8.1 DEFINITION

A random process (or stochastic process) is an infinite indexed collection of random variables

\[ \{X(t) : t \in T\}, \]

defined over a common probability space. The index parameter \( t \) is typically time, but can also be a spatial dimension. Random processes are used to model random experiments that evolve in space/time:

- Received sequence/waveform at the output of a communication channel
- Packet arrival times at a node in a communication network
- Thermal noise in a resistor
- Scores of an NBA team in consecutive games
- Daily price of a stock
- Winnings or losses of a gambler
- Contents in memory cells

We are interested in several questions involving random processes.

- Dependencies of the random variables of the process. How do future received values depend on past received values? How do future prices of a stock depend on its past values?
- Long-term averages. What is the proportion of time a queue is empty? What is the average noise power at the output of a circuit?
- Extreme or boundary events. What is the probability that a link in a communication network is congested? What is the probability that the maximum power in a power distribution line is exceeded? What is the probability that a gambler will lose all his capital?
- Estimation/detection. How best can one recover a signal from a noisy waveform? How can we estimate a DNA sequence from noisy reads?
A random process can be viewed as a function $X(t, \omega)$ of two variables, the time $t \in \mathcal{T}$ and the outcome $\omega \in \Omega$, where $\Omega$ is the space of the underlying random experiment. There are thus two ways to view it. First, for fixed $t$, $X(t, \omega)$ is a random variable over $\Omega$. In this view, the random process is an index collection of random variables. Second, for fixed $\omega$, $X(t, \omega)$ is a deterministic function of $t$, called a sample function, as illustrated in Figure 8.1. In this view, the random process is a randomly drawn function in $t$.

**Figure 8.1.** Sample functions of a random process

### 8.2 DISCRETE-TIME RANDOM PROCESSES

A random process is said to be discrete-time if $\mathcal{T}$ is a countably infinite set, e.g., $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{Z} = \{\ldots, -2, -1, 0, +1, +2, \ldots\}$. In this case, the process is denoted by $X_n$, for $n \in \mathbb{N}$, a countably infinite set, and is simply an infinite sequence of random variables. A sample function for a discrete-time process is called a sample sequence or sample path. A discrete-time process can comprise discrete, continuous, or mixed random variables.
**Example 8.1.** Let $Z \sim \text{Unif}[0, 1]$ and define the discrete time process

$$X_n = Z^n, \quad n = 1, 2, \ldots$$

The sample paths are illustrated in Figure 8.2. The first-order pdf of the process, that is, the sequence of pdfs of $X_n$ is

$$f_{X_n}(x) = \frac{1}{n^{X^{(n-1)/n}}} = \frac{1}{n} x^{\frac{1}{n} - 1}, \quad x \in [0, 1],$$

which can be easily found by differentiating $P\{X_n \leq x\} = P\{Z \leq x^{1/n}\}$ w.r.t. $x$.

---

**Figure 8.2.** Sample paths of the random process in Example 8.1

In the above example, we specified the random process by describing the set of sample paths and explicitly providing a probability measure over the set of events (subsets of sample paths). This way of specifying a random process has very limited applicability, and is suited only for very simple processes. A discrete-time random process is in general specified (directly or indirectly) by specifying all its $k$-th order cdfs (pdfs, pmfs), i.e., the joint cdf (pdf, pmf) of the samples

$$X_{n_1}, X_{n_2}, \ldots, X_{n_k}$$
for every order \( k \) and for every set of \( k \) points \( n_1 < n_2 < \cdots < n_k \in \mathcal{N} \). The Kolmogorov extension theorem, the proof of which is beyond this course, guarantees that the entire process can be defined this way.

In the following, we discuss several classes of discrete-time random processes.

### 8.2.1 IID Processes

We say that \( \{X_n : n \in \mathcal{N}\} \) is an IID process if the random variables \( X_1, X_2, \ldots \) are independent and identically distributed (i.i.d.).

**Example 8.2 (Bernoulli process).** \( X_1, X_2, \ldots \) i.i.d. \( \text{Bern}(p) \).

**Example 8.3 (Discrete-time white Gaussian noise).** \( X_1, X_2, \ldots \) i.i.d. \( \text{N}(0, N) \).

Here we specified the \( n \)-th order pmfs (pdfs) of the processes by specifying the first-order pmf (pdf) and stating that the random variables are independent. It would be quite difficult to provide the specifications for an IID process by specifying the probability measure over the subsets of the sample.

