Digital Communications III (ECE 154C)
Introduction to Coding and Information Theory

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Spring 2014
Coding With Distortion
Coding with Distortion

\[ \epsilon^2 = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \mathbb{E}(x(t) - \hat{x}(t))^2 dt = M.S.E. \]
Discrete-Time Signals

If signals are bandlimited, one can sample at nyquist rate and convert continuous-time problem to discrete-time problem. This sampling is part of the A/D converter.

\[
\epsilon^2 = \lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} \mathbb{E}(x_i - \hat{x}_i)^2
\]
A/D Conversion and D/A Conversion
A/D Conversion

- Assume a random variable $X$ which falls into the range $(X_{\text{min}}, X_{\text{max}})$.
- The goal is for $X$ to be converted into $k$ binary digits. Let $M = 2^k$.
- The usual A/D converter first subdivides the interval $(X_{\text{min}}, X_{\text{max}})$ into $M$ equal sub-intervals.
- Subintervals are of width $\Delta = (X_{\text{max}} - X_{\text{min}})/M$.
- Shown below for the case of $k = 3$ and $M = 8$.

- We call the $i^{th}$ sub-interval, $\mathcal{R}_i$. 
D/A Conversion

- Assume that if $X$ falls in the region $\mathbb{R}_i$, i.e. $x \in \mathbb{R}_i$
- D/A converter uses as an estimate of $X$, the value $\hat{X} = Y$ which is the center of the $i^{th}$ region.
- The mean-squared error between $X$ and $\hat{X}$ is

$$
\epsilon^2 = \mathbb{E}[(X - \hat{X})^2] = \int_{X_{\min}}^{X_{\max}} (X - \hat{X})^2 f_x(x) \, dx
$$

where $f_x(x)$ is the probability density function of the random variable $X$.
- Let $f_X|\mathbb{R}_i(x)$ be the conditional density function of $X$ given that $X$ falls in the region $\mathbb{R}_i$. Then

$$
\epsilon^2 = \sum_{i=1}^{M} P[x \in \mathbb{R}_i] \int_{x \in \mathbb{R}_i} (x - y_i)^2 f_{x|\mathbb{R}_i}(x) \, dx
$$
D/A Conversion

- Note that for \( i = 1, 2, \ldots, M \)
  \[
  \sum_{i=1}^{M} P[x \in \mathcal{R}_i] = \Delta \quad \text{and} \quad \int_{x \in \mathcal{R}_i} f_{X|\mathcal{R}_i}(x)\,dx = 1.
  \]

- Make the further assumption that \( k \) is large enough so that \( f_{X|\mathcal{R}_i}(x) \) is a constant over the region \( \mathcal{R}_i \).

- Then \( f_{X|\mathcal{R}_i}(x) = \frac{1}{\Delta} \) for all \( i \), and
  \[
  \int_{x \in \mathcal{R}_i} (x - y_i)^2 f_{X|\mathcal{R}_i}(x)\,dx = \frac{1}{\Delta} \int_{a}^{b} (x - \left(\frac{b - a}{2}\right))^2\,dx
  \]
  \[
  = \frac{1}{\Delta} \int_{-\Delta/2}^{\Delta/2} (x - 0)^2\,dx
  \]
  \[
  = \frac{1}{\Delta} \cdot \frac{2}{3} \cdot \left(\frac{\Delta}{2}\right)^3 = \frac{\Delta^2}{12}
  \]
D/A Conversion

- In other words, $\epsilon^2 = \sum_{i=1}^{M} P[x \in \mathcal{R}_i] \cdot \frac{\Delta^2}{12} = \frac{\Delta^2}{12}$

- If $X$ has variance $\sigma_x^2$, the signal-to-noise ratio of the A to the D (& D to A) converter is often defined as $\left(\frac{\sigma_x^2}{\frac{\Delta^2}{12}}\right)$

- If $X_{\text{min}}$ is equal to $-\infty$ and/or $X_{\text{max}} = +\infty$, then the last and first intervals can be infinite in extent.

- However $f_x(x)$ is usually small enough in those intervals so that the result is still approximately the same.
SCALAR QUANTIZATION of
GAUSSIAN SAMPLES
Scalar Quantization

**ENCODER:**

| $x \leq -3b$ | 000 | $0 < x < b$ | 100 |
| $-3b < x \leq -2b$ | 001 | $b < x \leq 2b$ | 101 |
| $-2b \leq x \leq -b$ | 010 | $2b < x \leq 3b$ | 110 |
| $-b \leq x \leq 0$ | 011 | $3b < x$ | 111 |

**DECODER:**

| 000 | $-3.5b$ | 100 | +.5b |
| 001 | $-2.5b$ | 101 | +1.5b |
| 010 | $-1.5b$ | 110 | +2.5b |
| 011 | $-.5b$ | 111 | +3.5b |
Optimum Scalar Quantizer

