Weekly Exercise Set #1

1. Using the definition of $\sigma$-algebra, show that $A_1, A_2, \ldots \in \mathcal{F}$ implies that $\cap_{i=1}^{\infty} A_i \in \mathcal{F}$.

2. Using the axioms of probability, show that $A \subseteq B$ implies that $P(A) \leq P(B)$.

3. Independence. Show that the events $A$ and $B$ are independent if $P(A|B) = P(A|B^c)$.

4. Conditional probabilities. Let $P(A) = 0.8$, $P(B^c) = 0.6$, and $P(A \cup B) = 0.8$. Find
   (a) $P(A^c|B^c)$.
   (b) $P(B^c|A)$.

5. Let $A, B$ be two events with $P(A) \geq 0.5$ and $P(B) \geq 0.75$. Show that $P(A \cap B) \geq 0.25$.

6. Two distinguishable dice are tossed. The number of dots facing up on each die is counted and noted, recording the number of dots on them.
   (a) Find the sample space.
   (b) Find the set $A$ corresponding to the event that the total number of dots showing is even.
   (c) Find the set $B$ corresponding to the event that both dice are even.
   (d) Does $A$ imply $B$ or does $B$ imply $A$? Find $A \cap B^c$ and describe this event in words.
   (e) Let $C$ be the event that the number of dots on the two dice differ by one. Find $A \cap C$.

7. Monty Hall. Gold is placed behind one of three curtains. A contestant chooses one of the curtains, Monty Hall (the game host) opens one of the unselected empty curtains. The contestant has a choice either to switch his selection to the third curtain or not.
   (a) What is the sample space for this random experiment? (Hint: An outcome consists of the curtain with gold, the curtain chosen by the contestant, and the curtain chosen by Monty.)
   (b) Assume that placement of the gold behind the three curtains is random, the contestant choice of curtains is random and independent of the gold placement, and that Monty Hall’s choice of an empty curtain is random among the alternatives. Specify the probability measure for this random experiment and use it to compute the probability of winning the gold if the contestant decides to switch.

8. Negative evidence. Suppose that the evidence of an event $B$ increases the probability of a criminal’s guilt; that is, if $A$ is the event that the criminal is guilty, then $P(A|B) \geq P(A)$. Does the absence of the event $B$ decrease the criminal’s probability of being guilty? In other words, is $P(A|B^c) \leq P(A)$? Prove or provide a counterexample.
9. **Binary communication channel.** Consider the binary communication channel discussed in class with \( p(1|0) = 0.1 \) and \( p(0|1) = 0.2 \). Assume that the inputs are equiprobable.

(a) Find the probability that the output is 0.
(b) Find the probability that the input was 0 given that the output is 1.

10. **Ternary communication channel.** Consider a communication channel with three inputs and three outputs, depicted in the Figure 1. Suppose that the input symbols 0,1, and 2 occur with probability 1/2, 1/4, and 1/4, respectively.

(a) Find the probabilities of the output symbols.
(b) Suppose that a 1 is observed as an output. What is the probability that the input was 0? 1? 2?

Your answers should be in terms of the conditional error probability \( \epsilon \).

![Figure 1: Ternary communication channel.](image)

11. **Geometric pairs.** Consider a probability space consisting of the sample space

\[
\Omega = \{1, 2, 3, \ldots\}^2 = \{(i, j) : i, j \in \mathbb{N}\},
\]

i.e., all pairs of positive integers, where the set of events is the power set of \( \Omega \) and the probability measure on points in the sample space is

\[
P((i, j)) = p^2(1 - p)^{i+j-2}, \quad 0 < p < 1.
\]

(a) Find \( P(\{(i, j) : i \geq j\}) \).
(b) Find \( P(\{(i, j) : i + j = k\}) \).
(c) Find \( P(\{(i, j) : i \text{ is an odd number}\}) \).
(d) Describe an experiment whose outcomes \((i, j)\) have the probabilities \( P((i, j)) \).
12. Juror’s fallacy. Suppose that $P(A|B) \geq P(A)$ and $P(A|C) \geq P(A)$. Is it always true that $P(A|B, C) \geq P(A)$? Prove or provide a counterexample.

13. Polya’s urn. Suppose we have an urn containing one red ball and one blue ball. We draw a ball at random from the urn. If it is red, we put the drawn ball plus another red ball into the urn. If it is blue, we put the drawn ball plus another blue ball into the urn. We then repeat this process. At the $n$-th stage, we draw a ball at random from the urn with $n + 1$ balls, note its color, and put the drawn ball plus another ball of the same color into the urn.

(a) Find the probability that the first ball is red.
(b) Find the probability that the second ball is red.
(c) Find the probability that the first three balls are all red.
(d) Find the probability that two of the first three balls are red.