### 8.2.2 Random Walk

Let \( Z_1, Z_2, \ldots, Z_n, \ldots \) be i.i.d., where

\[
Z_n = \begin{cases} 
+1 & \text{w.p. } 1/2, \\
-1 & \text{w.p. } 1/2.
\end{cases}
\]

The (symmetric) random walk is defined by

\[
X_0 = 0, \\
X_n = \sum_{i=1}^{n} Z_i = X_{n-1} + Z_n, \quad n = 1, 2, \ldots.
\]

Again this process is specified by (indirectly) specifying all \( n \)-th order pmfs. The sample path for a random walk is a sequence of integers as illustrated in Figure 8.3. The first-order pmf is \( P\{X_n = k\} \) as a function of \( n \). Note that

\[
k \in \{-n, -(n-2), \ldots, -2, 0, +2, \ldots, +(n-2), +n\} \quad \text{for } n \text{ even}, \\
k \in \{-n, -(n-2), \ldots, -1, +1, +3, \ldots, +(n-2), +n\} \quad \text{for } n \text{ odd}.
\]

Now if we let \( a \) be the number of +1’s in \( n \) steps and \( n - a \) be the number of −1’s, then

\[
k = a - (n - a) = 2a - n,
\]

or equivalently, \( a = n + k/2 \). Thus

\[
P\{X_n = k\} = P((n + k)/2 \text{ heads in } n \text{ independent coin tosses}) = \left( \frac{n}{n+k} \right) \cdot 2^{-n} \quad \text{if } n - k \text{ is even}.
\]

For example, \( P\{X_5 = 3\} = 5/32 \) and \( P\{X_{10} = 6\} = 45/1024 \).
8.2 Discrete-Time Random Processes

8.2.3 Markov Processes

A random process \( \{X_n\} \) is said to be Markov if the “future and the past are conditionally independent given the present.” Mathematically, this can be rephrased in several ways. For example, if the random variables \( X_1, X_2, \ldots \) are discrete, then the process is Markov if

\[
P_{X_{n+1}|X_1, X_2, \ldots, X_n}(x_{n+1}|x_1, x_2, \ldots, x_n) = P_{X_{n+1}|X_n}(x_{n+1}|x_n)
\]

for every \( n \) and every \((x_1, x_2, \ldots, x_{n+1})\). Discrete-time discrete Markov processes are often called Markov chains.

Example 8.4. IID processes are Markov.

Example 8.5. The random walk process is Markov. To see this, consider

\[
P\{X_{n+1} = x_{n+1} | X_1 = x_1, \ldots, X_n = x_n\} = P\{X_n + Z_{n+1} = x_{n+1} | X_1 = x_1, \ldots, X_n = x_n\}
\]

\[
= P\{Z_{n+1} = x_{n+1} - x_n | X_1 = x_1, \ldots, X_n = x_n\}
\]

\[
= P\{Z_{n+1} = x_{n+1} - x_n\}
\]

\[
= P\{Z_{n+1} = x_{n+1} - x_n | X_n = x_n\}
\]

\[
= P\{X_n + Z_{n+1} = x_{n+1} | X_n = x_n\}
\]

\[
= P\{X_{n+1} = x_{n+1} | X_n = x_n\},
\]

where the equalities in (a) and (b) follow since \( Z_{n+1} \) is independent of \((X_1, \ldots, X_n)\).

Figure 8.3. A sample path of the random walk.
8.2.4 Independent Increment Processes

A random process \( \{X_n\} \) is said to be independent increment if the increments

\[
X_{n_1}, X_{n_2} - X_{n_1}, \ldots, X_{n_k} - X_{n_{k-1}}
\]

are independent for all \( k \) and all \( n_1 < n_2 < \cdots < n_k \).

