Digital Communications III (ECE 154C)
Introduction to Coding and Information Theory

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These lecture notes were originally developed by late Prof. J. K. Wolf.
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Asymptotic Optimality
We have seen that for any U.D. \( n \)-ary code corresponding to the \( N^{th} \) extension of the I.I.D. Source \( S \), for every \( N = 1, 2, \ldots \),

\[
\frac{L_N}{N} \geq H_n(S).
\]

→ How do you get the result for general \( N \)?

- Next we show that for an \( n \)-ary Huffman Code corresponding to the \( N^{th} \) extension of the I.I.D. Source \( S \)

\[
H_n(S) + \frac{1}{N} > \frac{L_N}{N} \geq H_n(S)
\]

- But this implies that Huffman coding is \textit{asymptotically optimal}, i.e.

\[
\lim_{N \to \infty} \frac{L_N}{N} \rightarrow H_n(S)
\]
Sketch of the Proof

- From McMillan Inequality, we have that one can construct a U.D. code such that \( l_i = \left\lceil \log_n \frac{1}{p_i} \right\rceil \). In other words, we can construct a U.D. code for which we have

\[
H_n(S) + 1 > L \geq H_n(S).
\]

- Why?
Asymptotic Optimality of Huffman Coding
Examples

- Example I
- Example II
- Example III

Optimality Proof
Huffman Codes are asymptotically optimal

Ex 1: $n = 2, (.9, .09, .01)$ \[\Rightarrow H_2(S) = .516\]

<table>
<thead>
<tr>
<th>Prob</th>
<th>$\log_2 \frac{1}{p_i}$</th>
<th>$l_i = \left\lceil \log_2 \frac{1}{p_i} \right\rceil$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>.9</td>
<td>.152</td>
</tr>
<tr>
<td>B</td>
<td>.09</td>
<td>3.47</td>
</tr>
<tr>
<td>C</td>
<td>.01</td>
<td>6.67</td>
</tr>
</tbody>
</table>

$\bar{L} = 1.33$

Note that $H_n(S) \leq \bar{L} < H_n(S) + 1$

$H_2(S) = .516 \leq 1.33 < 1.516$

A better code (this is actually a Huffman Code):

<table>
<thead>
<tr>
<th>A</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>10</td>
</tr>
<tr>
<td>C</td>
<td>11</td>
</tr>
</tbody>
</table>

$\bar{L} = 1.1$
Huffman Codes are asymptotically optimal

Ex 2: $n = 2, (.19, .19, .19, .19, .19.05) \Rightarrow H_2(S) = 2.492$

\[
\begin{array}{|c|c|c|}
\hline
\text{Prob} & \log_2 \frac{1}{p_i} & l_i = \left\lceil \log_2 \frac{1}{p_i} \right\rceil \\
\hline
A & .19 & 2.396 & 3 \\
B & .19 & 2.396 & 3 \\
C & .19 & 2.396 & 3 \\
D & .19 & 2.396 & 3 \\
E & .19 & 2.396 & 3 \\
F & .05 & 4.322 & 5 \\
\hline
\end{array}
\]

Huffman Code:

- A 00
- D 101
- B 01
- E 110
- C 100
- F 111

$\bar{L} = 3.10$

$\bar{L} = 2.62$
Huffman Codes are asymptotically optimal

### Example

**Ex 3:** \( n = 3, (0.19, 0.19, 0.19, 0.19, 0.05) \Rightarrow H_2(S') = 2.492 \)

<table>
<thead>
<tr>
<th>Prob ( p_i )</th>
<th>( \log_3 \frac{1}{p_i} )</th>
<th>( l_i = \left\lceil \log_3 \frac{1}{p_i} \right\rceil )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A 0.19</td>
<td>1.51</td>
<td>2</td>
</tr>
<tr>
<td>B 0.19</td>
<td>1.51</td>
<td>2</td>
</tr>
<tr>
<td>C 0.19</td>
<td>1.51</td>
<td>2</td>
</tr>
<tr>
<td>D 0.19</td>
<td>1.51</td>
<td>2</td>
</tr>
<tr>
<td>E 0.19</td>
<td>1.51</td>
<td>2</td>
</tr>
<tr>
<td>F 0.05</td>
<td>2.73</td>
<td>3</td>
</tr>
</tbody>
</table>

**Huffman Code:**

- A 0
- B 10
- C 11
- D 12
- E 20
- F 21

\( \overline{L} = 1.57 \)

\( \overline{L} = 1.81 \)
Performance of Huffman Codes
Asymptotic Optimality of Huffman Codes

Theorem: For an I.I.D. source $S$, a Huffman Code exists with code alphabet size $n$ for the $N^{th}$ extension of this source such that

$$H_n(S) \leq \frac{L_N}{N} < H_n(S) + 1$$

Proof: Earlier (slide 3) we have seen that a U.D. code exists for the $N^{th}$ extension of the source such that

$$H_n(S^N) \leq \frac{L_N}{N} < H_n(S^N) + 1$$

But $H_n(S^N) = NH_n(S)$. So all we will need is to show that for a given fixed source with fixed alphabet Huffman Code is at least as good as the U.D. code.

Q.E.D.

Next we prove the fact we have stated without proof that, for any fixed and given alphabet, a Huffman code is optimal.
We only state the results in case of binary codes, even though similar lines of arguments can be used to establish $n$-ary codes. We only need to consider instantaneous codes!

1. If $p_i \leq p_j$, then $l_i \geq l_j$
   **Proof:** Otherwise switching code words will reduce $\overline{L}$.

2. There is no single code word of length $l^* \leq \max l_i$.
   **Proof:** If there were shorten it by one digit, it will still not be a prefix of any other code word and will shorten $\overline{L}$.

3. The code words of length $l^*$, they occur in pairs in which the code words in each pair agree in all but the last digit. **Proof:** If not, shorten the code word for which is not the case by one digit and it will not be the prefix of any other code word. This will shorten $\overline{L}$. 

Properties of an Optimal (Compact) Binary Code
Optimality of Binary Huffman Codes

\[ \overline{L}_{j-1} = \overline{L}_j + P_{\alpha 1} + P_{\alpha 2} \] since the code at \((j - 1)\) is same as the code at \((j)\) except for two words that have length one more.

We now show that if code \(C_j\) is optimal, then code \(C'_{j-1}\) must also be optimal. Suppose not; and there were a better code at \((j - 1)\). Call it's average length \(\overline{L'}_{j-1} < \overline{L}_{j-1}\). But the two code words with probabilities \(P_{\alpha 0}\) & \(P_{\alpha 1}\) are identical in all but the last digit. Form a new code at \(j\) that has the identical prefix as the code word for \(P_{\alpha}\). This code will have average length \(\overline{L'}_j = \overline{L'}_{j-1} - (P_{\alpha 1} + P_{\alpha 2})\) so that \(\overline{L'}_{j-1} = \overline{L}_{j-1}\). But this can't be the case if \(C_j\) was optimal.