1. **Stationary Gauss-Markov process.** Let

\[ X_0 \sim N(0, a) \]
\[ X_n = \frac{1}{2} X_{n-1} + Z_n, \quad n \geq 1, \]

where \( Z_1, Z_2, Z_3, \ldots \) are i.i.d. \( N(0, 1) \) independent of \( X_0 \).

(a) Find \( a \) such that \( X_n \) is stationary. Find the mean and autocorrelation functions of \( X_n \).

(b) (Difficult.) Consider the sample mean \( S_n = \frac{1}{n} \sum_{i=1}^{n} X_i, \quad n \geq 1 \). Show that \( S_n \) converges to the process mean in probability even though the sequence \( X_n \) is not i.i.d. (A stationary process for which the sample mean converges to the process mean is called **mean ergodic**.)

**Solution:**

(a) We are asked to find \( a \) such that \( \mathbb{E}(X_n) \) is independent of \( n \) and \( R_X(n_1, n_2) \) depends only on \( n_1 - n_2 \). For \( X_n \) to be stationary, \( \mathbb{E}(X_n^2) \) must be independent of \( n \). Thus

\[ \mathbb{E}(X_n^2) = \frac{1}{4} \mathbb{E}(X_{n-1}^2) + \mathbb{E}(Z_n^2) + \mathbb{E}(X_{n-1} Z_n) = \frac{1}{4} \mathbb{E}(X_n^2) + 1. \]

Therefore, \( a = \mathbb{E}(X_0^2) = \mathbb{E}(X_n^2) = \frac{4}{3} \). Using the method of problem 5, we can easily verify that \( \mathbb{E}(X_n) = 0 \) for every \( n \) and that

\[ R_X(n_1, n_2) = \mathbb{E}(X_{n_1} X_{n_2}) = \frac{4}{3} 2^{-|n_1 - n_2|}. \]

(b) To prove convergence in probability, we first prove convergence in mean square and then use the fact that mean square convergence implies convergence in probability.

\[ \mathbb{E}(S_n) = \mathbb{E} \left( \frac{1}{n} \sum_{i=1}^{n} X_i \right) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(X_i) = \frac{1}{n} \sum_{i=1}^{n} 0 = 0. \]
To show convergence in mean square we show that $\text{Var}(S_n) \to 0$ as $n \to \infty$.

$$\text{Var}(S_n) = \text{Var}\left(\frac{1}{n} \sum_{i=1}^{n} X_i\right) = \mathbb{E}\left(\left(\frac{1}{n} \sum_{i=1}^{n} X_i\right)^2\right) \quad \text{(since } \mathbb{E}(X_i) = 0)$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} R_X(i, j) = \frac{4}{3n^2} \left( n + 2 \sum_{i=1}^{n-1} (n - i)2^{-i} \right)$$

$$\leq \frac{4}{3n} \left( 1 + 2 \sum_{i=1}^{n-1} 2^{-i} \right) \leq \frac{4}{3n} \left( 1 + 2 \sum_{i=1}^{\infty} 2^{-i} \right) = \frac{4}{n}.$$  

Thus $S_n$ converges to the process mean, even though the sequence is not i.i.d.

2. **AM modulation.** Consider the AM modulated random process

$$X(t) = A(t) \cos(2\pi t + \Theta),$$

where the amplitude $A(t)$ is a zero-mean WSS process with autocorrelation function $R_A(\tau) = e^{-1/2|\tau|}$, the phase $\Theta$ is a Unif\([0, 2\pi]\) random variable, and $A(t)$ and $\Theta$ are independent. Is $X(t)$ a WSS process? Justify your answer.