Example 8.6. The random walk is an independent increment process since

\[
X_{n_1} = \sum_{i=1}^{n_1} Z_i,
\]

\[
X_{n_2} - X_{n_1} = \sum_{i=n_1+1}^{n_2} Z_i,
\]

\[\vdots\]

\[
X_{n_k} - X_{n_{k-1}} = \sum_{i=n_{k-1}+1}^{n_k} Z_i
\]

are independent (as functions of independent random vectors). The independent increment property makes it easy to find the \( n \)-th order pmfs of the random walk process. For example,

\[
P\{X_3 = 3, X_{10} = 6, X_{20} = 10\} = P\{X_3 = 3, X_{10} - X_5 = 3, X_{20} - X_{10} = 4\}
\]

\[
= P\{X_3 = 3\} P\{X_{10} - X_5 = 3\} P\{X_{20} - X_{10} = 4\}
\]

\[
= P\{X_3 = 3\} P\{X_5 = 3\} P\{X_5 = 3\} P\{X_{10} = 4\}
\]

\[
= \binom{5}{3} \cdot \frac{5}{4}^3 \cdot \frac{5}{4}^2 \cdot \frac{10}{7}^2 \cdot 10
\]

\[
= 3000 \cdot 2^{-20}.
\]

In general if a process is independent increment, then it is also Markov. To see this, let \( \{X_n\} \) be an independent increment process. Then

\[
P\{X_{n+1} = x_{n+1} \mid X_1 = x_1, \ldots, X_n = x_n\}
\]

\[
= P\{X_{n+1} - X_n = x_{n+1} - x_n \mid X_1 = x_1, X_2 - X_1 = x_2 - x_1, \ldots, X_n - X_{n-1} = x_n - x_{n-1}\}
\]

\[
= P\{X_{n+1} - X_n = x_{n+1} - x_n\}
\]

\[
= P\{X_{n+1} - X_n = x_{n+1} - x_n \mid X_n = x_n\}
\]

\[
= P\{X_{n+1} = x_{n+1} \mid X_n = x_n\},
\]

where the equalities in (a) and (b) follows by the independent increment property of the process. The converse is not necessarily true, e.g., IID processes are Markov but not independent increment.
8.2.5 Gauss–Markov Process

Let $Z_1, Z_2, \ldots$ be i.i.d. $\sim N(0, N)$, i.e., $\{Z_n\}$ be a white Gaussian noise (WGN) process. The Gauss–Markov process is a first-order autoregressive process defined by

$$
X_1 = Z_1, \\
X_n = \alpha X_{n-1} + Z_n, \quad n = 2, 3, \ldots
$$

where $\alpha$ is a parameter such that $|\alpha| < 1$. This process can be generated by passing a WGN process through a discrete-time linear time invariant system, as we will see in Lecture #9 in more detail. The Gauss–Markov process is Markov. It is not, however, independent increment.

8.3 CONTINUOUS-TIME RANDOM PROCESSES

A random process is continuous time if $T$ is a continuous set, e.g., $\mathbb{R} = (-\infty, \infty)$ or $\mathbb{R}^+ = [0, \infty)$.

Example 8.7 (Sinusoidal signal with random phase). Let

$$
X(t) = \alpha \cos(\omega t + \Theta), \quad t \geq 0,
$$

where $\Theta \sim \text{Unif}[0, 2\pi]$ and $\alpha$ and $\omega$ are constants. The sample functions are illustrated in Figure 8.4. The first-order pdf of the process is the pdf of $X(t) = \alpha \cos(\omega t + \Theta)$. In Problem 3.7, we found it to be

$$
f_{X(t)}(x) = \frac{1}{\alpha \pi \sqrt{1 - (x/\alpha)^2}}, \quad -\alpha < x < +\alpha.
$$

Note that the pdf is independent of $t$.

A continuous-time random process $\{X(t): t \geq 0\}$ is said to be independent increment if $X(t+s) - X(t)$ is independent of $\{X(u): 0 \leq u \leq t\}$ for every $s, t \geq 0$. In particular, $X(t_1), X(t_2) - X(t_1), \ldots, X(t_k) - X(t_{k-1})$ are independent for every $k$ and $t_1 < t_2 < \cdots < t_k$. A continuous-time random process $\{X(t): t \geq 0\}$ is said to be Markov if the future $\{X(t+s): s > 0\}$ is conditionally independent of the past $\{X(u): 0 \leq u \leq t\}$ given the present $X(t)$. In particular, $X(t_{k+1})$ is conditionally independent of $(X(t_1), \ldots, X(t_k))$ given $X(t_k)$ for every $k$ and $t_1 < t_2 < \cdots < t_{k+1}$. As in the discrete-time case, an independent increment process is Markov, but the converse does not hold.

