Solutions to Exercise Set #4

1. *Optical communication channel.* Let the signal input to an optical channel be given by

\[
X = \begin{cases} 
1 & \text{with probability } \frac{1}{2} \\
10 & \text{with probability } \frac{1}{2}.
\end{cases}
\]

The conditional pmf of the output of the channel \(Y\{|X = 1\} \sim \text{Poisson}(1)\), i.e., Poisson with intensity \(\lambda = 1\), and \(Y\{|X = 10\} \sim \text{Poisson}(10)\).

Show that the MAP rule reduces to

\[
D(y) = \begin{cases} 
1, & y < y^* \\
10, & \text{otherwise.}
\end{cases}
\]

Find \(y^*\) and the corresponding probability of error.

**Solution:** We have

\[
p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x)p_X(x)}{\sum_{x \in \mathcal{X}} p_{Y|X}(y|x)p_X(x)}.
\]

For \(x = 1\), this gives

\[
p_{X|Y}(1|y) = \frac{p_{Y|X}(y|1)p_X(1)}{\sum_{x \in \mathcal{X}} p_{Y|X}(y|x)p_X(x)}
= \frac{e^{-1}}{\sum_{x \in \mathcal{X}} p_{Y|X}(y|x)p_X(x)}
= \frac{e^{-1}}{e^{-1} + e^{-10}10^y}
= \frac{e^{-1}}{e^{-1} + e^{-10}10^y}.
\]

Similarly,

\[
p_{X|Y}(10|y) = \frac{e^{-10}10^y}{e^{-1} + e^{-10}10^y}.
\]

Thus, the MAP rule will give \(D(y) = 1\) if \(p_{X|Y}(1|y) > p_{X|Y}(10|y)\), i.e., if \(e^{-1} > e^{-10}10^y\), i.e., if \(10^y < e^{9}\), i.e., if \(y < \frac{9}{\log_e 10}\).
Thus, the MAP rule is given by
\[
D(y) = \begin{cases} 
1, & y < \frac{9}{\log_e 10}, \\
10, & \text{otherwise}.
\end{cases}
\]

The probability of error is given by
\[
P(Y > \frac{9}{\log_e 10} | X = 1)P(X = 1) + P(Y < \frac{9}{\log_e 10} | X = 10)P(X = 10)
= \frac{1}{2} \left[ P\left( \text{Poisson}(1) > \frac{9}{\log_e 10} \right) + P\left( \text{Poisson}(10) < \frac{9}{\log_e 10} \right) \right]
= \frac{1}{2} \left[ 1 - \sum_{j=0}^{3} \frac{e^{-1}}{j!} + \sum_{j=0}^{3} \frac{e^{-10}10^j}{j!} \right]
= .0147
\]

2. **Iocane or Sennari.** An absent-minded chemistry professor forgets to label two identically looking bottles. One bottle contains a chemical named “Iocane” and the other bottle contains a chemical named “Sennari”. It is well known that the radioactivity level of “Iocane” has the $U[0, 1]$ distribution, while the radioactivity level of “Sennari” has the $\text{Exp}(1)$ distribution.

(a) Let $X$ be the radioactivity level measured from one of the bottles. What is the optimal decision rule (based on the measurement $X$) that maximizes the chance of correctly identifying the content of the bottle?

(b) What is the associated probability of error?

**Solution:**

(a) Let $Y = \begin{cases} 
0, & \text{if Iocane bottle is chosen} \\
1, & \text{if Sennari bottle is chosen}.
\end{cases}$ Then, $P(Y = 1) = P(Y = 0) = 1/2$, $X|\{Y = 0\} \sim U[0, 1]$ and $X|\{Y = 1\} \sim \text{Exp}(1)$.

Thus,
\[
P(Y = 0|X = x) = \frac{f_X|Y(x|0)P(Y = 0)}{\sum_{y \in \{0, 1\}} f_X|Y(x|y)P(Y = y)}
= \frac{1_{[0,1]}(x)}{1_{[0,1]}(x) + 1_{[0,\infty)}(x)e^{-x}}
= \begin{cases} 
0, & x > 1 \\
\frac{1}{1 + e^{-x}}, & x \in [0, 1].
\end{cases}
\]

Thus, $P(Y = 0|X = x) > P(Y = 1|X = x) \iff P(Y = 0|X = x) > 1/2 \iff \{e^{-x} > 1\} \cup \{x \in (0, 1)\}.$
Thus, the optimal decision rule is to declare “Iocane” if \( X \in (0, 1) \) (or \([0, 1]\)), and to declare “Sennari” otherwise.