**Solution:** $X(t)$ is wide-sense stationary if $EX(t)$ is independent of $t$ and if $R_X(t_1, t_2)$ depends only on $t_1 - t_2$. Consider

$$EX(t) = E[A(t) \cos(\omega t + \Theta)]$$

$$= E[A(t)]E[\cos(\omega t + \Theta)] \quad \text{by independence}$$

$$= 0,$$

and

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)]$$

$$= E[A(t_1) \cos(\omega t_1 + \Theta)A(t_2) \cos(\omega t_2 + \Theta)]$$

$$= E[A(t_1)A(t_2) \cos(\omega t_1 + \Theta) \cos(\omega t_2 + \Theta)]$$

$$= E[A(t_1)A(t_2) \cdot \cos(\omega t_1 + \Theta) \cos(\omega t_2 + \Theta)] \quad \text{by independence}$$

$$= R_A(t_1 - t_2)E[\cos(\omega t_1 + \Theta) \cos(\omega t_2 + \Theta)]$$

$$= R_A(t_1 - t_2) \mathbb{E}\left[\frac{1}{2} \left( \cos(\omega(t_1 + t_2)) + 2\Theta + \cos(\omega(t_1 - t_2)) \right)\right]$$

$$= \frac{1}{2} R_A(t_1 - t_2) \mathbb{E}\left(\begin{pmatrix} \cos(\omega(t_1 + t_2)) \cos(2\Theta) \\ -\sin(\omega(t_1 + t_2)) \sin(2\Theta) \\ +\cos(\omega(t_1 - t_2)) \end{pmatrix}\right)$$


\[
\frac{1}{2} R_A(t_1 - t_2) \begin{pmatrix}
E \cos(\omega(t_1 + t_2)) \cdot E \cos(2\Theta) \\
- E \sin(\omega(t_1 + t_2)) \cdot E \sin(2\Theta) \\
+ E \cos(\omega(t_1 - t_2))
\end{pmatrix}
\]

which is a function of \(t_1 - t_2\) only. Hence \(X(t)\) is wide-sense stationary.

3. LTI system with WSS process input. Let \(Y(t) = h(t) * X(t)\) and \(Z(t) = X(t) - Y(t)\) as shown in the Figure 1.

(a) Find \(S_Z(f)\).
(b) Find \(E(Z^2(t))\).

Your answers should be in terms of \(S_X(f)\) and the transfer function \(H(f) = \mathcal{F}[h(t)]\).

\[
\begin{array}{c}
X(t) \\
\downarrow \\
h(t) \\
\uparrow \\
Y(t) \\
\downarrow \\
+ \\
\downarrow \\
\uparrow \\
Z(t)
\end{array}
\]

Figure 1: LTI system.

Solution:

(a) To find \(S_Z(f)\), we first find the autocorrelation function

\[
R_Z(\tau) = E(Z(t)Z(t + \tau))
= E((X(t) - Y(t))(X(t + \tau) - Y(t + \tau)))
= R_X(\tau) + R_Y(\tau) - R_{XY}(\tau) - R_{XY}(-\tau)
= R_X(\tau) + R_Y(\tau) - R_{XY}(\tau) - R_{XY}(-\tau).
\]

Now, taking the Fourier Transform, we get

\[
S_Z(f) = S_X(f) + S_Y(f) - S_{XY}(f) - S_{XY}(-f)
= S_X(f) + |H(f)|^2 S_X(f) - H(-f)S_X(f) - H(f)S_X(f)
= S_X(f) (1 + |H(f)|^2 - 2\text{Re}[H(f)])
= S_X(f) |1 - H(f)|^2.
\]
(b) To find the average power of $Z(t)$, we find the area under $S_Z(f)$

$$E(Z^2(t)) = \int_{-\infty}^{\infty} |1 - H(f)|^2 S_X(f) \, df.$$ 

4. **Echo filtering.** A signal $X(t)$ and its echo arrive at the receiver as $Y(t) = X(t) + X(t - \Delta) + Z(t)$. Here the signal $X(t)$ is a zero-mean WSS process with power spectral density $S_X(f)$ and the noise $Z(t)$ is a zero-mean WSS with power spectral density $S_Z(f) = N_0/2$, uncorrelated with $X(t)$.

(a) Find $S_Y(f)$ in terms of $S_X(f)$, $\Delta$, and $N_0$.

