Solutions to Exercise Set #2

1. Probabilities from a cdf. Let $X$ be a random variable with the cdf shown below.

\[ F(x) \]

Find the probabilities of the following events.

(a) $\{X = 2\}$.
(b) $\{X < 2\}$.
(c) $\{X = 2\} \cup \{0.5 \leq X \leq 1.5\}$.
(d) $\{X = 2\} \cup \{0.5 \leq X \leq 3\}$.

Solution:

(a) There is a jump at $X = 2$, so we have

\[
P\{X = 2\} = P\{X \leq 2\} - P\{X < 2\} = F(2) - F(2^-) = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.
\]

(b) $P\{X < 2\} = F(2^-) = \frac{1}{3}$. 
(c) since \{X = 2\} and \{0.5 \leq X \leq 1.5\} are two disjoint events,
\[
P(\{X = 2\} \cup \{0.5 \leq X \leq 1.5\}) = P\{X = 2\} + P\{0.5 \leq X \leq 1.5\}
\]
\[
= \frac{1}{3} + F(1.5) - F(0.5^-)
\]
\[
= \frac{1}{3} + \frac{1}{3} - \frac{1}{3} \times 0.5^2
\]
\[
= \frac{7}{12}.
\]

(d) We have
\[
P(\{X = 2\} \cup \{0.5 \leq X \leq 3\}) = P\{0.5 \leq X \leq 3\} - P\{X = 2\}
\]
\[
= F(3) - F(0.5^-)
\]
\[
= \frac{5}{6} - \frac{1}{3} \times 0.5^2
\]
\[
= \frac{3}{4}.
\]

2. Gaussian probabilities. Let \(X \sim N(1000, 400)\). Express the following in terms of the \(Q\) function.

(a) \(P\{0 < X < 1020\}\).

(b) \(P\{X < 1020|X > 960\}\).

Solution: Using the fact that \(\frac{X-\mu}{\sigma} \sim N(0,1)\), thus \(F(x) = \Phi(\frac{x-\mu}{\sigma}) = 1 - Q(\frac{x-\mu}{\sigma})\).

(a) We have
\[
P\{0 < X < 1020\} = Q\left(\frac{0 - 1000}{20}\right) - Q\left(\frac{1020 - 1000}{20}\right) = Q(-50) - Q(1).
\]

(b) We have
\[
P\{X < 1020|X > 960\} = \frac{P\{960 < X < 1020\}}{P\{X > 960\}}
\]
\[
= \frac{Q\left(\frac{960 - 1000}{20}\right) - Q\left(\frac{1020 - 1000}{20}\right)}{Q\left(\frac{960 - 1000}{20}\right)}
\]
\[
= \frac{Q(-2) - Q(1)}{Q(-2)}.
\]
3. **Laplacian.** Let $X \sim f(x) = \frac{1}{2}e^{-|x|}$.

(a) Sketch the cdf of $X$.

(b) Find $P\{|X| \leq 2 \text{ or } X \geq 0\}$.

(c) Find $P\{|X| + |X - 3| \leq 3\}$.

(d) Find $P\{X \geq 0 \mid X \leq 1\}$.

**Solution:**

(a) We have

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{2}e^{-|u|} \, du = \begin{cases} \frac{1}{2}e^{x}, & \text{if } x < 0 \\ 1 - \frac{1}{2}e^{-x}, & \text{if } x \geq 0 \end{cases}.$$ 

![CDF of X](image)

**Figure 1: cdf of X**

(b) We have

$$P\{|X| \leq 2 \text{ or } X \geq 0\} = P\{X \geq -2\} = 1 - P\{X < -2\} = 1 - \int_{-\infty}^{-2} \frac{1}{2}e^{-|x|} \, dx = 1 - \frac{1}{2}e^{-2}.$$
(c) We have
\[ P\{|X| + |X - 3| \leq 3\} = P\{0 \leq X \leq 3\} \]
\[ = \int_0^3 \frac{1}{2} e^{-|x|} \, dx \]
\[ = \frac{1}{2} - \frac{1}{2} e^{-3}. \]

