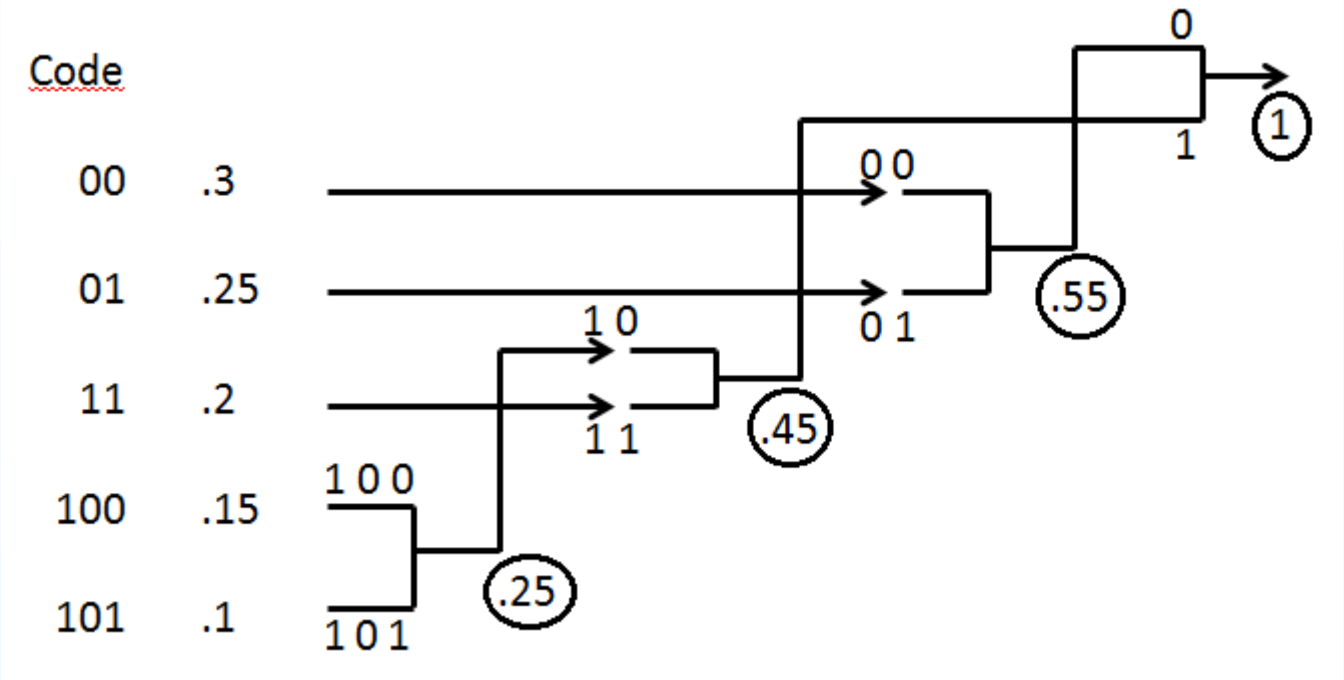
1. $\{.1, .2, .15, .3, .25\} \overset{order}{\rightarrow} \{.3, .25, .2, .15, .1\}$

# Huffman Coding: Example



## Example Continued

1. $\{.1, .2, .15, .3, .25\} \stackrel{order}{\rightarrow} \{.3, .25, .2, .15, .1\}$