(b) Find the best linear filter to estimate $X(t)$ from $\{Y(s)\}_{-\infty < s < \infty}$.

**Solution:**

(a) We can write $Y(t) = g(t) \ast X(t) + Z(t)$ where $g(t) = \delta(t) + \delta(t - \Delta)$.

Thus, $S_Y(f) = |G(f)|^2 S_X(f) + S_Z(f) = |1 + e^{-j2\pi \Delta f}|^2 S_X(f) + \frac{N_0}{2}$.

(b) Since $S_{YX}(f) = (1 + e^{-j2\pi \Delta f}) S_X(f)$,

$$\hat{X}(t) = h(t) \ast Y(t),$$

where the linear filter $h(t)$ has the transfer function

$$H(f) = \frac{S_{XY}(f)}{S_Y(f)} = \frac{S_{YX}(-f)}{S_Y(f)} = \frac{(1 + e^{j2\pi \Delta f}) S_X(f)}{|1 + e^{-j2\pi \Delta f}|^2 S_X(f) + \frac{N_0}{2}}.$$

5. **Discrete-time LTI system with white noise input.** Let $\{X_n : -\infty < n < \infty\}$ be a discrete-time white noise process, i.e., $E(X_n) = 0$, $-\infty < n < \infty$, and

$$R_X(n) = \begin{cases} 1 & n = 0, \\ 0 & \text{otherwise}. \end{cases}$$

The process is filtered using a linear time invariant system with impulse response

$$h(n) = \begin{cases} \alpha & n = 0, \\ \beta & n = 1, \\ 0 & \text{otherwise}. \end{cases}$$

Find $\alpha$ and $\beta$ such that the output process $Y_n$ has

$$R_Y(n) = \begin{cases} 2 & n = 0, \\ 1 & |n| = 1, \\ 0 & \text{otherwise}. \end{cases}$$
Solution: We are given that $R_X(n)$ is a discrete-time unit impulse. Therefore

$$R_Y(n) = h(n) \ast R_X(n) \ast h(-n) = h(n) \ast h(-n).$$

The impulse response $h(n)$ is the sequence $(\alpha, \beta, 0, 0, \ldots)$. The convolution with $h(-n)$ has only finitely many nonzero terms.

$$R_Y(0) = 2 = h(0) \ast h(0) = \alpha^2 + \beta^2$$
$$R_Y(+1) = 1 = h(1) \ast h(-1) = \alpha \beta$$
$$R_Y(-1) = 1 = R_Y(1)$$

This pair of equations has two solutions: $\alpha = +1$ and $\beta = +1$ or $\alpha = -1$ and $\beta = -1$.

6. Finding time of flight. Finding the distance to an object is often done by sending a signal and measuring the time of flight, the time it takes for the signal to return (assuming speed of signal, e.g., light, is known). Let $X(t)$ be the signal sent and $Y(t) = X(t - \delta) + Z(t)$ be the signal received, where $\delta$ is the unknown time of flight. Assume that $X(t)$ and $Z(t)$ (the sensor noise) are uncorrelated zero mean WSS processes. The estimated crosscorrelation function of $Y(t)$ and $X(t)$, $R_{YX}(t)$ is shown in Figure 2. Find the time of flight $\delta$.

![Figure 2: Crosscorrelation function.](image)

Solution: Consider

$$R_{YX}(\tau) = E(Y(t + \tau)X(t)) = E((X(t - \delta + \tau) + Z(t + \tau))X(t)) = R_X(\tau - \delta).$$

Now since the maximum of $|R_X(\alpha)|$ is achieved for $\alpha = 0$, by inspection of the given $R_{YX}$ we get that $5 - \delta = 0$. Thus $\delta = 5$.