(d) We have
\[ P\{X \geq 0 \mid X \leq 1\} = \frac{P\{0 \leq X \leq 1\}}{P\{X \leq 1\}} = \frac{F_X(1) - F_X(0^-)}{F_X(1)} = \frac{1/2 - 1/2e^{-1}}{1 - 1/2e^{-1}} = \frac{1 - e^{-1}}{2 - e^{-1}}. \]

4. Lognormal distribution. Let \( X \sim N(0, \sigma^2) \). Find the pdf of \( Y = e^X \) (known as the lognormal pdf).

Solution: \( Y = e^X > 0 \) implies \( f_Y(y) = 0 \) if \( y \leq 0 \). For \( y > 0 \)
\[ P(Y \leq y) = P(e^X \leq y) = P(X \leq \ln(y)) = F_X(\ln(y)) \]
taking derivative with respect to \( y \),
\[ f_Y(y) = \frac{1}{y} f_X(\ln(y)) = \frac{1}{y} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(\ln(y))^2}{2\sigma^2}} \quad \text{for} \; y > 0. \]

5. Distance to the nearest star. Let the random variable \( N \) be the number of stars in a region of space of volume \( V \). Assume that \( N \) is a Poisson r.v. with pmf
\[ p_N(n) = \frac{e^{-\rho V} (\rho V)^n}{n!}, \quad \text{for} \; n = 0, 1, 2, \ldots, \]
where \( \rho \) is the "density" of stars in space. We choose an arbitrary point in space and define the random variable \( X \) to be the distance from the chosen point to the nearest star. Find the pdf of \( X \) (in terms of \( \rho \)).

Solution: The trick in this problem, as in many others, is to find a way to connect events regarding \( X \) with events regarding \( N \). In our case, for \( x \geq 0 \):
\[ F_X(x) = P\{X \leq x\} \]
\[ = 1 - P\{X > x\} \]
\[ = 1 - P\{\text{No stars within distance} \; x\} \]
\[ = 1 - P\{N = 0 \; \text{in sphere centered at origin of radius} \; x\} \]
\[ = 1 - e^{-\rho \frac{4}{3} \pi x^3}. \]
Now differentiating, we get
\[ f_X(x) = 4\pi \rho x^2 e^{-\rho x^4/\pi}. \]

For \( x < 0 \), both the cdf and the pdf are zero everywhere.

6. **Random phase signal.** Let \( Y(t) = \sin(\omega t + \Theta) \) be a sinusoidal signal with random phase \( \Theta \sim U[-\pi, \pi] \). Find the pdf of the random variable \( Y(t) \) (assume here that both \( t \) and the radial frequency \( \omega \) are constant). Comment on the dependence of the pdf of \( Y(t) \) on time \( t \).

**Solution:** We can easily see (by plotting \( y \) vs. \( \theta \)) that for \( y \in (-1, 1) \)
\[
P(Y \leq y) = P(\sin(\omega t + \Theta) \leq y) \\
= P(\sin(\Theta) \leq y) \\
= \frac{2 \left( \sin^{-1}(y) + \frac{\pi}{2} \right)}{2\pi} \\
= \frac{\sin^{-1}(y) + 1}{2}.
\]

By differentiating with respect to \( y \), we get
\[ f_Y(y) = \frac{1}{\pi \sqrt{1 - y^2}}. \]

Note that \( f_Y(y) \) does not depend on time \( t \), i.e., is time invariant (or stationary) (more on this later in the course).

7. **Quantizer.** Let \( X \sim \text{Exp}(\lambda) \), i.e., an exponential random variable with parameter \( \lambda \) and \( Y = \lfloor X \rfloor \), i.e., \( Y = k \) for \( k \leq X < k + 1, \ k = 0, 1, 2, \ldots \) Find the pmf of \( Y \). Define the quantization error \( Z = X - Y \). Find the pdf of \( Z \).