$$\overline{L} = .25 + .45 + .55 + 1 = 2.25$$

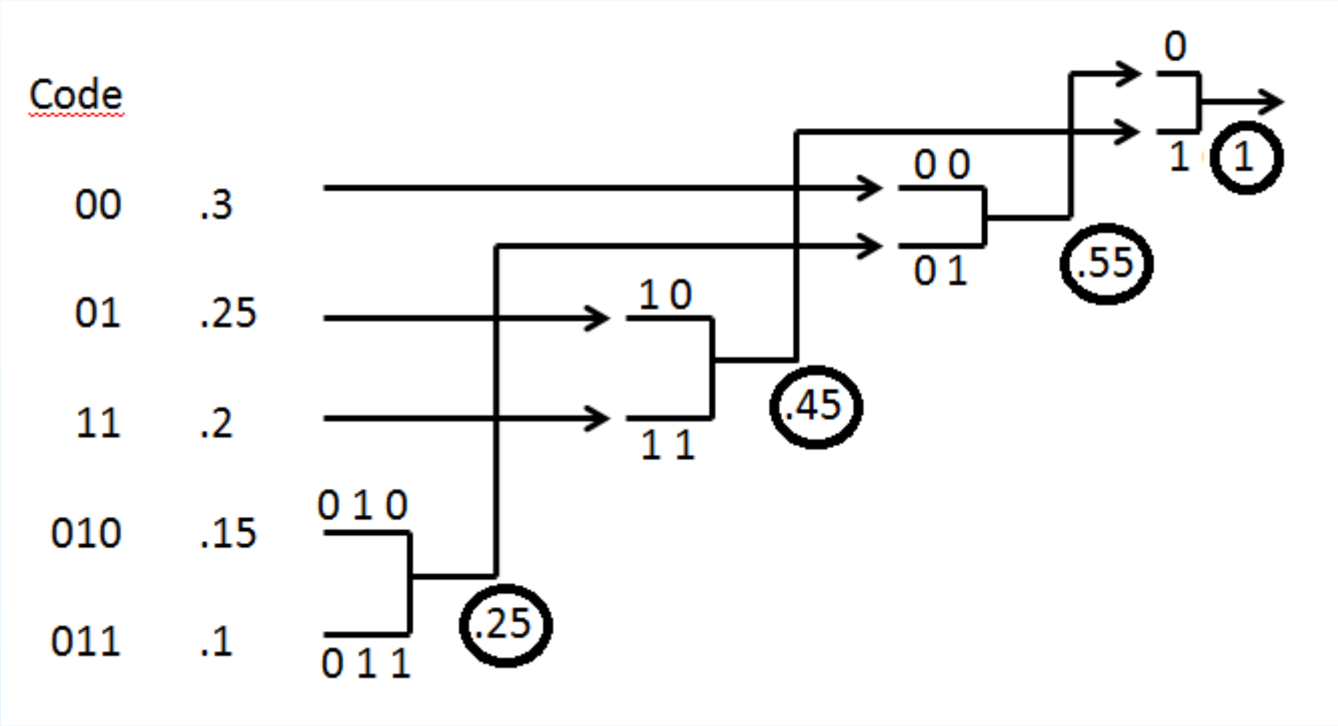
# Huffman Coding: Example



## Example Continued

1. $\{.1, .2, .15, .3, .25\} \overset{order}{\rightarrow} \{.3, .25, .2, .15, .1\}$

