Cumulative distribution function (cdf)

The probability of an (arbitrary) random variable \( X \) is specified by:

\[
P(\{ X \in (a, b) \}) = P(\{ a < X \leq b \}), \quad \forall a, b
\]

Equivalently, it suffices to specify its cdf:

\[
F_X(x) = P(\{ X \leq x \}), \quad \forall x \in \mathbb{R}
\]

Note that:

\[
P(\{ X \in (a, b) \}) = F_X(b) - F_X(a)
\]

Note:

\[
P(\{ X < a \}) = P(\bigcup_{b < a} \{ X \leq b \}) = \lim_{b \uparrow a} F_X(b)
\]

Properties of cdf

1. \( F_X(x) \geq 0 \)
2. \( F_X(x) \) is monotonically non-decreasing
3. \( \lim_{x \to -\infty} F_X(x) = 0 \), \( \lim_{x \to \infty} F_X(x) = 1 \)
4. \( F_X(x) \) is right-continuous:
   \[
   F_X(a^+) = \lim_{x \downarrow a} F_X(x) = F_X(a)
   \]
   \((x \to a^+)(approaching \ a \ from \ the \ right \ side)\)
5. \( P(\{ X = a \}) = P(\{ X < a \}) - P(\{ X < a \}) = F_X(a) - F_X(a^-)
   \]
   \[
   = F_X(a) - \lim_{x \uparrow a} F_X(x)
   \]
   \((x \to a^-)\)
(6) For any event $A$, $P(\{X \in A\})$ can be computed by $F_X(x)$.

(7) If $X$ is discrete, then $F_X(x)$ consists of countably many steps.

Definition

A random variable $X$ is continuous if $F_X(x)$ is continuous.

Definition

$X$ is mixed if it is neither discrete nor continuous.

If $F_X(x)$ is continuous and differentiable (except for countably many points), then $X$ has a probability density function (pdf or density) $f_X(x)$ such that:

$$F_X(x) = \int_{-\infty}^{x} f_X(u) \, du$$

And if $F_X(x)$ is differentiable at $x$, then:

$$\frac{dF_X(x)}{dx} = f_X(x)$$

$$f_X(x) = \frac{d}{dx} F_X(x) = \lim_{\Delta x \to 0} \frac{F_X(x + \Delta x) - F_X(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{P(\{x < X \leq x + \Delta x\})}{\Delta x}$$

Properties

(1) $f_X(x) \geq 0$ (since $F_X(x)$ is non-decreasing)

(2) $\int_{-\infty}^{\infty} f_X(x) \, dx = 1 \Leftrightarrow F_X(\infty) = 1$
(3) For any event \( A \):
\[
P(\{X \in A\}) = \int_A f_X(x) \, dx
\]

For example:
\[
P(\{X \in (a, b)\}) = P(\{X \in (a, b)\}) = P(\{X \in [a, b]\}) = \int_a^b f_X(x) \, dx
\]

Note:
(i) \( f_X(x) \) can be greater than 1 (pdf is not probability)
(ii) Some EE texts use delta function to denote jumps in the cdf.

For example:
if \( X \sim \text{Bern}(p) \)

\[
F_X(x)
\]

\[
\frac{1}{1-p}
\]

\[
X \sim U[a, b]
\]

\[
\frac{1}{b-a}
\]

Models quantization noise

(2) Exponential:
\( X \sim \text{Exp}(\lambda), \lambda > 0 \)

\[
f_X(x) = \lambda e^{-\lambda x}
\]

Models service/inter-arrival times
**Memoryless property**

For any \( a, t > 0 \):

\[
P(X > a + t | X > a) = P(X > t)
\]

Proof:

\[
P(X > t) = \int_{t}^{\infty} \lambda e^{-\lambda x} \, dx = e^{-\lambda t}
\]

\[
P(X > a + t | X > a) = \frac{P(X > a + t, X > a)}{P(X > a)} = \frac{P(X > a + t)}{P(X > a)} = \frac{e^{-\lambda (a + t)}}{e^{-\lambda a}} = e^{-\lambda t}
\]

**Relationship between exponential and Poisson**

Let \( N \) be the number of packet arrivals during the time interval \((0, t]\) at rate \( \lambda \), i.e. \( P_N(n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \), \( n = 0, 1, 2 \ldots \) (suppose this is true for every \( t \)).

Let \( X \) be the time until the first packet arrival. Find the pdf of \( X \).

\[0 \quad \text{suppose 1st packet arrives here} \quad t\]

\[
\{X \leq t\} \subset \{N \geq 1\} \text{ or } \{X \leq t\} = \{N \geq 1\} \\
\Rightarrow P(\{X \leq t\}) = P(\{N \geq 1\}) = 1 - e^{-\lambda t} \\
\Rightarrow \{X > t\} = \{N = 0\}
\]

\( F_X(t) \): pdf of \( X \) is:

\[
f_X(t) = \frac{\lambda e^{-\lambda t}}{t} \quad , \quad t > 0
\]

**Gaussian (normal) distribution**:

\( X \sim N(\mu, \sigma^2) \)

Model for noise: thermal noise and shot noise.

\[
f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}
\]