Practice Midterm Examination (Winter 2017)

There are 3 problems, each problem with multiple parts. Your answer should be as clear and readable as possible. Please justify any claim that you make.

The following facts might be useful:

\[ \int t^k \, dt = \frac{t^{k+1}}{k+1} + C, \]
\[ \int \ln t \, dt = t \ln t - t + C, \]

and for \(|x| < 1\),

\[ \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \]
\[ \sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}, \]
\[ \sum_{n=2}^{\infty} n(n-1)x^{n-2} = \frac{2}{(1-x)^3}. \]

1. MMSE estimation (40 pts). Let \( X \sim N(\mu, P) \) be the input to the additive Gaussian noise channel with two outputs

\[ Y_1 = X + Z_1, \]
\[ Y_2 = X + Z_2, \]

where \( Z_1 \sim N(0, N_1) \) and \( Z_2 \sim N(0, N_2) \) are independent of each other and of \( X \).

(a) Find the estimate of \( X \) in the form

\[ \hat{X} = a_1 Y_1 + a_2 Y_2 \]

that minimizes the mean square error \( E[(X - \hat{X})^2] \).

(b) Find the MSE of the estimate found in part (a).
(c) Find the estimate of $X$ in the form

$$\hat{X} = a_1Y_1^2 + a_2Y_2^2 + a_{12}Y_1Y_2 + b_1Y_1 + b_2Y_2 + c$$

that minimizes the MSE.

(d) Find the MSE of the estimate found in part (c).

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2. Random number of coin flips (30 pts). Suppose that we flip a coin with bias $p$ independently $N$ times, where $N \sim \text{Geom}(p)$ is random. Let $X$ be the number of heads, that is, $X \mid \{N = n\} \sim \text{Binom}(n, p)$.

(a) Find $E[X]$.

(b) Find $\text{Var}(X)$.

(c) Find $P\{X = 0\}$.

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3. Convergence (10 pts). Suppose that $X_1, X_2, \ldots$ are i.i.d. $\text{Unif}[0, 1]$ random variables. Find the limit

$$\lim_{n \to \infty} \left( \prod_{i=1}^{n} X_i \right)^{1/n}$$

in probability.
1. Inequalities (18 points). Let $X$ be a random variable, with $E[X] = 0$ and $\text{Var}[X] = 1$. For each of the following pairs of quantities $A$ vs. $B$, indicate their relationship with $\leq$, $=,$ or $\geq$. That is: $A \leq B$, $A = B$, or $A \geq B$. If the correct answer is $A = B$, no credit will be given for the other options.

Justify your answers.

(a) (i) $E[e^{2X^4}]$ vs. $e^{2E[X^4]}$
(ii) $E\left[\frac{1}{X}\right]$ vs. $\frac{1}{E[\frac{1}{X}]}$
(iii) $P(X^4 \leq 4)$ vs. $P(X^4 \geq 4)$

Now let $X$ and $Y$ be random variables. Assume that $E[X] = E[Y] = 0$ and $\text{Var}[X] = \text{Var}[Y] = 1$. Follow the same instructions as in part (a).

(b) (i) $\text{Var}(X)$ vs. $E[\text{Var}(X|Y)]$
(ii) $E[(X - E[X|Y])(Y - E[X|Y])]$ vs. 0
(iii) $\text{Var}(X + Y) - 2\text{Cov}(X, Y)$ vs. 1

2. Big bank (22 points). You are an active customer at a big bank. The bank schedule calls for a different number of tellers at various times. The number of tellers when you visit each morning is $N$, where $N \sim \text{Geom}(p)$, i.e., $P(N = n) = (1 - p)^{n-1}p$, for a fixed $0 < p < 1$. (Since banking is a business, naturally the probability of there being $n$ tellers decreases as a function of $n$.)

You are always the first in line at the bank each morning, while the tellers are busy preparing to serve customers. The preparation times of the tellers, who are indexed by $i = 1, 2, \ldots$, are independent exponentially distributed random variables $X_i \sim \text{Exp}(\lambda)$, for all $i \geq 1$, and also independent of $N$.

Let $Y$ be your waiting time until a teller is ready to serve you.

(a) (7 points) Determine the pdf $f_{Y|N}(y|n)$ of your waiting time given that the number of tellers is $N = n$.

(b) (7 points) Determine the mean $E[Y]$ of your waiting time.

(c) (8 points) Determine the variance $\text{Var}(Y)$ of your waiting time.
3. **Linear Prediction (20 points).** Let \( \mathbf{X} = [X_2, X_2, X_3]^\top \) be a random vector, with mean zero and covariance matrix given by

\[
\Sigma_{\mathbf{X}} = \begin{bmatrix}
1 & \beta & \beta^2 \\
\beta & 1 & \beta \\
\beta^2 & \beta & 1
\end{bmatrix}
\]

where \(|\beta| < 1\).

(a) (6 points) Find the minimum mean-square error (MMSE) linear estimate of \( X_3 \) given \( X_1 \), and determine the corresponding mean-square error (MSE).

(b) (6 points) Find the minimum mean-square error (MMSE) linear estimate of \( X_3 \) given \( X_2 \), and determine the corresponding mean-square error (MSE).

(c) (6 points) Find the minimum mean-square error (MMSE) linear estimate of \( X_3 \) given \( X_1 \) and \( X_2 \), and determine the corresponding mean-square error (MSE).

(d) (2 points) What do your results tell you about the relative usefulness of \( X_1 \) and \( X_2 \) in linear prediction of \( X_3 \)?