- Let us construct boundaries $b_i$, $(b_0 = -\infty, b_M = +\infty$ and quantization symbols $a_i$ such that

$$b_{i-1} \leq x < b_i \rightarrow \hat{x} = a_i \quad i = 1, 2, ..., M$$

- The question is how to optimize $\{b_i\}$ and $\{a_i\}$ to minimize distortion $\varepsilon^2$

$$\varepsilon^2 = \sum_{i=1}^{M} \int_{b_{i-1}}^{b_i} (x - a_i)^2 f_x(x) \, dx$$
Optimum Scalar Quantizer

- To optimize $\epsilon^2 = \sum_{i=1}^{M} \int_{b_{i-1}}^{b_i} (x - a_i)^2 f_x(x) dx$ we take derivatives and putting them equal to zero:

$$\frac{\delta \epsilon^2}{\delta a_j} = 0$$

$$\frac{\delta \epsilon^2}{\delta b_j} = 0$$

- And use Leibnitz’s Rule:

$$\frac{\delta}{\delta t} \int_{a(t)}^{b(t)} f(x, t) dx = f(b(t), t) \frac{\delta b(t)}{\delta t}$$

$$- f(a(t), t) \frac{\delta a(t)}{\delta t}$$

$$+ \int_{a(t)}^{b(t)} \frac{\delta}{\delta t} f(x, t) dt$$
Optimum Scalar Quantizer

\[
\frac{\delta}{\delta b_j} \left( \sum_{i=1}^{M} \int_{b_{i-1}}^{b_i} (x - a_i)^2 f_x(x) \, dx \right) = \\
\frac{\delta}{\delta b_j} \int_{b_{j-1}}^{b_j} (x - a_j)^2 f_x(x) \, dx + \frac{\delta}{\delta b_j} \int_{b_j}^{b_{j+1}} (x - a_{j+1})^2 f_x(x) \, dx \\
= (b_j - a_j)^2 f_x(x) \bigg|_{x=b_j} - (b_j - a_{j+1})^2 f_x(x) \bigg|_{x=b_j} = 0
\]

\[
b_j^2 - 2a_j b_j + a_j^2 = b_j^2 - 2b_j a_{j+1} + a_{j+1}^2 \\
2b_j (a_{j+1} - a_j) = a_{j+1}^2 - a_j^2
\]

\[
b_j = \frac{a_{j+1} + a_j}{2} \quad (1)
\]
Optimum Scalar Quantizer

\[
\frac{\delta}{\delta a_j} \left( \sum_{i=1}^{M} \int_{b_{i-1}}^{b_i} b_i(x - a_i)^2 f_x(x) \, dx \right) = -2 \int_{b_{j-1}}^{b_j} (x - a_j) f_x(x) \, dx = 0
\]

\[
a_j \int_{b_{j-1}}^{b_j} f_x(x) \, dx = \int_{b_{j-1}}^{b_j} x f_x(x) \, dx
\]

\[
a_j = \frac{\int_{b_{j-1}}^{b_j} x f_x(x) \, dx}{\int_{b_{j-1}}^{b_j} f_x(x) \, dx} \quad (II)
\]
Optimum Scalar Quantizer

- Note that the \( \{b_i\} \) can be found from (I) once the \( \{a_i\} \) is known.
  - In fact, the \( \{b_i\} \) are the midpoints of the \( \{a_i\} \).
- The \( \{a_i\} \) can also be solved from (II) once the \( \{b_i\} \) are known.
  - The \( \{a_i\} \) are centroids of the corresponding regions.
- Thus one can use a computer to iteratively solve for the \( \{a_i\} \) and the \( \{b_i\} \)
  1. One starts with an initial guess for the \( \{b_i\} \).
  2. One uses (II) to solve for the \( \{a_i\} \).
  3. One uses (I) to solve for the \( \{b_i\} \).
  4. One repeats steps 2 and 3 until the \( \{a_i\} \) and the \( \{b_i\} \) "stop changing".
Comments on Optimum Scalar Quantizer

1. This works for any $f_x(x)$

2. If $f_x(x)$ only has a finite support one adjusts $b_0$ & $b_M$ to be the limits of the support.

3. One needs to know $\sum_{\alpha} f_x(x) dx$ and $\sum_{\alpha} x f_x(x) dx$ (true for any $f_x(x)$)

4. For a Gaussian, we can integrate by parts or let $y = x^2$

$$\int_{\alpha}^{\beta} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} x^2} dx = Q(\beta) - Q(\alpha) \int_{\alpha}^{\beta} x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} x^2} dx = ...$$
Comments on Optimum Scalar Quantizer

5. If $M = 2^a$ one could use $a$ binary digits to represent the quantized value. However since the quantized values are not necessarily equally likely, one could use a HUFFMAN CODE to use fewer binary digits (on the average).