**Solution:** For \( k < 0 \), \( p_Y(k) = 0 \). Elsewhere
\[
p_Y(k) = P(Y = k) \\
= P\{k \leq X < k + 1\} \\
= F_X(k + 1) - F_X(k) \\
= \left(1 - e^{-\lambda(k+1)}\right) - \left(1 - e^{-\lambda k}\right) \\
= e^{-\lambda k} - e^{-\lambda(k+1)} \\
= e^{-\lambda k} \left(1 - e^{-\lambda}\right).
\]
Since $Z = X - Y = X - \lfloor X \rfloor$ is the fractional part of $X$, $f_Z(z) = 0$ for $z < 0$ or $z \geq 1$. For $0 \leq z < 1$, we have
\[
F_Z(z) = P(Z \leq z) = \sum_{k=0}^{\infty} P(k \leq X \leq k + z) \\
= \sum_{k=0}^{\infty} e^{-\lambda k} - e^{-\lambda (k+z)} \\
= \frac{1 - e^{-\lambda z}}{1 - e^{-\lambda}}.
\]
By differentiating with respect to $z$, we get
\[
f_Z(z) = \frac{\lambda e^{-\lambda z}}{1 - e^{-\lambda}}
\]
for $0 \leq z < 1$.
Refer to Figure 2 for a graphical explanation of the above.
a) \( f_X(x) \) and \( \Delta z \) width bands that go from \( k+z \) to \( k+z+\Delta z \). 

b) \( p_Y(k) \) and areas of regions I, II, III, IV, V, etc...
8. *Gambling*. Alice enters a casino with one unit of capital. She looks at her watch to generate a uniform random variable $U \sim \text{unif}[0, 1]$, then bets the amount $U$ on a fair coin flip. Her wealth is thus given by the r.v.

\[
X = \begin{cases} 
1 + U, & \text{with probability } 1/2, \\
1 - U, & \text{with probability } 1/2.
\end{cases}
\]

Find the cdf of $X$.

**Solution:** First note that $U \in [0, 1]$ with probability one, so $X \in [0, 2]$ with probability one.

Hence, $F_X(x) = 0$ for $x < 0$, and $F_X(x) = 1$ for $x \geq 2$.

We note that $1 - U$ also follows the uniform distribution on $[0, 1]$, while $1 + U$, which is simply a shifted version of $U$, follows the uniform distribution on $[1, 2]$. Thus, it is intuitively clear that $X \sim \text{unif}[0, 2]$. In order to formally show this, we proceed as follows.

For $0 \leq x < 1$, we have

\[
F_X(x) = P(X \leq x) = P(X \leq x \mid \text{Alice wins }) P(\text{Alice wins }) + P(X \leq x \mid \text{Alice loses }) P(\text{Alice loses })
\]

\[
= \frac{1}{2} \left[ P(1 + U \leq x) + P(1 - U \leq x) \right]
\]

\[
= \frac{1}{2} \left[ P(U \leq x - 1) + P(U \geq 1 - x) \right]
\]

\[
= \frac{1}{2} \left[ 0 + (1 - (1 - x)) \right]
\]

(since $x < 1$, we have $x - 1 < 0$ and so the first probability is zero)

\[
= \frac{x}{2}.
\]
For $1 \leq x < 2$, we have

\[
F_X(x) = P(X \leq x) = P(X \leq x | Alice \text{ wins }) P( Alice \text{ wins }) + P(X \leq x | Alice \text{ loses }) P( Alice \text{ loses })
\]

\[
= \frac{1}{2} \left[ P(1 + U \leq x) + P(1 - U \leq x) \right]
\]

\[
= \frac{1}{2} \left[ P(U \leq x - 1) + P(U \geq 1 - x) \right]
\]

\[
= \frac{1}{2} \left( x - 1 \right) + 1
\]

(since $x \geq 1$, we have $1 - x \leq 0$ and so the second probability is one)

\[
= \frac{x}{2}.
\]

Thus, $F_X(x) = \begin{cases} 
0, & x < 0 \\
\frac{x}{2}, & 0 \leq x < 2 \\
1, & x \geq 2.
\end{cases}$

Thus $X \sim \text{unif}[0, 2]$.

9. **Nonlinear processing.** Let $X \sim \text{Unif}[-1, 1]$. Define the random variable

\[
Y = \begin{cases} 
X^2 + 1, & \text{if } |X| \geq 0.5 \\
0, & \text{otherwise}
\end{cases}
\]

Find and sketch the cdf of $Y$.

**Solution:** First we note that $Y \geq 0$ and thus for $y < 0$, $F_Y(y) = P(Y \leq y) = 0$.

It can be easily shown that $|X| \sim \text{Unif}[0, 1]$.

We have $P(Y = 0) = P(|X| < 0.5) = 1/2$. 