Unlike discrete-time random processes, a continuous-time random process cannot be fully specified by $k$-th order distributions alone. For example, knowing $k$-th order distributions for every $k$ does not answer whether the process is continuous or not. Thus, for continuous-time random processes, we often need additional properties (such as continuity).

In the following, we discuss a few famous continuous-time random processes.
Random Processes

8.3.1 Brownian Motion
A random process \( \{ W(t) : t \geq 0 \} \) is said to be a Brownian motion (or Wiener process) if

- \( W(0) = 0 \),
- \( \{ W(t) \} \) is independent increment with \( W(t) - W(s) \sim N(0, t - s) \) for every \( t > s \), and
- \( W(t) \) is continuous for \( t \geq 0 \) almost surely.

The \( k \)-th order pdf can be easily computed from the independent increment property. For example,

\[
\begin{align*}
    f_{W(t_1), W(t_2), W(t_3)}(w_1, w_2, w_3) &= f_{W(t_1)}(w_1) f_{W(t_2) | W(t_1)}(w_2 | w_1) f_{W(t_3) | W(t_2)}(w_3 | w_2) \\
    &= f_{W(t_1)}(w_1) f_{W(t_2 - t_1)}(w_2 - w_1) f_{W(t_3 - t_2)}(w_3 - w_2).
\end{align*}
\]

8.3.2 Poisson Process
A random process \( \{ N(t) : t \geq 0 \} \) is said to be Poisson with rate \( \lambda \) if

- \( N(0) = 0 \) and

Figure 8.4. Sample functions of the sinusoidal signal with random phase.
• \( \{N(t)\} \) is independent increment with \( N(t) - N(s) \sim \text{Poisson}(\lambda(t - s)) \) for every \( t > s \).

A sample path is shown in Figure 8.5. Here \( t_1, t_2, \ldots \) are the **arrival times** or the **wait times** of the events. The differences \( t_1, t_2 - t_1, \ldots \) are called the **interarrival times** of the events.

Recalling from Example 3.6, the first arrival time \( T_1 \) can be specified by the relationship

\[
\{ T_1 > t \} = \{ N(t) = 0 \},
\]

which implies that \( T_1 \) is an \( \text{Exp}(\lambda) \) random variable. More generally, the interarrival times \( T_1, T_2 - T_1, T_3 - T_2, \ldots \) are i.i.d. \( \text{Exp}(\lambda) \).

### 8.4 MEAN AND AUTOCORRELATION FUNCTIONS

For a random process \( X(t) \) the first and second order moments are

- **mean function**: \( \mu_X(t) = \mathbb{E}[X(t)] \) for \( t \in \mathcal{T} \).
- **autocorrelation function**: \( R_X(t_1, t_2) = \mathbb{E}[X(t_1)X(t_2)] \) for \( t_1, t_2 \in \mathcal{T} \).

The **autocovariance function** of a random process is defined as

\[
C_X(t_1, t_2) = \mathbb{E}[(X(t_1) - \mathbb{E}[X(t_1)))(X(t_2) - \mathbb{E}[X(t_2))]
= R_X(t_1, t_2) - \mu_X(t_1)\mu_X(t_2).
\]

**Example 8.8.** For an IID process \( X_n \)

\[
\begin{align*}
\mu_X(n) &= \mathbb{E}[X_1], \\
R_X(n_1, n_2) &= \mathbb{E}[X_{n_1}X_{n_2}] = \begin{cases} 
\mathbb{E}[X_1^2] & n_1 = n_2, \\
\mathbb{E}[X_1]^2 & n_1 \neq n_2.
\end{cases}
\end{align*}
\]
Example 8.9. For the random phase signal process in Example 8.7:

\[ \mu_X(t) = \mathbb{E}[\alpha \cos(\omega t + \Theta)] = \int_0^{2\pi} \frac{\alpha}{2\pi} \cos(\omega t + \theta) d\theta = 0, \]

\[ R_X(t_1, t_2) = \mathbb{E}[X(t_1)X(t_2)] = \int_0^{2\pi} \frac{\alpha^2}{2\pi} \cos(\omega t_1 + \theta) \cos(\omega t_2 + \theta) d\theta \]

\[ = \int_0^{2\pi} \frac{\alpha^2}{4\pi} \left[ \cos(\omega(t_1 + t_2) + 2\theta) + \cos(\omega(t_1 - t_2)) \right] d\theta \]

\[ = \frac{\alpha^2}{2} \cos(\omega(t_1 - t_2)) \]

