n = 298 million combinations

\( \text{an tickets are sold. (say} \, a = 2) \)

What is probability that there is no winner?

\[ n \text{ winners} = \text{Binom} \left( a \, n, 1 \right) \]

\[ p(\text{no winner}) = \frac{p(\text{Binom}(a \, n, 1) = 0)}{n} \rightarrow p(\text{Poisson}(a) = 0) \]

Really \( n \to \infty \) large

\[ = e^{-a} \]

When \( a = 1 \), then \( p(\text{no winner}) = e^{-1} = 37\% \)

\( a = 2 \), \( p(\text{no winner}) = e^{-2} = 14\% \)

\( a = 3 \), \[ \frac{n}{e^{3}} = e^{-3} = 5\% \]

- **Cumulative Distribution Function**: (cdf)

  The probability of any R.V. is fully specified by

  \[ P \left( x \in (a, b] \right) = P \left( a < x \leq b \right) = a, b \]

  (If known for every \( a, b \), we know probability for every Borel set)

  Equivalently, it suffices to specify \( \text{cdf} \).

  \[ F_x(x) = P \left( x \leq x \right) \forall x \in \mathbb{R} \]

  Note: i) \( P \left( \{ x \in (a, b) \} \right) = \frac{b - a}{a - b} \)

  \[ = F_x(b) - F_x(a) \]

  ii) \( P \left( x < a \right) = P \left( \bigcup_{b \text{ is rational, } b < a} (x < b) \right) = \lim_{b \to a} F_x(b) \)
Properties of cdf (distribution):

1) $F_x(x) \geq 0$
2) $F_x(x)$ is monotonically non-decreasing
3) $\lim_{x \to -\infty} F_x(x) = 0$ \& $\lim_{x \to \infty} F_x(x) = 1$
4) $F_x(x)$ is right continuous, i.e.,
   
   $F_x(a^+) = \lim_{x \to a} F_x(x) = F_x(a)$

5) $P(x = a) = F_x(a) - F_x(a^-) = F_x(a) - F_x(a)$

6) For any event $A$, $P(x \in A)$ can be determined by $F_x(x)$

7) If RV $X$ is discrete, then $F_x(x)$ is a staircase (consists of countably many steps).

Continuous RV:
A RV 'X' is continuous if its cdf is continuous.

'X' is mixed if it is neither discrete nor continuous.

If $F_x(x)$ is continuous & differentiable (except for countably many points), then by the theory of integration, 'X' has a probability density function (pdf or density) $f_x(x)$ such that

$F_x(x) = \int_{-\infty}^{x} f_x(u) \, du$
If $F_X(x)$ is differentiable at $x$, then
\[
\frac{d}{dx} F_X(x) = f_X(x)
\]

pdf does not have to be continuous for a RV to be continuous.

\[
f_X(x) = \frac{d}{dx} F_X(x) = \lim_{\Delta x \to 0} \frac{F_X(x+\Delta x)-F_X(x)}{\Delta x}
\]

\[
= \lim_{\Delta x \to 0} \frac{P(x < x \leq x+\Delta x)}{\Delta x}
\]

Properties:

1) $f_X(x) \geq 0$ ($F_X(x)$ is non-decreasing, not probability.)

2) $\int_{-\infty}^{\infty} f_X(x) \, dx = 1 = F_X(\infty)$

3) For any event $A$,
\[
P(A) = \int f_X(a) \, da
\]

eg. $P(x \in [a, b]) = P(x \in (a, b)) = P(x \in [a, b]) - P(x \in (a, b)) = \int_a^b f_X(a) \, da$

Notes:

1) $f_X(x)$ can be greater than 1, pdf is NOT probability.

2) Some EE texts use delta function to denote jumps in the cdf.

eg. if $X \sim \text{Bern}(p)$

\[
\begin{align*}
F_X(x) &= \begin{cases} \\
0 & x < 0 \\
p & 0 \leq x < 1 \\
1 & x \geq 1 
\end{cases} \\
f_X(x) &= \begin{cases} \\
0 & x < 0 \\
p & 0 \leq x < 1 \\
-1 & x \geq 1 
\end{cases}
\end{align*}
\]
Examples:
1) Uniform: \( X \sim U[a,b] \)
\[ f_X(x) = \frac{1}{b-a} \quad \text{for } a \leq x \leq b \]
Models quantization noise.

2) Exponential:
\( X \sim \text{Exp}(\lambda) \), \( \lambda > 0 \)
\[ f_X(x) = \lambda e^{-\lambda x} \quad \text{for } x \geq 0 \]
Models service/inter-arrival time.

Memoryless Property:
For any \( a, t \geq 0 \),
\[ P(X > a + t \mid X > a) = P(X > t) \]
Proof:
\[ P(X > t) = \int_t^{\infty} \lambda e^{-\lambda x} \, dx = e^{-\lambda t} \]
\[ P(X > a + t \mid X > a) = \frac{P(X > a + t, X > a)}{P(X > a)} = \frac{P(X > a + t)}{P(X > a)} = e^{-\lambda(a+t)} \]
\[ = e^{-\lambda a} \]

Exponential RV is memoryless.

Relation between exponential & Poisson:
Let RV 'N' denotes the number of packet arrivals during the time interval \((0,t)\) at rate \( \lambda \), i.e.,
\[ p_n(n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \quad n = 0, 1, \ldots \]

Let 'X' be the time until the first packet arrival. Find pdf of 'X'.

\[ t \quad \Uparrow \quad X \quad \text{1st arrival} \]

\[ P(X \leq t) = P(N \geq 1) \]

\[ P(X \leq t) = P(N \geq 1) \quad \text{III} \]

\[ P(X \leq t) = P(N \geq 1) \quad \text{IV} \]

\[ F_X(t) = 1 - e^{-\lambda t} \]

\[ f_X(t) = \frac{d}{dt} F_X(t) = \lambda e^{-\lambda t}, \quad t \geq 0 \]

Exponential

3) Gaussian (Normal)

\[ X \sim N(\mu, \sigma^2) \]

Mean Variance

Model for thermal & shot noise

\[ f_X(x) = \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \]