Exercise Set #6

1. **Markov chain.** Assume that the continuous random variables $X_1$ and $X_3$ are independent given $X_2$. Show that $f(x_1, x_2, x_3) = f(x_1)f(x_2|x_1)f(x_3|x_2) = f(x_3)f(x_2|x_3)f(x_1|x_2)$.

2. **Covariance matrices.** Which of the following matrices can be a covariance matrix? Justify your answer either by constructing a random vector $X$, as a function of the i.i.d zero mean unit variance random variables $Z_1, Z_2,$ and $Z_3$, with the given covariance matrix, or by establishing a contradiction.

   - (a) \[
   \begin{bmatrix}
   1 & 2 \\
   0 & 2
   \end{bmatrix}
   \]
   - (b) \[
   \begin{bmatrix}
   2 & 1 \\
   1 & 2
   \end{bmatrix}
   \]
   - (c) \[
   \begin{bmatrix}
   1 & 1 & 1 \\
   1 & 2 & 2 \\
   1 & 2 & 3
   \end{bmatrix}
   \]
   - (d) \[
   \begin{bmatrix}
   1 & 1 & 2 \\
   1 & 2 & 3 \\
   2 & 3 & 3
   \end{bmatrix}
   \]

3. The correlation matrix $C$ for a random vector $X$ is the matrix whose entries are $c_{ij} = E(X_iX_j)$. Show that it has the same properties as the covariance matrix, i.e., that it is real, symmetric, and positive semidefinite definite.

4. **Gaussian random vector.** Given a Gaussian random vector $X \sim \mathcal{N}(\mu, \Sigma)$, where $\mu = (1, 5, 2)^T$ and $\Sigma = 
   \begin{bmatrix}
   1 & 1 & 0 \\
   1 & 4 & 0 \\
   0 & 0 & 9
   \end{bmatrix}
   .

   - (a) Find the pdfs of
     i. $X_1$,
     ii. $X_2 + X_3$,
     iii. $2X_1 + X_2 + X_3$,
     iv. $X_3$ given $(X_1, X_2)$, and
     v. $(X_2, X_3)$ given $X_1$.

   - (b) What is $P\{2X_1 + X_2 - X_3 < 0\}$? Express your answer using the $Q$ function.

   - (c) Find the joint pdf on $Y = AX$, where
     \[
     A = \begin{bmatrix}
     2 & 1 & 1 \\
     1 & -1 & 1
     \end{bmatrix}
     .
     
5. **Gaussian Markov chain.** Let $X, Y,$ and $Z$ be jointly Gaussian random variables with zero mean and unit variance, i.e., $E(X) = E(Y) = E(Z) = 0$ and $E(X^2) = E(Y^2) = 1$.
\( \mathbb{E}(Z^2) = 1 \). Let \( \rho_{X,Y} \) denote the correlation coefficient between \( X \) and \( Y \), and let \( \rho_{Y,Z} \) denote the correlation coefficient between \( Y \) and \( Z \). Suppose that \( X \) and \( Z \) are conditionally independent given \( Y \).

(a) Find \( \rho_{X,Z} \) in terms of \( \rho_{X,Y} \) and \( \rho_{Y,Z} \).

(b) Find the MMSE estimate of \( Z \) given \((X,Y)\) and the corresponding MSE.

6. **Sufficient statistic.** The bias of a coin is a random variable \( P \sim \mathcal{U}[0,1] \). Let \( Z_1, Z_2, \ldots, Z_{10} \) be the outcomes of 10 coin flips. Thus \( Z_i \sim \text{Bern}(P) \) and \( Z_1, Z_2, \ldots, Z_{10} \) are conditionally independent given \( P \). If \( X \) is the total number of heads, then \( X|\{P = p\} \sim \text{Binom}(10, p) \). Assuming that the total number of heads is 9, show that
\[
 f_{P|Z_1, Z_2, \ldots, Z_{10}}(p|z_1, z_2, \ldots, z_{10}) = f_{P|X}(p|9)
\]
is independent of the order of the outcomes.

7. **Packet switching.** Let \( N \) be the number of packets per unit time arriving at a network switch. Each packet is routed to output port 1 with probability \( p \) and to output port 2 with probability \( 1 - p \), independent of \( N \) and of other packets. Let \( X \) be the number of packets per unit time routed to output port 1. Thus
\[
 X = \begin{cases} 
 0 & N = 0 \\
 \sum_{i=1}^{N} Z_i & N > 0 
\end{cases}
\]
where \( Z_i = \begin{cases} 
 1 & \text{packet } i \text{ routed to Port 1} \\
 0 & \text{packet } i \text{ routed to Port 2},
\end{cases} \)

and \( Z_1, Z_2, \ldots, Z_N \) are conditionally independent given \( N \). Suppose that \( N \sim \text{Poisson}(\lambda) \), i.e., has Poisson pmf with parameter \( \lambda \).

(a) Find the mean and variance of \( X \).

(b) Find the pmf of \( X \) and the pmf of \( N - X \).

8. **Nonlinear estimator.** Consider a channel with the observation \( Y = XZ \), where the signal \( X \) and the noise \( Z \) are uncorrelated Gaussian random variables. Let \( \mathbb{E}[X] = 1 \), \( \mathbb{E}[Z] = 2 \), \( \sigma_X^2 = 5 \), and \( \sigma_Z^2 = 8 \).

(a) Using the fact that \( \mathbb{E}(W^3) = \mu^3 + 3\mu\sigma^2 \) and \( \mathbb{E}(W^4) = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4 \) for \( W \sim \mathcal{N}(\mu, \sigma^2) \), find the mean and covariance matrix of \([X^T Y Y^2]^T\).

(b) Find the MMSE linear estimate of \( X \) given \( Y \) and the corresponding MSE.

(c) Find the MMSE linear estimate of \( X \) given \( Y^2 \) and the corresponding MSE.

(d) Find the MMSE linear estimate of \( X \) given \( Y \) and \( Y^2 \) and the corresponding MSE.

(e) Compare your answers in parts (b) through (d). Is the MMSE estimate of \( X \) given \( Y \) (namely, \( \mathbb{E}(X|Y) \)) linear?
9. **Prediction.** Let $\mathbf{X}$ be a random process with zero mean and covariance matrix

$$
\Sigma_X = \begin{bmatrix}
1 & \alpha & \alpha^2 & \cdots & \alpha^{n-1} \\
\alpha & 1 & \alpha & & \\
\alpha^2 & \alpha & 1 & & \\
& \ddots & & & \\
\alpha^{n-1} & \cdots & 1 & & 
\end{bmatrix}
$$

for $|\alpha| < 1$. $X_1, X_2, \ldots, X_{n-1}$ are observed, find the best linear MSE estimate (predictor) of $X_n$. Compute its MSE.

10. **Minimum waiting time.** Let $X_1, X_2, \ldots$ be i.i.d. exponentially distributed random variables with parameter $\lambda$, i.e., $f_{X_i}(x) = \lambda e^{-\lambda x}$, for $x \geq 0$.

   (a) Does $Y_n = \min\{X_1, X_2, \ldots, X_n\}$ converge in probability as $n$ approaches infinity?

   (b) If it converges, what is the limit?

   (c) What about $Z_n = nY_n$?

11. **Roundoff errors.** The sum of a list of 100 real numbers is to be computed. Suppose that these numbers are rounded off to the nearest integer so that each number has an error that is uniformly distributed in the interval $(-0.5, 0.5)$. Use the central limit theorem to estimate the probability that the total error in the sum of the 100 numbers exceeds 6.