Example 8.10. For the random walk,

\[ \mu_X(n) = \mathbb{E}\left[ \sum_{i=1}^{n} Z_i \right] = \sum_{i=1}^{n} 0 = 0. \]

To compute the autocorrelation function, first assume that \( n_1 \leq n_2 \) and consider

\[ R_X(n_1, n_2) = \mathbb{E}[X_{n_1}X_{n_2}] = \mathbb{E}[X_{n_1}(X_{n_2} - X_{n_1} + X_{n_1})] = \mathbb{E}[X_{n_1}^2] = n_1. \]

In general,

\[ R_X(n_1, n_2) = \min\{n_1, n_2\}. \]

Example 8.11. For the Gauss–Markov process,

\[ \mu_X(n) = \mathbb{E}[X_n] = \mathbb{E}[\alpha X_{n-1} + Z_n] \]

\[ = \alpha \mathbb{E}[X_{n-1}] + \mathbb{E}[Z_n] \]

\[ = \alpha \mathbb{E}[X_{n-1}] \]

\[ = \alpha^{n-1} \mathbb{E}[Z_1] = 0. \]

To find the autocorrelation function, assume first that \( n_1 < n_2 \). Then,

\[ X_{n_2} = \alpha^{n_2-n_1} X_{n_1} + \sum_{i=0}^{n_2-n_1-1} \alpha^i Z_{n_2-i}. \]

Thus,

\[ R_X(n_1, n_2) = \mathbb{E}[X_{n_1}X_{n_2}] = \alpha^{n_2-n_1} \mathbb{E}[X_{n_1}^2] + 0, \]

since \( X_{n_1} \) and \( Z_{n_2-i} \) are independent, zero mean for \( 0 \leq i \leq n_2 - n_1 - 1 \). Next, to find
8.5 Gaussian Random Processes

$E[X_{n_1}^2]$, consider

$$
E[X_{n_1}^2] = N,
E[X_{n_i}^2] = E[(\alpha X_{n_{i-1}} + Z_{n_i})^2]
= \alpha^2 E[X_{n_{i-1}}^2] + N
= \frac{1 - \alpha^{2n_1}}{1 - \alpha^2} N.
$$

Therefore, in general,

$$
R_X(n_1, n_2) = \alpha^{|n_2-n_1|} \frac{1 - \alpha^{2 \min(n_1,n_2)}}{1 - \alpha^2} N.
$$

8.5 GAUSSIAN RANDOM PROCESSES

A random process is said to be Gaussian if

$$
[X(t_1), X(t_2), \ldots, X(t_k)]^T
$$

is a Gaussian random vector for every $k$ and $t_1 < t_2 < \cdots < t_k$.

**Example 8.12.** The discrete time WGN process is Gaussian.

**Example 8.13.** The Gauss–Markov process is Gaussian. Indeed, since $X_1 = Z_1$ and $X_k = \alpha X_{k-1} + Z_k$ with $Z_1, Z_2, \ldots$ i.i.d. $N(0, N)$, we have

$$
\begin{bmatrix}
X_1 \\
X_2 \\
X_3 \\
\vdots \\
X_n
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & \cdots & 0 & 0 \\
\alpha & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha^{n-2} & \alpha^{n-3} & \cdots & 1 & 0 \\
\alpha^{n-1} & \alpha^{n-2} & \cdots & \alpha & 1
\end{bmatrix}
\begin{bmatrix}
Z_1 \\
Z_2 \\
Z_3 \\
\vdots \\
Z_n
\end{bmatrix},
$$

which is a linear transformation of a Gaussian random vector and is therefore Gaussian itself.

**Example 8.14.** The Brownian motion is Gaussian (why?).

Since the joint pdf for a Gaussian random vector is specified by its mean and covariance matrix, a discrete-time Gaussian random process is specified by its mean $\mu_X(t)$ and autocorrelation $R_X(t_1, t_2)$ functions. For the continuous-time case, mean and autocorrelation functions determine every finite-order distributions, although additional properties (such as continuity) are needed to fully specify the Gaussian random process.
8.1. *Symmetric random walk.* Let $X_n$ be a random walk defined by

\[ X_0 = 0, \]
\[ X_n = \sum_{i=1}^{n} Z_i, \]

where $Z_1, Z_2, \ldots$ are i.i.d. with $P\{Z_1 = -1\} = P\{Z_1 = 1\} = \frac{1}{2}$.