9
For $y > 0$, we have

$$F_Y(y) = P(Y \leq y)$$
$$= P(Y = 0) + P(0 < Y \leq y)$$
$$= 1/2 + P(|X| \geq 0.5, X^2 + 1 \leq y)$$
$$= 1/2 + P(|X| \in (0.5, \sqrt{y-1}])$$

$$= \begin{cases} 
 1/2, & \sqrt{y-1} \leq 0.5 \\
 1/2 + \sqrt{y-1} - 1/2, & \text{otherwise}
\end{cases}$$

$$= \begin{cases} 
 1/2, & 0 < y < 1.25 \\
 \sqrt{y-1}, & 1.25 \leq y < 2.
\end{cases}$$

Collecting the results, we have

$$F_Y(y) = \begin{cases} 
 0, & y < 0 \\
 1/2, & 0 \leq y < 1.25 \\
 \sqrt{y-1}, & 1.25 \leq y < 2 \\
 1, & y \geq 2.
\end{cases}$$

The cdf is plotted in Figure 3

10. Geometric with conditions. Let $X$ be a geometric random variable with pmf

$$p_X(k) = p(1-p)^{k-1}, \quad k = 1, 2, \ldots.$$ 

Find and plot the conditional pmf $p_X(k|A) = P\{X = k|X \in A\}$ if:

(a) $A = \{X > m\}$ where $m$ is a positive integer.

(b) $A = \{X < m\}$.

(c) $A = \{X$ is an even number}$.$

Comment on the shape of the conditional pmf of part (a).

Solution:
Figure 2: a) pdf of X, b) pmf of Y, c) pdf of Z

Figure 3: cdf of Y
(a) We have

\[
P(A) = \sum_{n=m+1}^{\infty} p(1-p)^{n-1}
\]

\[
= \sum_{n=0}^{\infty} p(1-p)^{n+m}
\]

\[
= p(1-p)^m \sum_{n=0}^{\infty} (1-p)^n
\]

\[
= (1-p)^m.
\]

For \( k \leq m \), \( p_X(k|A) = 0 \). For \( k > m \),

\[
p_X(k|A) = P\{X = k|X > m\}
\]

\[
= \frac{P\{X = k\}}{P\{X > m\}}
\]

\[
= \frac{p(1-p)^{k-1}}{(1-p)^m}
\]

\[
= p(1-p)^{k-m-1}.
\]

(b) We have

\[
P(A) = \sum_{n=0}^{m-2} p(1-p)^n
\]

\[
= p \frac{1 - (1-p)^{m-1}}{1 - (1-p)}
\]

\[
= 1 - (1-p)^{m-1}.
\]

For \( k \geq m \) or \( k \leq 0 \), \( p_X(k|A) = 0 \). For \( 0 < jk < m \),

\[
p_X(k|A) = P\{X = k|X < m\}
\]

\[
= \frac{P\{X = k\}}{P\{X < m\}}
\]

\[
= \frac{p(1-p)^{k-1}}{1 - (1-p)^{m-1}}.
\]
(c) We have

\[
P(A) = \sum_{n \text{ even}}^\infty p(1 - p)^{n-1}
\]

\[
= \sum_{n' = 0}^\infty p(1 - p)((1 - p)^2)^n'
\]

\[
= \frac{p(1 - p)}{1 - (1 - p)^2}
\]

\[
= \frac{1 - p}{2 - p}.
\]

For \( k \) odd, \( P_X(k|A) = 0 \). For \( k \) even,

\[
p_X(k|A) = \frac{P\{X = k|X \text{ is even}\}}{P\{X = k\}}
\]

\[
= \frac{P\{X = k\}}{P\{X \text{ is even}\}}
\]

\[
= \frac{p(1 - p)^{k-1}}{P(A)}
\]

\[
= p(2 - p)(1 - p)^{k-2}.
\]

Plots are shown in Figure 4. The shape of the conditional pmf in part (a) shows that the geometric random variable is memoryless:

\[
p_X(x|X > k) = p_X(x - k), \quad \text{for } x \geq k.
\]

Note that in all three parts \( p_X(x) \) is defined for all \( x \). This is required.
Figure 4: Plots of the conditional pmf's using $p = \frac{1}{4}$ and $m = 5$. 