Solutions to Midterm Examination #1
(Prepared by TA Nadim Ghaddar)

There are 3 problems, each problem with multiple parts. Your answer should be as clear and readable as possible. Please justify any claim that you make.

1. Thinning (50 pts). Suppose that we flip a coin with bias $p$ independently $N$ times, where $N$ is a Poisson random variable with pmf

$$p_N(n) = \frac{e^{-\lambda} \lambda^n}{n!}, \quad n = 0, 1, 2, \ldots,$$

and count the number of heads as $H$ and the number of tails as $T$.

(a) Find the conditional pmf $p_{H|N}(h|n)$ of $H$ given $\{N = n\}$.
(b) Find the pmf $p_H(h)$ of $H$.
(c) Find the conditional pmf $p_{N|H}(n|h)$ of $N$ given $\{H = h\}$.
(d) Find the joint pmf $p_{H,T}(h,t)$ of $H$ and $T$.
(e) Are $H$ and $T$ independent? Justify your answer.

Solution:

(a) Note that $H|\{N = n\} \sim \text{Binom}(n, p)$. Hence,

$$p_{H|N}(h|n) = \binom{n}{h} p^h (1 - p)^{n-h}, \quad 0 \leq h \leq n$$

(b) For $h \geq 0$,

$$p_H(h) = \sum_{n=h}^{\infty} p_N(n)p_{H|N}(h|n)$$

$$= \sum_{n=h}^{\infty} \frac{e^{-\lambda} \lambda^n}{n!} \binom{n}{h} p^h (1 - p)^{n-h}$$

$$= \frac{e^{-\lambda} p^h}{h!} \sum_{n=h}^{\infty} \frac{\lambda^{(n-h)}}{(n-h)!} (1 - p)^{(n-h)}$$

$$= \frac{e^{-\lambda} (\lambda p)^h}{h!} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} (1 - p)^n$$

$$= \frac{e^{-\lambda} p^h (\lambda p)^h}{h!}.$$
Hence, $H \sim \text{Poisson}(\lambda p)$.

(c) Consider

$$p_{N|H}(n|h) = \frac{p_N(n)p_{H|N}(h|n)}{p_H(h)} = \frac{e^{-\lambda \lambda^h}(n) p_h(1-p)^{n-h}}{e^{-\lambda \lambda^h} h!}$$

for $n \geq h$.

(d) Note that

$$p_{H,T}(h, t) = \mathbb{P}\{H = h, T = t\} = \mathbb{P}\{H = h, N = h + t\} = p_N(h + t) p_{H|N}(h|h + t)$$

$$= e^{-\lambda \lambda^{h+t}} \binom{h + t}{h} p^h (1-p)^t$$

$$= \left( \frac{e^{-\lambda p} (\lambda p)^h}{h!} \right) \left( \frac{e^{-\lambda(1-p)} (\lambda(1-p))^t}{t!} \right)$$

for $h, t \geq 0$.

(e) Yes. To see this, note that the joint pmf $p_{H,T}(h, t)$ is the product of two Poisson pmfs. Thus, $H$ and $T$ are independent, although they are conditionally dependent given $N$ (as $T = H - N$).

2. **Sum and minimum (40 pts).** Let $(X, Y) \sim f_{X,Y}(x, y)$, where

$$f_{X,Y}(x, y) = \begin{cases} 2, & x \geq 0, y \geq 0, x + y \leq 1, \\ 0, & \text{otherwise}, \end{cases}$$

that is, $(X, Y)$ is uniformly distributed over the triangle formed by the points $(0, 0)$, $(0, 1)$, and $(1, 0)$. Let

$$S = X + Y \quad \text{and} \quad V = \min(X, Y).$$

(a) Find the pdf $f_S(s)$ of $S$.
(b) Find the pdf $f_V(v)$ of $V$.
(c) Find the joint pdf $f_{S,V}(s, v)$ of $S$ and $V$.  

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(d) Find the probability \( P\{S - V \leq 1/2 \} \).