(a) Find $P\{X_{10} = 10\}$.

(b) Approximate $P\{-10 \leq X_{100} \leq 10\}$ using the central limit theorem.

(c) Find $P\{X_n = k\}$.

8.2. *Absolute-value random walk.* Consider the symmetric random walk $X_n$ in the previous problem. Define the absolute value random process $Y_n = |X_n|$.

(a) Find $P\{Y_n = k\}$.

(b) Find $P\{\max_{1 \leq i < 20} Y_i = 10 \mid Y_{20} = 0\}$.

8.3. *Sampled random walk.* Let $\{X_n\}$ be the (standard) symmetric random walk, i.e.,

\[ X_0 = 0, \]
\[ X_n = \sum_{i=1}^{n} Z_i, \quad n = 1, 2, \ldots, \]

where $Z_1, Z_2, \ldots$ are i.i.d. with $P\{Z_1 = -1\} = P\{Z_1 = 1\} = 1/2$. Let $\{Y_n\}$ be a sampled version of $\{X_n\}$ defined by

\[ Y_n = X_{2n}, \quad n = 0, 1, 2, \ldots. \]

(a) Is $\{Y_n\}$ independent increment? Justify your answer.

(b) Is $\{Y_n\}$ Markov? Justify your answer.

(c) Find $E[Y_3 \mid Y_2]$.

8.4. *Discrete-time Wiener process.* Let $Z_n$, $n \geq 0$, be a discrete time white Gaussian noise process, i.e., $Z_1, Z_2, \ldots$ are i.i.d $N(0, 1)$. Define the process $X_n$, $n \geq 1$, such that $X_0 = 0$, and $X_n = X_{n-1} + Z_n$, for $n \geq 1$.

(a) Is $X_n$ an independent increment process? Justify your answer.

(b) Is $X_n$ a Gaussian process? Justify your answer.

(c) Find the mean and autocorrelation functions of $X_n$.

(d) Specify the first-order pdf of $X_n$.

(e) Specify the joint pdf of $X_3, X_5,$ and $X_8$. 
(f) Find $E(X_{20} | X_1, X_2, \ldots, X_{10})$.

(g) Given $X_1 = 4, X_2 = 2,$ and $0 \leq X_3 \leq 4$, find the minimum MSE estimate of $X_4$.

8.5. **Wiener process.** Recall the following definition of the (standard) Wiener process:

- $W(0) = 0$,
- $\{W(t)\}$ is independent increment with $W(t) - W(s) \sim N(0, t - s)$ for all $t > s$,
- $P[\omega : W(\omega, t) \text{ is continuous in } t] = 1$.

Let $W_1(t)$ and $W_2(t)$ be independent Wiener processes.

(a) Find the mean and the variance of

$$X(t) = \frac{1}{\sqrt{2}} (W_1(t) + W_2(t))$$

Is $\{X(t)\}$ a Wiener process? Justify your answer.

(b) Find $E[X(t) | W(1) = 0]$ for $t \in [0, 1]$.

(c) Find $E[(W(t))^2 | W(1) = 0]$ for $t \in [0, 1]$.

(d) Find $E[W(t_1) W(t_2) | W(1) = 0]$ for $t_1, t_2 \in [0, 1]$.

8.6. **Brownian bridge.** Let $\{W(t)\}_{t=0}^{\infty}$ be the standard Brownian motion (Wiener process). Recall that the process is independent-increment with $W(0) = 0$ and

$$W(t) - W(s) \sim N(0, t - s), \quad 0 \leq s < t.$$ 

In the following, we investigate several properties of the process conditioned on $\{W(1) = 0\}$.

(a) Find the conditional distribution of $W(1/2)$ given $W(1) = 0$.

(b) Find $E[W(t) | W(1) = 0]$ for $t \in [0, 1]$.

(c) Find $E[(W(t))^2 | W(1) = 0]$ for $t \in [0, 1]$.

(d) Find $E[W(t_1) W(t_2) | W(1) = 0]$ for $t_1, t_2 \in [0, 1]$.

8.7. **A random process.** Let $X_n = Z_{n-1} + Z_n$ for $n \geq 1$, where $Z_0, Z_1, Z_2, \ldots$ are i.i.d. $\sim N(0, 1)$.

