Solutions to Exercise Set #4

1. Suppose that a pair of dice are thrown, each independently taking on the values 
\( \{1, 2, 3, 4, 5, 6\} \) with equal probabilities. Let \( X \) denote the output of the first die and \( Y \) denote the output of the second die. Let \( Z = X + Y \) and \( Q = Z \mod 2 \). Compute the following entropies.

(a) \( H(X), H(Y), H(Z) \).
(b) \( H(X, Z), H(Z|X), H(X|Z) \).
(c) \( H(Q, X), H(Q|X), H(X|Q) \).
(d) \( H(Q, Z), H(Q|Z), H(Z|Q) \).
(e) \( H(X, Y, Z), H(Z|X, Y), H(X, Y|Z) \).
(f) \( H(X, Y, Z, Q), H(X, Y, Z|Q), H(Q|X, Y, Z) \).

Solution:

(a) We have \( H(X) = H(Y) = \log 6 = 2.585 \). By considering all the combinations of \( X \) and \( Y \), we have

\[
H(Z) = \frac{6}{36} \log 6 + 2 \left( \frac{1}{36} \log 36 + \frac{2}{36} \log 18 + \frac{3}{36} \log 12 + \frac{4}{36} \log 9 + \frac{5}{36} \log \frac{36}{5} \right) = 3.27.
\]

(b) We have

\[
H(X|Z) = H(X, Z) - H(Z) = H(Z|X) + H(X) - H(Z) = H(X + Y|X) + H(X) - H(Z) = H(Y) + H(X) - H(Z) = 1.9.
\]

Hence

\[
H(X, Z) = H(X|Z) + H(Z) = 5.17
\]

and

\[
H(Z|X) = H(X, Z) - H(X) = 2.585.
\]
(c) Since given $X$ is even, the binary random variable $Q = 0$ if $Y$ is even and $Q = 1$ if $Y$ is odd. Similarly for the case when $X$ is odd. Thus $H(Q|X) = 1$. Note that since $H(Q) = 1$, $X$ and $Q$ are independent of each other. Thus

$$H(Q, X) = H(Q|X) + H(X) = 3.585$$

and

$$H(X|Q) = H(Q, X) - H(Q) = 2.585.$$ 

(d) Since $Q$ is a function of $Z$, we have $H(Q|Z) = 0$. Thus

$$H(Z, Q) = H(Q|Z) + H(Z) = H(Z) = 3.274$$

and

$$H(Z|Q) = H(Z, Q) - H(Q) = 2.274.$$ 

(e) Since $Z = X + Y$, we have $H(Z|X, Y) = 0$ and $H(Y|X, Z) = 0$. Note that $H(X, Y) = H(X) + H(Y)$ since $X$ and $Y$ are independent, hence

$$H(X, Y, Z) = H(Z|X, Y) + H(X, Y) = H(X, Y) = H(X) + H(Y) = 5.17$$

and


(f) Since $H(Q|X, Y, Z) = 0$, we have


Thus

$$H(X, Y, Z|Q) = H(X, Y, Z, Q) - H(Q) = 5.17 - 1 = 4.17.$$ 

2. Consider the cascade of two binary channels shown below.

(a) Draw a single channel with binary input and binary output that is equivalent to this cascade.

(b) Repeat part (a) when the order of the two channels is reversed.

(c) Show that a cascade of two binary symmetric channels is also a binary symmetric channel.
Solution:

(a) The cascaded channel is the following

\[
1 - e = p(Y_2 = 1|X_1 = 1) = (1 - a)(1 - c) + ad, \\
e = p(Y_2 = 0|X_1 = 1) = (1 - a)c + a(1 - d), \\
f = p(Y_2 = 1|X_1 = 0) = (1 - b)d + b(1 - c), \\
1 - f = p(Y_2 = 0|X_1 = 0) = (1 - b)(1 - d) + bc.
\]

(b) The cascaded channel is the following

\[
1 - e' = p(Y_1 = 1|X_2 = 1) = (1 - c)(1 - a) + cb, \\
e' = p(Y_1 = 0|X_2 = 1) = c(1 - b) + (1 - c)a, \\
f' = p(Y_1 = 1|X_2 = 0) = d(1 - a) + (1 - d)b, \\
1 - f' = p(Y_1 = 0|X_2 = 0) = (1 - d)(1 - b) + da.
\]

(c) Let \( a = b \) and \( c = d \) in part (a), it is easy to see that the cascaded channel is a binary symmetric channel. Similarly, by letting \( a = b \) and \( c = d \) in part (b), we get a binary symmetric channel as well. Thus, the order of cascade doesn’t matter.