(b) The probability of error is given by

\[
\begin{align*}
P\left(X > 1 \mid Y = 0\right)P(Y = 0) + P\left(X < 1 \mid Y = 1\right)P(Y = 1) \\
= \frac{1}{2} \left[ P\left(U[0, 1] > 1\right) + P\left(\text{Exp}(1) < 1\right)\right] \\
= \frac{1}{2} \left[ \int_{0}^{1} e^{-x}dx \right] \\
= \frac{1 - e^{-1}}{2}.
\end{align*}
\]

3. Radar signal detection. The signal for a radar channel \( S = 0 \) if there is no target and a random variable \( S \sim \mathcal{N}(0, P) \) if there is a target. Both occur with equal probability. Thus

\[
S = \begin{cases} 
0, & \text{with probability } \frac{1}{2} \\
X \sim \mathcal{N}(0, P), & \text{with probability } \frac{1}{2}.
\end{cases}
\]

The radar receiver observes \( Y = S + Z \), where the noise \( Z \sim \mathcal{N}(0, N) \) is independent of \( S \). Find the optimal decoder for deciding whether \( S = 0 \) or \( S = X \) and its probability of error? Provide your answer in terms of intervals of \( y \) and provide the boundary points of the intervals in terms of \( P \) and \( N \).

**Solution:** Since \( Z \) is independent of \( X \) and \( Z \sim \mathcal{N}(0, N) \) and \( X \sim \mathcal{N}(0, P) \), we have that if \( S = X \), \( Y = S + Z = X + Z \sim \mathcal{N}(0, P + N) \).

Let \( Z = \begin{cases} 
0, & S = 0 \\
1, & S = X.
\end{cases} \) Then, \( Y\{Z = 0\} \sim \mathcal{N}(0, N) \), and \( Y\{Z = 1\} \sim \mathcal{N}(0, P + N) \).

Thus,

\[
P(Z = 0 \mid Y = y) = \frac{f_{Y \mid Z}(y|0)P(Z = 0)}{f_{Y \mid Z}(y|0)P(Z = 0) + f_{Y \mid Z}(y|1)P(Z = 1)}
= \frac{f_{Y \mid Z}(y|0)}{f_{Y \mid Z}(y|0) + f_{Y \mid Z}(y|1)}
= \frac{1}{\sqrt{2\pi N}} \exp \left[ -\frac{y^2}{2N} \right]
= \frac{1}{\sqrt{2\pi N}} \exp \left[ -\frac{y^2}{2N} \right] + \frac{1}{\sqrt{2\pi (P + N)}} \exp \left[ -\frac{y^2}{2(P + N)} \right].
\]
Thus,

\[ \Pr(Z = 0|Y = y) > \Pr(Z = 1|Y = y) \]
\[ \iff \Pr(Z = 0|Y = y) > 1/2 \]
\[ \iff \frac{1}{\sqrt{2\pi N}} \exp\left(-\frac{y^2}{2N}\right) > \frac{1}{\sqrt{2\pi (P + N)}} \exp\left(-\frac{y^2}{2(P + N)}\right) \]
\[ \iff \exp\left(-\frac{y^2}{2(P + N)} + \frac{y^2}{2N}\right) < \sqrt{1 + \frac{P}{N}} \]
\[ \iff y^2 < \frac{N(P + N)}{P} \log \left(1 + \frac{P}{N}\right). \]

Thus, the optimal decoding rule is to declare

\[ S = 0 \text{ if } y \in \left(-\sqrt{N\left(1 + \frac{N}{P}\right)} \log \left(1 + \frac{P}{N}\right), \sqrt{N\left(1 + \frac{N}{P}\right)} \log \left(1 + \frac{P}{N}\right)\right), \text{ and} \]
\[ S = X \text{ otherwise.} \]

Denoting \( \sqrt{N\left(1 + \frac{N}{P}\right)} \log \left(1 + \frac{P}{N}\right) \) by \( y^* \), the probability of error is given by

\[ \Pr\left(Y \notin (-y^*, y^*) \bigg| Z = 0\right) \Pr(Z = 0) + \Pr\left(Y \in (-y^*, y^*) \bigg| Z = 1\right) \Pr(Z = 1) \]
\[ = \frac{1}{2} \left[ \Pr\left(\mathcal{N}(0, N) \notin (-y^*, y^*)\right) + \Pr\left(\mathcal{N}(0, P + N) \in (-y^*, y^*)\right) \right] \]
\[ = \frac{1}{2} \left(2Q\left(\frac{y^*}{\sqrt{N}}\right) + \Phi\left(\frac{y^*}{\sqrt{P + N}}\right) - \Phi\left(\frac{-y^*}{\sqrt{P + N}}\right) \right) \]
\[ = Q\left(\frac{y^*}{\sqrt{N}}\right) + \Phi\left(\frac{y^*}{\sqrt{P + N}}\right) - \frac{1}{2}. \]
\[ = \frac{1}{2} + Q\left(\frac{y^*}{\sqrt{N}}\right) - Q\left(\frac{y^*}{\sqrt{P + N}}\right). \]