7. Finding impulse response of LTI system. To find the impulse response $h(t)$ of an LTI system (e.g., a concert hall), i.e., to identify the system, white noise $X(t)$, $-\infty < t < \infty$, is applied to its input and the output $Y(t)$ is measured. Given the input and output sample functions, the crosscorrelation $R_{YX}(\tau)$ is estimated. Show how $R_{YX}(\tau)$ can be used to find $h(t)$.
Solution: Since white noise has a flat psd, the crosspower spectral density of the input \( X(t) \) and the output \( Y(t) \) is just the transfer function of the system scaled by the psd of the white noise.

\[
S_{YX}(f) = H(f)S_X(f) = H(f) \frac{N_0}{2}
\]

\[
R_{YX}(\tau) = \mathcal{F}^{-1}(S_{YX}(f)) = \frac{N_0}{2}h(\tau)
\]

Thus to estimate the impulse response of a linear time invariant system, we apply white noise to its input, estimate the crosscorrelation function of its input and output, and scale it by \( 2/N_0 \).

8. Generating a random process with a prescribed power spectral density. Let \( S(f) \geq 0, \) for \( -\infty < f < \infty \), be a real and even function such that

\[
\int_{-\infty}^{\infty} S(f)df = 1.
\]

Define the random process

\[
X(t) = \cos(2\pi Ft + \Theta),
\]

where \( F \sim S(f) \) and \( \Theta \sim U[-\pi, \pi] \) are independent. Find the power spectral density of \( X(t) \). Interpret the result.

Solution: We have

\[
E[X(t)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} \cos(2\pi ft + \theta) S(f) df d\theta
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} 0 \cdot S(f) df
\]

\[
= 0.
\]

Also,

\[
E[X(t)X(t + \tau)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} \cos(2\pi ft + \theta) \cos(2\pi ft + 2\pi f \tau + \theta) S(f) df d\theta
\]

\[
= \frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} \left( \cos(4\pi ft + 2\pi f \tau + 2\theta) \cos(2\pi f \tau) \right) S(f) df d\theta
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} \cos(2\pi f \tau) S(f) df,
\]
which is a function only of \( \tau \). Thus, \( X(t) \) is WSS and has autocorrelation function

\[
R_{XX}(\tau) = \frac{1}{2} \int_{-\infty}^{\infty} \cos(2\pi f \tau) S(f) df.
\]

Now, \( S(f) \) is even and hence, \( S(f) \sin(2\pi f \tau) \) is odd. Thus, \( \int_{-\infty}^{\infty} \sin(2\pi f \tau) S(f) df = 0 \), and hence,

\[
\int_{-\infty}^{\infty} S(f)e^{i2\pi f \tau} df = \int_{-\infty}^{\infty} \cos(2\pi f \tau) S(f) df + i \int_{-\infty}^{\infty} \sin(2\pi f \tau) S(f) df
\]

\[
= \int_{-\infty}^{\infty} \cos(2\pi f \tau) S(f) df
\]

\[
= 2R_{XX}(\tau).
\]

Thus, \( R_{XX}(\tau) = \frac{1}{2} \mathcal{F}^{-1}(S(f)) \).

Hence, the power spectral density of \( X(t) \) is given by \( S_X(f) = \frac{1}{2} S(f) \).

9. **Integrators.** Let \( Y(t) \) be a short-term integration of a WSS process \( X(t) \):

\[
Y(t) = \frac{1}{T} \int_{t-T}^{t} X(u) du.
\]

Find \( S_Y(f) \) in terms of \( S_X(f) \).

**Solution:** It is easy to see that the system that generates \( Y(t) \) from \( X(t) \) is linear and time-invariant. Writing \( \delta(t) \) in place of \( X(t) \), the impulse response of the system can then be obtained as

\[
h(t) = \frac{1}{T} \int_{t-T}^{t} \delta(u) du
\]

\[
= \begin{cases} 
\frac{1}{T}, & 0 \leq t \leq T \\
0, & \text{otherwise.}
\end{cases}
\]

Alternatively, we can find \( h(t) \) by attempting to write \( Y(t) \) as \( Y(t) = \int_{\infty}^{\infty} h(\tau) X(t-\tau) d\tau \). We have

\[
Y(t) = \frac{1}{T} \int_{t-T}^{t} X(u) du
\]

\[
= \frac{1}{T} \int_{0}^{T} X(t-\tau) d\tau.
\]
This shows, as before, that \( h(\tau) = \begin{cases} 
\frac{1}{T}, & 0 \leq \tau \leq T \\
0, & \text{otherwise}. 
\end{cases} \)