Or



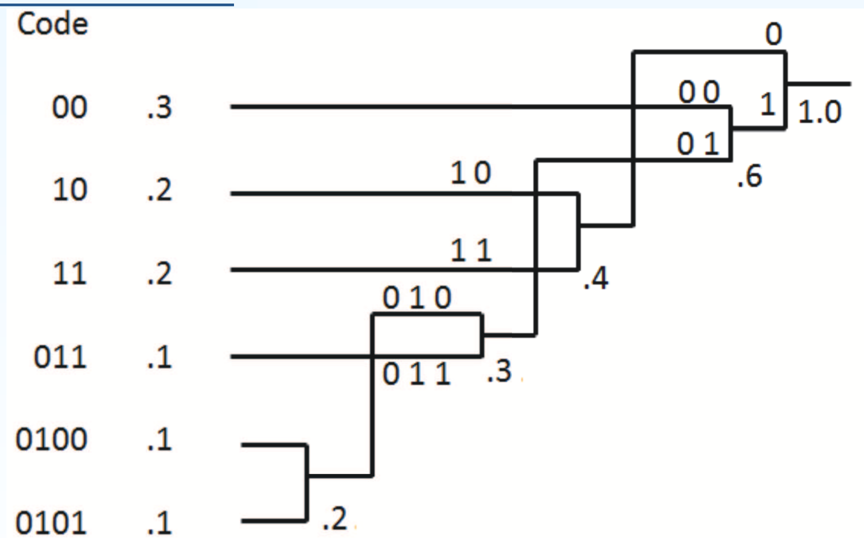

$$\overline{L} = .25 + .45 + .55 + 1 = 2.25$$

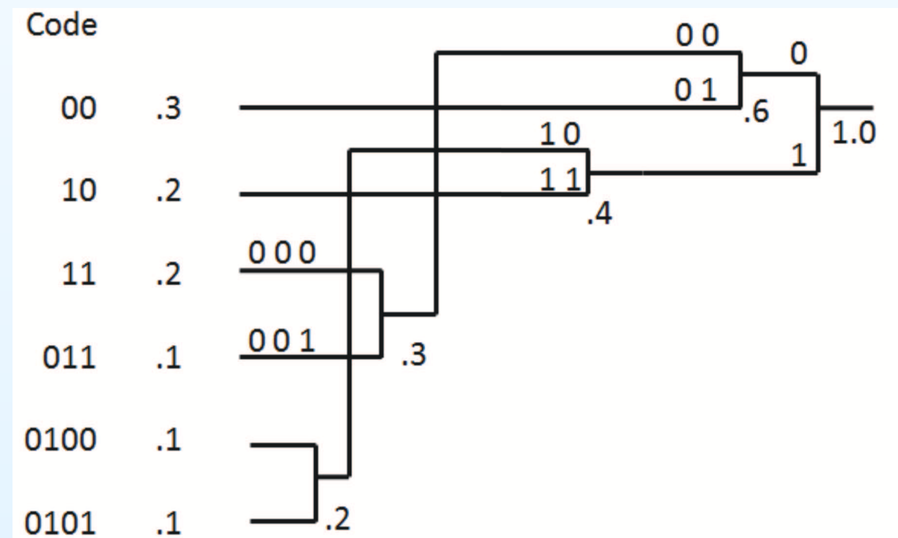
# Huffman Coding: Tie Breaks



In the last example, the two ways of breaking the tie led to two different codes with the same set of code lengths. This is not always the case — Sometimes we get different codes with different code lengths.

EXAMPLE:



# Huffman Coding: Optimal Average Length

- Binary Huffman Code will have the shortest average length as compared with any U.D. Code for set of probabilities (No U.D. will have a shorter average length).
  - The proof that a *Binary Huffman Code* is optimal — that is, has the shortest average code word length as compared with any U.D. code for that the same set of probabilities — is omitted.
  - However, we would like to mention that the proof is based on the fact that in the process of constructing a *Huffman Code* for that set of probabilities other codes are formed for other sets of probabilities, all of which are optimal.

## Shannon-Fano Codes

SHANNON - FANO CODES is another binary coding technique to construct U.D. codes (not necessarily optimum!)

1. Order probabilities in decreasing order.
2. Partition into 2 sets that one as close to equally probable as possible. Label top set with a "‘0’" and bottom set with a "‘1’".
3. Continue using step 2 over and over

# Shannon-Fano Codes

SHANNON - FANO CODES is another binary coding technique to construct U.D. codes (not necessarily optimum!)

1. Order probabilities in decreasing order.
2. Partition into 2 sets that one as close to equally probable as possible. Label top set with a "'0"' and bottom set with a "'1"'.
3. Continue using step 2 over and over

$$
\begin{array}{llll}
.5 \Rightarrow & 0 & & \\
.2 \Rightarrow & 1 & 0 & \\
.15 \Rightarrow & 1 & 1 & 0 \\
.15 \Rightarrow & 1 & 1 & 1
\end{array}
\qquad
\begin{array}{llllll}
.4 \Rightarrow & 0 & & & & \\
.3 \Rightarrow & 1 & 0 & & & \\
.1 \Rightarrow & 1 & 1 & 0 & 0 & \\
.05 \Rightarrow & 1 & 1 & 0 & 1 & \\
.05 \Rightarrow & 1 & 1 & 1 & 0 & \\
.05 \Rightarrow & 1 & 1 & 1 & 1 & 0 \\
.05 \Rightarrow & 1 & 1 & 0 & 1 & 1
\end{array}
$$

$$\overline{L} = ? \qquad \overline{L} = 2.3$$

Compare with Huffman coding. Same length as Huffman code?!

# More Examples

# Binary Huffman Codes

Construct binary Huffman and Shannon-Fano codes where:

EXAMPLE 1: $(p_1, p_2, p_3, p_4) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8})$

EXAMPLE 2: Consider the examples on the previous slide and construct binary Huffman codes.

# Large Alphabet Size



Example 3: Consider a binary Source $\{A, B\}$ $(p_1, p_2) = (.9, .1)$. Now construct a series of *Huffman Codes* and series of Shannon-Fano Codes, by encoding $N$ source symbols at a time for $N = 1, 2, 3, 4$.

# Shannon-Fano codes are suboptimal !



Example 3: Construct a Shannon-Fano code:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | $2 \times .25$ | 0 | 0 | | | | |
| | $2 \times .20$ | 0 | 1 | | | | |
| $.6$ | $3 \times .15$ | 1 | 0 | 0 | | | |
| $.7$ | $3 \times .10$ | 1 | 0 | 1 | | | |
| $.75$ | $4 \times .05$ | 1 | 1 | 0 | 0 | | |
| $.8$ | $5 \times .05$ | 1 | 1 | 0 | 1 | 0 | |
| $.85$ | $5 \times .05$ | 1 | 1 | 0 | 1 | 1 | |
| $.9$ | $4 \times .05$ | 1 | 1 | 1 | 0 | | |
| | $5 \times .04$ | 1 | 1 | 1 | 1 | 0 | |
| | $6 \times .03$ | 1 | 1 | 1 | 1 | 1 | 0 |
| | $6 \times .03$ | 1 | 1 | 1 | 1 | 1 | 1 |

$$\overline{L} = 3.11$$

Compare this with a binary Huffman Code.