Solution:

(a) We have

\[
F_S(s) = P\{S \leq s\} = P\{X + Y \leq s\} = \int_0^s \int_0^{s-x} f_{X,Y}(x,y) \, dy \, dx = \int_0^s 2(s-x) \, dx = s^2,
\]

which is equal to twice the area of the shaded region in Figure 1(a). Therefore, \( f_S(s) = 2s, \ 0 \leq s \leq 1 \).

(b) Note that

\[
P\{V > v\} = P\{\min(X,Y) > v\} = 1 - P\{X > v, Y > v\} = (1 - 2v)^2,
\]

which is twice the area of the shaded region in Figure 1(b). Therefore,

\[
F_V(v) = 1 - P\{V > v\} = 1 - (1 - 2v)^2 = 4v(1 - v), \quad 0 \leq v \leq \frac{1}{2},
\]

and \( f_V(v) = 4 - 8v, \ 0 \leq v \leq 1/2 \).

(c) We have

\[
F_{S,V}(s,v) = P\{X + Y \leq s, \min(X,Y) \leq v\} = P\{X + Y \leq s\} - P\{X + Y \leq s, \min(X,Y) > v\} = F_S(s) - P\{X + Y \leq s, \min(X,Y) > v\}
\]

Now, for \( 0 \leq v \leq s/2 \),

\[
P\{X + Y \leq s, \min(X,Y) > v\} = \int_0^v \int_v^{s-y} 2 \, dx \, dy + \int_v^s \int_0^{s-y} 2 \, dx \, dy = \int_0^v 2(-y + s - v) \, dy + \int_v^s 2(-y + s) \, dy = (s - 2v)^2,
\]
which is twice the area of the shaded region in Figure 1(c). Therefore,

\[ F_{S,V}(s, v) = s^2 - (s - 2v)^2 = 4v(s - v), \quad s \in [0, 1], v \in [0, s/2], \]

and \( f_{S,V} = 4, \ s \in [0, 1], v \in [0, s/2], \) which is uniformly distributed over the shaded region in Figure 1(d).

(d) From part (c), we have

\[
P\{S - V \leq 1/2\} = \int_0^{1/2} \int_{2v}^{v+1/2} 4dsdv
= \int_0^{1/2} 2 - 4v dv
= 1/2.
\]

Alternatively,

\[
P\{S - V \leq 1/2\} = P\{\max(X, Y) \leq 1/2\}
= P\{X \leq 1/2, Y \leq 1/2\}
= \int_0^{1/2} \int_0^{1/2} f_{X,Y}(x, y)dxdy
= 1/2.
\]

3. Gaussian mixture (30 pts). Let \( X \sim N(1, 3) \) and \( Y \sim N(-1, 2) \) be independent Gaussian random variables. Let

\[
Z = \begin{cases} 
X, & \text{with probability } 1/3, \\
Y, & \text{with probability } 2/3.
\end{cases}
\]

(a) Find the pdf \( f_Z(z) \).
(b) Is the random variable \( Z \) Gaussian? Justify your answer.
(c) Find \( P\{Z \geq 0\} \) in terms of the Q function.

Solution:

(a) We have

\[
F_Z(z) = P\{Z \leq z\}
= \frac{1}{3} P\{X \leq z\} + \frac{2}{3} P\{Y \leq z\}
\]
Therefore,

\[ f_Z(z) = \frac{1}{3} f_X(z) + \frac{2}{3} f_Y(z) \]
\[ = \frac{1}{3} \frac{1}{\sqrt{6\pi}} e^{-\frac{(z-1)^2}{6}} + \frac{2}{3} \frac{1}{\sqrt{4\pi}} e^{-\frac{(z+1)^2}{4}} \]

(b) No. Since \( f_Z(z) \) cannot be written as a Gaussian pdf, \( Z \) is not a Gaussian random variable.

(c) We have

\[ P\{Z \geq 0\} = \frac{1}{3} P\{X \geq 0\} + \frac{2}{3} P\{Y \geq 0\} \]
\[ = \frac{1}{3} \left[ \frac{1}{2} + Q\left(\frac{1}{\sqrt{3}}\right) \right] + \frac{2}{3} Q\left(\frac{1}{\sqrt{2}}\right) \]