\[ X \sim \text{Binom}(n, \frac{2}{n}) \]
\[ P_X(k) = \binom{n}{k} \left(\frac{2}{n}\right)^k \left(1 - \frac{2}{n}\right)^{n-k} \]
\[ \xrightarrow{n \to \infty} \frac{2^k}{k!} e^{-\frac{2}{n}} \quad \text{(pmf of Poisson}(\lambda)) \]

\[ \text{Binom}(\frac{1}{\lambda}, \lambda) \xrightarrow{\lambda \to \infty} \text{Poisson}(\lambda) \]

**Cumulative Distribution Function (cdf)**

The probability of an arbitrary r.v. \( X \) is specified by
\[ P(\{X \in [a,b]\}) = P(\{a < X \leq b\}) \quad \forall a, b \]

Equivalently, it suffices to specify its cdf
\[ F_X(x) = P(\{X \leq x\}) \quad \forall x \in \mathbb{R} \]

**Note:**
\[ P(\{X \in (a,b]\}) = F_X(b) - F_X(a) \]

**Note:**
\[ P(\{X < a\}) = P(\{U \{X \leq b\}) = \lim_{b \to a} F_X(b) \]

**Properties of cdf (distribution)**

1. \( F_X(x) \geq 0 \)
2. \( F_X(x) \) is (monotonically) non-decreasing
3. \( \lim_{x \to -\infty} F_X(x) = 0 \), \( \lim_{x \to \infty} F_X(x) = 1 \)
4. \( F_X(x) \) is right continuous, i.e. \( F_X(a^+) = \lim_{x \to a^+} F_X(x) = F_X(a) \)
5. \( P(\{X = a\}) = P(\{X < a\}) - P(\{X \leq a\}) = F_X(a) - F_X(a^-) \)
6. For any event \( A \), \( P(\{X \in A\}) \) can be determined by \( F_X(x) \)
7. If \( X \) is discrete, then \( F_X(x) \) consists of countably many steps.
**Def** A random variable $X$ is **continuous** if $F_x(x)$ is continuous.

**Def** $X$ is **mixed** if it is neither discrete nor continuous.

If $F_x(x)$ is continuous and differentiable (except for countably many points) then $X$ has a probability density function (pdf or density) $f_x(x)$ such that

$$F_x(x) = \int_{-\infty}^{x} f_x(u) \, du$$

And if $F_x(x)$ is differentiable at $x$, then

$$f_x(x) = \frac{d}{dx} F_x(x) = \lim_{\Delta x \to 0} \frac{F_x(x + \Delta x) - F_x(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{P(x < X \leq x + \Delta x)}{\Delta x}$$

**Properties**

1. $f_x(x) \geq 0$ (if $f_x(x)$ is non-decreasing)
2. $\int_{-\infty}^{\infty} f_x(x) \, dx = 1$ $\Rightarrow$ $F_x(\infty)
3. For any event $A$, $P(\{x \in A\}) = \int_A f_x(x) \, dx$

For example, if $X$ is continuous,

$$P(\{x \in (a, b)\}) = P(\{x \in (a, b)\}) = P(\{x \in (a, b)\}) = \int_a^b f_x(x) \, dx$$

**Note**

1. $f_x(x)$ can be greater than 1 (pdf is NOT probability)
2. Some EE texts use delta functions to denote jumps in the CDF

For example, if $X \sim \text{Bern}(p)$:

$$f_x(x) \rightarrow \begin{cases} 0 & \text{if } x < 0 \\ 1-p & \text{if } x = 1 \\ p & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

**Examples of continuous r.v.s**

1. **Uniform**: $X \sim U[a, b]$

2. **Exponential**: $X \sim \text{Exp}(-\lambda)$, $\lambda > 0$

Models service/inter-arrival times

**Memoryless Property**

For any $a, t > 0$, $P(X > a + t \mid X > a) = P(X > t)$

**pdf**

$$P(x > t) = \int_{t}^{\infty} \lambda e^{-\lambda x} \, dx = e^{-\lambda t}$$

$$P(x > a + t \mid x > a) = \frac{P(x > a + t, x > a)}{P(x > a)} = \frac{P(x > a + t)}{P(x > a)} = \frac{e^{-\lambda (a+t)}}{e^{-\lambda a}} = e^{-\lambda t}$$
Relationship between Exponential and Poisson

Let $N$ be the number of packet arrivals during interval $(0, t]$ at rate $\lambda$, i.e.,

$$P_N(n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \quad n = 0, 1, 2, \ldots$$

Let $X$ be the time until the first packet arrival. Find the pdf of $X$.

$$P(\{X \leq t\}) = P(\{N \geq 1\}) = 1 - e^{-\lambda t}$$

The pdf of $X$ is $f_X(t) = \lambda e^{-\lambda t}$, $t > 0$

(3) Gaussian (normal): $X \sim N(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Model for thermal and shot noise