(a) Find the mean and autocorrelation functions of $\{X_n\}$.

(b) Is $\{X_n\}$ Gaussian? Justify your answer.

(c) Find $E(X_3 | X_1, X_2)$.

(d) Find $E(X_3 | X_2)$.

(e) Is $\{X_n\}$ Markov? Justify your answer.
Random Processes

(f) Is \( \{X_n\} \) independent increment? Justify your answer.

8.8. Moving average process. Let \( X_n = \frac{1}{2} Z_{n-1} + Z_n \) for \( n \geq 1 \), where \( Z_0, Z_1, Z_2, \ldots \) are i.i.d. \( \sim \) N(0, 1). Find the mean and autocorrelation function of \( X_n \).

8.9. Autoregressive process. Let \( X_0 = 0 \) and \( X_n = \frac{1}{2} X_{n-1} + Z_n \) for \( n \geq 1 \), where \( Z_1, Z_2, \ldots \) are i.i.d. \( \sim \) N(0, 1). Find the mean and autocorrelation function of \( X_n \).

8.10. Random binary waveform. In a digital communication channel the symbol “1” is represented by the fixed duration rectangular pulse

\[
g(t) = \begin{cases} 
1 & \text{for } 0 \leq t < 1 \\
0 & \text{otherwise},
\end{cases}
\]

and the symbol ”0” is represented by \(-g(t)\). The data transmitted over the channel is represented by the random process

\[
X(t) = \sum_{k=0}^{\infty} A_k g(t - k), \quad \text{for } t \geq 0,
\]

where \( A_0, A_1, \ldots \) are i.i.d random variables with

\[
A_i = \begin{cases} 
+1 & \text{w.p. } \frac{1}{2} \\
-1 & \text{w.p. } \frac{1}{2}.
\end{cases}
\]

(a) Find its first and second order pmfs.
(b) Find the mean and the autocorrelation function of the process \( X(t) \).

8.11. Arrow of time. Let \( X_0 \) be a Gaussian random variable with zero mean and unit variance, and \( X_n = \alpha X_{n-1} + Z_n \) for \( n \geq 1 \), where \( \alpha \) is a fixed constant with \( |\alpha| < 1 \) and \( Z_1, Z_2, \ldots \) are i.i.d. \( \sim \) N(0, 1 - \( \alpha^2 \)), independent of \( X_0 \).

(a) Is the process \( \{X_n\} \) Gaussian?
(b) Is \( \{X_n\} \) Markov?
(c) Find \( R_X(n, m) \).
(d) Find the (nonlinear) MMSE estimate of \( X_{100} \) given \( (X_1, X_2, \ldots, X_{99}) \).
(e) Find the MMSE estimate of \( X_{100} \) given \( (X_{101}, X_{102}, \ldots, X_{199}) \).
(f) Find the MMSE estimate of \( X_{100} \) given \( (X_1, \ldots, X_{99}, X_{101}, \ldots, X_{199}) \).

8.12. Convergence of random processes. Let \( \{N(t)\}_{t=0}^{\infty} \) be a Poisson process with rate \( \lambda \). Recall that the process is independent increment and \( N(t) - N(s), 0 \leq s < t \), has the pmf

\[
p_{N(t)-N(s)}(n) = \frac{e^{-\lambda(t-s)}(\lambda(t-s))^n}{n!}, \quad n = 0, 1, \ldots
\]

Define

\[
M(t) = \frac{N(t)}{t}, \quad t > 0.
\]
(a) Find the mean and autocorrelation function of \( \{ M(t) \}_{t \geq 0} \).

(b) Does \( \{ M(t) \}_{t \geq 0} \) converge in mean square as \( t \to \infty \), that is,
\[
\lim_{t \to \infty} E[(M(t) - M)^2] = 0
\]
for some random variable (or constant) \( M \)? If so, what is the limit?

Now consider \( L(t) = \frac{1}{t} \int_0^t \frac{N(s)}{s} \, ds, \quad t > 0 \).

(c) Does \( \{ L(t) \}_{t \geq 0} \) converge in mean square as \( t \to \infty \)? If so, what is the limit?
(Hint: \( \int \frac{1}{x} \, dx = \ln x + C \), \( \int \ln x \, dx = x \ln x - x + C \), and \( \lim_{x \to 0} x \ln x = 0 \).)