Thus, the frequency response is given by

\[
H(f) = \frac{1}{T} \int_0^T e^{-i2\pi ft} dt
= \frac{1}{T} \left( \frac{e^{-i2\pi fT} - 1}{-i2\pi f} \right)
= \frac{e^{-i\pi fT}}{\pi fT} \left( \frac{e^{i\pi fT} - e^{-i\pi fT}}{2i} \right)
= e^{-i\pi fT} \frac{\sin(\pi fT)}{\pi fT}.
\]

Thus,

\[
S_Y(f) = S_X(f) |H(f)|^2
= S_X(f) \frac{\sin^2(\pi fT)}{\pi^2 f^2 T^2}.
\]

**Alternative Method:**

We have

\[
\text{E}[Y(t)Y(t+\tau)] = \frac{1}{T^2} \text{E} \left[ \int_{t-T}^{t+\tau} \int_{t+\tau-t}^{t+\tau} X(u)X(v) dv du \right]
= \frac{1}{T^2} \text{E} \left[ \int_{t-T}^{t} \int_{t-T}^{t} X(u)X(v+\tau) dv du \right]
= \frac{1}{T^2} \int_{t-T}^{t} \int_{t-T}^{t} R_{XX}(v+\tau-u) dv du.
\]

Writing \( w = u - v \) in the integral, we see that \(-T \leq w \leq T\), and for each fixed \( w, t-T+w \leq u \leq t+w \). Thus,

\[
\text{E}[Y(t)Y(t+\tau)] = \frac{1}{T^2} \int_{-T}^{T} \int_{\max(t-T,t-T+w)}^{\min(t,t+w)} R_{XX}(\tau-w) dudw
= \frac{1}{T^2} \int_{-T}^{T} R_{XX}(\tau-w) \left( \min(t,t+w) - \max(t-T,t-T+w) \right) dw
= \int_{-T}^{T} R_{XX}(\tau-w) g(w) dw,
\]

8
where

\[ g(w) = \frac{1}{T^2} \left( \min(t, t + w) - \max(t - T, t - T - w) \right) \]

\[ = \begin{cases} 
  \frac{T + w}{T^2}, & -T \leq w \leq 0 \\
  \frac{T - w}{T^2}, & 0 \leq w \leq T \\
  T - \frac{|w|}{T^2}.
\end{cases} \]

Thus, \( R_{YY}(\tau) = R_{XX}(\tau) * g(\tau) \), and hence

\[ S_Y(f) = S_X(f) \int_{-\infty}^{\infty} g(t)e^{-i2\pi ft} dt \]

\[ = S_X(f) \frac{1}{T^2} \int_{-T}^{T} (T - |t|)e^{-i2\pi ft} dt \]

\[ = S_X(f) \frac{1}{T^2} \int_{-T}^{T} (T - |t|) \cos(2\pi ft) dt \]

\[ = S_X(f) \left( \frac{\sin(2\pi fT)}{\pi fT} - \frac{1}{4\pi^2 f^2 T^2} \int_{-2\pi fT}^{2\pi fT} |u| \cos u du \right) \]

\[ = S_X(f) \left( \frac{\sin(2\pi fT)}{\pi fT} - \frac{1}{4\pi^2 f^2 T^2} \cdot 2\left( 2\pi fT \sin(2\pi fT) + \cos(2\pi fT) - \cos 0 \right) \right) \]

\[ = \frac{S_X(f)}{2\pi^2 f^2 T^2} \left( 1 - \cos(2\pi fT) \right) \]

\[ = \frac{S_X(f)}{2\pi^2 f^2 T^2} \left( 2\sin^2(\pi fT) \right) \]

\[ = S_X(f) \frac{\sin^2(\pi fT)}{\pi^2 f^2 T^2}. \]