4. **Two envelopes.** An amount \( A \) is placed in one envelope and the amount \( 2A \) is placed in another envelope. The amount \( A \) is fixed but unknown to you. The envelopes are shuffled and you are given one of the envelopes at random. Let \( X \) denote the amount you observe in this envelope. Designate by \( Y \) the amount in the other envelope. Thus

\[ (X, Y) = \begin{cases} 
(A, 2A), & \text{with probability } \frac{1}{2}, \\
(2A, A), & \text{with probability } \frac{1}{2}.
\end{cases} \]

You may keep the envelope you are given, or you can switch envelopes and receive the amount in the other envelope.

(a) Find \( E(X) \) and \( E(Y) \).
(b) Find $E\left(\frac{X}{Y}\right)$ and $E\left(\frac{Y}{X}\right)$.
(c) Suppose you switch. What is the expected amount you receive?

**Solution:**

(a) The expected amount in the first envelope is

$$E(X) = \sum_{x \in \mathcal{X}} xp_X(x) = \frac{1}{2}A + \frac{1}{2}(2A) = \frac{3}{2}A.$$ 

The expected amount in the second envelope is $E(Y) = \frac{3}{2}A$.

(b) The expected factor by which the amount in the second envelope exceeds the amount in the first is

$$E\left(\frac{X}{Y}\right) = \sum_{(x,y) \in \mathcal{X} \times \mathcal{Y}} \left(\frac{x}{y}\right) p_{XY}(x,y) = \frac{1}{2}A + \frac{12A}{2A} = \frac{1}{4} + 1 = \frac{5}{4},$$

$$E\left(\frac{Y}{X}\right) = \sum_{(x,y) \in \mathcal{X} \times \mathcal{Y}} \left(\frac{y}{x}\right) p_{XY}(x,y) = \frac{1}{2}A + \frac{12A}{2A} = \frac{1}{4} + 1 = \frac{5}{4}.$$

(c) If you switch, the expected amount you will receive is $E(Y) = \frac{3}{2}A$. 

5. **Mean and variance.** Let $X$ and $Y$ be random variables with joint pdf

$$f_{X,Y}(x,y) = \begin{cases} 1 & \text{if } |x| + |y| \leq 1/\sqrt{2} \\ 0 & \text{otherwise} \end{cases}$$

Define the random variable $Z = |X| + |Y|$. Find the mean and variance of $Z$ without first finding the pdf of $Z$.

**Solution:** We have

$$E(Z) = E(|X|) + E(|Y|),$$

$$\text{Var}(Z) = E(Z^2) - (E(Z))^2 = E(X^2) + 2E(|XY|) + E(Y^2) - (E(|X|) + E(|Y|))^2.$$ 

So to find the mean and variance of $Z$ without first finding the pdf of $Z$, all we need to know are $E(|X|)$, $E(|Y|)$, $E(X^2)$, $E(Y^2)$, and $E(|XY|)$. We have

$$E(|X|) = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{-\frac{1}{\sqrt{2}} + |x|}^{\frac{1}{\sqrt{2}} - |x|} |x| f_{X,Y}(x,y) dy dx = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} |x| (\sqrt{2} - 2|x|) dx = \frac{\sqrt{2}}{6},$$

$$E(X^2) = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{-\frac{1}{\sqrt{2}} + |x|}^{\frac{1}{\sqrt{2}} - |x|} x^2 f_{X,Y}(x,y) dy dx = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} x^2 (\sqrt{2} - 2|x|) dx = \frac{1}{12},$$

$$E(|XY|) = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{-\frac{1}{\sqrt{2}} + |x|}^{\frac{1}{\sqrt{2}} - |x|} |xy| f_{X,Y}(x,y) dy dx = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} |x| \left(\frac{1}{\sqrt{2}} - |x|\right)^2 dx = \frac{1}{24}.$$
By symmetry, $E(|Y|) = E(|X|) = \frac{\sqrt{2}}{6}$, $E(Y^2) = E(X^2) = \frac{1}{12}$. Hence, we have

\[
E(Z) = E(|X|) + E(|Y|) = \frac{\sqrt{2}}{6} + \frac{\sqrt{2}}{6} = \frac{\sqrt{2}}{3},
\]

\[
\text{Var}(Z) = E(Z^2) - (E(Z))^2 = E(X^2) + 2E(|XY|) + E(Y^2) - (E(|X|) + E(|Y|))^2
\]

\[
= \frac{1}{12} + 2 \times \frac{1}{24} + \frac{1}{12} - \left( \frac{\sqrt{2}}{3} \right)^2 = \frac{1}{36}.
\]