6. After the $\{a_i\}$ and $\{b_i\}$ are known, one computes $\epsilon^2$ from

$$
\epsilon^2 = \sum_{i=1}^{M} \int_{b_{i-1}}^{b_i} (x - a_i)^2 f_x(x) dx
$$

7. For $M = 2$ and $f_x(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2} \frac{x^2}{\sigma^2}}$ we have

$$
b_0 = -\infty, b_1 = 0, b_2 = +\infty, \text{ and } a_2 = -a_1 = \sqrt{\frac{2\sigma^2}{\pi}}
$$

8. Also easy to show that $\epsilon^2 = (1 - \frac{2}{\pi})\sigma^2 = .3634\sigma^2$. 

Vector Quantization
Vector Quantization

- One can achieve a smaller $\epsilon^2$ by quantizing several samples at a time.
- We would then use regions in an $m$-dimensional space.

Shannon characterized this in terms of "rate-distortion formula" which tells us how small $\epsilon^2$ can be ($m \to \infty$).

- For a Gaussian source with one binary digit per sample,
  $\epsilon^2 \geq \frac{\sigma^2}{4} = 0.25\sigma^2$
  - This follows from the result on the next page.
  - Contrast this with scalar case: $\epsilon_s^2 = (1 - \frac{2}{\pi})\sigma^2 = 0.3634\sigma^2$. 

VQ: Discrete Memoryless Gaussian Source

- Let source produce i.i.d. Gaussian samples $X_1, X_2, \ldots$ where
  \[
  f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2} \frac{x^2}{\sigma^2}}
  \]

- Let the source encoder produce a sequence of binary digits at a rate of $R$ binary digits/source symbol.
  - In our previous terminology $R = \log M$

- Let the source decoder produce the sequence $\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_i, \ldots$ such that the mean-squared error between $\{X_i\}$ and $\{\hat{X}_i\}$ is $\epsilon^2$.
  \[
  \epsilon^2 = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\{ (X_i - \hat{X}_i)^2 \}
  \]
VQ: Discrete Memoryless Gaussian Source

- Then one can prove that for any such system

$$R \geq \frac{1}{2} \log_2 \left( \frac{\sigma^2}{\epsilon^2} \right) \quad \text{for} \quad \epsilon^2 \leq \sigma^2$$

  - Note that $R = 0$ for $\epsilon^2 \geq \sigma^2$. What does this mean?
  - Note that for $R = \log M = 1$,

$$1 \geq \frac{1}{2} \log_2 \left( \frac{\sigma^2}{\epsilon^2} \right) \Rightarrow 2 \geq \log_2 \left( \frac{\sigma^2}{\epsilon^2} \right)$$

$$\Rightarrow 4 \geq \frac{\sigma^2}{\epsilon^2} \Rightarrow \epsilon^2 \geq (1/4)\sigma^2$$

- This is an example of “Rate-Distortion Theory.”
Reduced Fidelity Audio Compression

- MP3
- CD
- MPEG-1 Layer 3

Lossy Source Coding
A/D & D/A
Scalar Quantization
Vector Quantization
Audio Compression
MP3 players use a form of audio compression called MPEG-1 Audio Layer 3.

It takes advantage of a psycho-acoustic phenomena whereby
- a loud tone at one frequency “masks” the presence of softer tones at neighboring frequencies;
- hence, these softer neighbouring tones need not be stored (or transmitted).

Compression efficiency of an audio compression scheme is usually described by the encoded bit rate (prior to the introduction of coding bits.)
### Reduced Fidelity Audio Compression

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- The CD has a bit rate of \((44.1 \times 10^3 \times 2 \times 16) = 1.41 \times 10^6\) bits/second.
  - The term \(44.1 \times 10^3\) is the sampling rate which is approximately the Nyquist frequency of the audio to be compressed.
  - The term 2 comes from the fact that there are two channels in a stereo audio system.
  - The term 16 comes from the 16-bit (or \(2^{16} = 65536\) level) A to D converter.
  - Note that a slightly higher sampling rate \(48 \times 10^3\) samples/second is used for a DAT recorder.
Reduced Fidelity Audio Compression

- Different standards are used in MP3 players.
- Several bit rates are specified in the MPEG-1, Layer 3 standard.
  - These are 32, 40, 48, 56, 64, 80, 96, 112, 128, 144, 160, 192, 224, 256 and 320 kilobits/sec.
  - The sampling rates allowed are 32, 44.1 and 48 kiloHz but the sampling rate of $44.1 \times 10^3$ Hz is almost always used.
- The basic idea behind the scheme is as follows.
  - A block of 576 time domain samples are converted into 576 frequency domain samples using a DFT.
  - The coefficients then modified using psycho-acoustic principles.
  - The processed coefficients are then converted into a bit stream using various schemes including Huffman Encoding.
  - The process is reversed at the receiver: bits $\rightarrow$ frequency domain coefficients $\rightarrow$ time domain samples.