6. **Tall trees.** Suppose that the average height of trees on campus is 20 feet. Argue that no more than half of the tree population is taller than 40 feet.

**Solution:** The average height of the trees in the population is 20 feet. So $\frac{1}{n} \sum_{i=1}^{n} h_i = 20$, where $n$ is the population size and $h_i$ is the height of the $i$-th tree. If more than half of the population is at least 40 feet tall, then the average will be greater than $\frac{1}{2} \cdot 40 = 20$ feet since each person is at least 0 feet tall. Thus no more than half of the population is 40 feet tall.

Alternatively, we can use the Markov inequality with respect to the fraction of population to obtain the same result.

7. **Random phase signal.** Let $Y(t) = \sin(\omega t + \Theta)$ be a sinusoidal signal with random phase $\Theta \sim \text{Unif}[-\pi, \pi]$. Assume here that $\omega$ and $t$ are constants. Find the mean and variance of $Y(t)$. Do they depend on $t$?

**Solution:** We have

\[
E(Y(t)) = \int_{-\pi}^{\pi} \frac{1}{2\pi} \sin(\omega t + \theta) d\theta = 0,
\]

\[
\text{Var}(Y(t)) = E(Y^2(t)) - (E(Y(t)))^2 = E(Y^2(t)) = \int_{-\pi}^{\pi} \frac{1}{2\pi} \sin^2(\omega t + \theta) d\theta
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2} - \frac{1}{2} \cos(2\omega t + 2\theta) d\theta
\]

\[
= \frac{1}{2}.
\]

So neither the mean or variance depend on $t$.

8. **Iterated expectation.** Let $\Lambda$ and $X$ be two random variables with

$\Lambda \sim f_\Lambda(\lambda) = \begin{cases} \frac{5}{3} \lambda^2, & 0 \leq \lambda \leq 1 \\ 0, & \text{otherwise} \end{cases}$

and $X|\{\Lambda = \lambda\} \sim \text{Exp}(\lambda)$. Find $E(X)$.

**Solution:** Since $X|\{\Lambda = \lambda\} \sim \text{Exp}(\lambda)$, we have

\[
E(X|\Lambda = \lambda) = \frac{1}{\lambda}.
\]
Thus,
\[ E[X] = E(E(X|\Lambda)) = E\left( \frac{1}{\Lambda} \right) = \int_{-\infty}^{\infty} \frac{1}{\lambda} f_\Lambda(\lambda) \, d\lambda = \int_{0}^{1} \frac{1}{\lambda} \cdot \frac{5}{3} \lambda^{\frac{3}{2}} \, d\lambda = \frac{5}{2}. \]

9. **Sum of packet arrivals.** Consider a network router with two types of incoming packets, wireline and wireless. Let the random variable \( N_1(t) \) denote the number of *wireline* packets arriving during time \((0, t]\) and let the random variable \( N_2(t) \) denote the number of *wireless* packets arriving during time \((0, t]\). Suppose \( N_1(t) \) and \( N_2(t) \) are independent Poisson with pmfs

\[
P\{N_1(t) = n\} = \frac{(\lambda_1 t)^n}{n!} e^{-\lambda_1 t} \quad \text{for } n = 0, 1, 2, \ldots
\]
\[
P\{N_2(t) = k\} = \frac{(\lambda_2 t)^k}{k!} e^{-\lambda_2 t} \quad \text{for } k = 0, 1, 2, \ldots.
\]

Let \( N(t) = N_1(t) + N_2(t) \) be the total number of packets arriving at the router during time \((0, t]\).

(a) Find the mean \( E(N(t)) \) and variance \( Var(N(t)) \) of the total number of packet arrivals.
(b) Find the pmf of \( N(t) \).
(c) Let the random variable \( Y \) be the time to receive the first packet of either type. Find the pdf of \( Y \).
(d) What is the probability that the first received packet is wireless?

**Solution:**

(a) Since \( N_1 \) and \( N_2 \) are independent,
\[
EN = EN_1 + EN_2 = \lambda_1 t + \lambda_2 t = (\lambda_1 + \lambda_2)t.
\]

and
\[
Var(N) = Var(N_1) + Var(N_2) = (\lambda_1 + \lambda_2)t.
\]

(b) In fact, as discussed in class, \( N(t) \) is Poisson itself. To see this,
\[
P\{N = n\} = \sum_{k=0}^{n} P\{N_1 = k\} P\{N_2 = n-k\}
\]
\[
= \sum_{k=0}^{n} \frac{(\lambda_1 t)^k}{k!} e^{-\lambda_1 t} \left(\frac{(\lambda_2 t)^{n-k}}{(n-k)!} e^{-\lambda_2 t}\right)
\]
\[
= \sum_{k=0}^{n} \binom{n}{k} \frac{(\lambda_1 t)^k}{k!} \frac{(\lambda_2 t)^{n-k}}{(n-k)!} \frac{e^{-\lambda_1 t} e^{-\lambda_2 t}}{n!}
\]
\[
= \frac{((\lambda_1 + \lambda_2)t)^n}{n!} e^{-\lambda_1 t - \lambda_2 t}
\]

for \( n = 0, 1, 2, \ldots \).
(c) Since \( N(t) \) is Poisson with parameter \((\lambda_1 + \lambda_2)t\), the time to the first packet is exponential with parameter \(\lambda_1 + \lambda_2\) (refer to the note below to see why). Therefore,

\[
f_Y(y) = \begin{cases} 
(\lambda_1 + \lambda_2)e^{-(\lambda_1 + \lambda_2)y}, & y \geq 0, \\
0, & \text{otherwise.}
\end{cases}
\]

Alternatively, we can let \( X_1 \) and \( X_2 \) denote the times until first packets of each type and see that \( Y = \min(X_1, X_2) \).

Note: Let \( N(t) \) denote the number of packets arriving during time \((0, t]\) and let \( X \) be the time of the arrival of the first packet. Then

\[
P\{X > t\} = P\{N(t) = 0\}
\]

If \( N(t) \) is a Poisson r.v. with parameter \( \lambda t \), then for \( t \geq 0 \) we have

\[
P\{X > t\} = P\{N(t) = 0\} = e^{-\lambda t}
\]

Therefore,

\[
F_X(t) = P\{X \leq t\} = 1 - P\{X > t\} = 1 - e^{-\lambda t}
\]

and

\[
f_X(t) = \begin{cases} 
\lambda e^{-\lambda t}, & t \geq 0, \\
0, & \text{otherwise.}
\end{cases}
\]

Hence, the time of the arrival of the first packet is exponential with parameter \( \lambda \).

(d) Let \( X_1 \) and \( X_2 \) denote the times until first packets of each type. According to the note in the previous part, \( X_1 \) and \( X_2 \) are exponential random variables with parameters \( \lambda_1 \) and \( \lambda_2 \) respectively. Therefore, the probability that the first packet is wireless is simply

\[
P\{X_2 < X_1\} = \frac{\lambda_2}{\lambda_1 + \lambda_2} \quad \text{(refer to First available teller problem).}
\]

10. **Conditioning on an event.** Let \( X \) be a r.v. with pdf

\[
f_X(x) = \begin{cases} 
2(1 - x) & \text{for } 0 \leq x \leq 1 \\
0 & \text{otherwise}
\end{cases}
\]

and let the event \( A = \{X \geq 1/3\} \). Find \( f_{X|A}(x) \), \( E(X|A) \), and \( \text{Var}(X|A) \).

**Solution:** By definition

\[
f_{X|A}(x) = \begin{cases} 
\frac{f_X(x)}{P(X \in A)} & x \in A \\
0 & \text{otherwise.}
\end{cases}
\]

Thus for the given \( X \) and \( A \),

\[
f_{X|A}(x) = \begin{cases} 
\frac{9}{2}(1 - x) & x \in A \\
0 & \text{otherwise.}
\end{cases}
\]
Now, using this conditional pdf

\[ E(X|A) = \int_{1/3}^{1} \frac{9}{2} x(1 - x) \, dx = \frac{5}{9}. \]

And

\[ E(X^2|A) = \int_{1/3}^{1} \frac{9}{2} x^2(1 - x) \, dx = \frac{1}{3}. \]

Thus, \( \text{Var}(X|A) = E(X^2|A) - (E(X|A))^2 = \frac{2}{81}. \)