Let $X$ be a Random Variable with Known cdf/pdf/pmf. Find the distribution of $Y = g(X)$.

General recipes

1) Suppose $X \sim p_X(x)$ is discrete.

\[ P_Y(y) = P(\{Y \leq y\}^c) = P(\{g(X) \leq y\}^c) \]
\[ = P(X \in \{x : g(x) \leq y\}^c) \]
\[ = \sum_{x : g(x) \leq y} p_X(x) \]

2) Suppose $X \sim f_X(x)$ is continuous then

\[ F_Y(y) = P(\{Y \leq y\}) = P(\{g(X) \leq y\}) \]
\[ = P(X \in \{x : g(x) \leq y\}) \]
\[ = \int_{x : g(x) \leq y} f_X(x) \, dx \]

\[ f_Y(y) = \frac{d}{dy} \int_{x : g(x) \leq y} f_X(x) \, dx \]

Examples:

1) Linear function $Y = aX + b$, $a \neq 0$

\[ F_Y(y) = P(\{Y \leq y\}) = P(\{aX + b \leq y\}) = \begin{cases} P(X \leq \frac{y-b}{a}) = F_X(\frac{y-b}{a}) & a > 0 \\ P(X > \frac{y-b}{a}) = 1 - F_X(\frac{y-b}{a}) & a < 0 \end{cases} \]

Take derivative:

\[ f_Y(y) = \begin{cases} \frac{1}{a} f_X(\frac{y-b}{a}) & a > 0 \\ -\frac{1}{a} f_X(\frac{y-b}{a}) & a < 0 \end{cases} \Rightarrow f_Y(y) = \frac{1}{|a|} f_X(\frac{y-b}{a}) \]

Note: Let $X \sim N(0,1)$ (Standard Normal) $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

and $Y = aX + b$, $a > 0$

\[ f_Y(y) = \frac{1}{a} f_X(\frac{y-b}{a}) = \frac{1}{\sqrt{2\pi} a} e^{-\frac{(y-b)^2}{2a^2}} = N(b, a^2) \]

\[ P{\Phi}(n) = P(\{X \leq x\}) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \, du \]

\[ Q(n) = 1 - P{\Phi}(n) = P(\{X > x\}) \]
2) Quadratic function: Let $X \sim f_X(x)$ and $Y = X^2$

Then for $y \geq 0$

$$F_Y(y) = P(Y \leq y)$$

$$= P(X^2 \leq y)$$

$$= P\left(\{-\sqrt{y} \leq X \leq \sqrt{y}\}\right)$$

$$= P\left(X \leq \sqrt{y}\right) - P\left(X \leq -\sqrt{y}\right)$$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

Taking derivative:

$$f_Y(y) = \frac{1}{2\sqrt{y}} \left[ f_X(\sqrt{y}) + f_X(-\sqrt{y}) \right] \quad y \geq 0$$

(3) Let $X$ be continuous with invertible $F(x)$

and $Y = F(X) \in [0, 1]$ Then

$$F_Y(y) = P(Y \leq y)$$

$$= P(F(X) \leq y)$$

$$= P\left(X \leq F^{-1}(y)\right)$$

$$= F\left(F^{-1}(y)\right)$$

$$F_Y(y) = y \quad \text{for } y \in [0, 1]$$

$$f_Y(y) = 1 \quad \text{for } y \in [0, 1]$$

$\Rightarrow Y \sim U[0, 1]$

4) Let $X \sim U[0, 1]$ and $Y = F^{-1}(X)$ for some invertible CDF $F$. Then:

$$F_Y(y) = P(Y \leq y)$$

$$= P(F^{-1}(X) \leq y)$$

$$= P\left(X \leq F(y)\right)$$

$$= F(Y)$$

$X_1, X_2, \ldots, X_n \text{ iid } \text{Bern}(\frac{1}{2})$

$X = X, X_1, X_2, \ldots, X_n \overset{\text{convergence in distribution}}{\rightarrow} U[0, 1]$
Two Random Variables

we say that random variables $X$ and $Y$ are defined on the same probability space and throughout, unless noted otherwise, random variables $X$ and $Y$ are assumed to be defined on the same probability space.

We will focus on three common cases:

1) $X$ and $Y$ are discrete
2) $X$ and $Y$ are continuous
3) $X$ is discrete and $Y$ is continuous (mixed)

Discrete random pair $(X,Y)$

The probability is fully specified by their joint pmf $P_{X,Y}(x,y) = P\left(\{x=x, y=y\}\right)$, $x \in X$, $y \in Y$.

Marginal pmf of $X$ (by the law of total probability):

$$P_X(x) = P\left(\{x=x\}\right) = \sum_{y \in Y} P\left(\{x=x, y=y\}\right) = \sum_{y \in Y} P_{X,Y}(x,y)$$

Chain Rule:

$$P_{X,Y}(x,y) = P_X(x) P_{Y|X}(y|x) \quad x \in X, y \in Y$$

Independence: we say $X$ and $Y$ are "independent" if

$$P_{X,Y}(x,y) = P_X(x) P_Y(y) \quad x \in X, y \in Y$$

or equivalently

$$P_{X,Y}(x,y) = P_X(x) \quad x \in X, y \in Y$$

law of total probability

$$P(A) = \sum_{x \in X} P(A, x \in X)$$

$$= \sum_{x \in X} P(A|x=x) P_x(x)$$

Another way of writing marginal pmf:

$$P_Y(y) = \sum_{x \in X} P_{Y|X}(y|x) P_X(x)$$
**Example (binary symmetric channel)**

\[ X \xrightarrow{1 \rightarrow P} Y \]

\[ P_{x}(1) = \alpha \quad P_{x}(0) = 1 - \alpha \]

\[ P_{y|x}(0|0) = 1 - P \]

\[ P_{y|x}(1|0) = P \]

\[ P_{y|x}(0|1) = P \]

\[ P_{y|x}(1|1) = 1 - P \]

Find \( P_{y}(y) \) and \( P_{x|y}(x|y) \)

**First:**

\[ P_{y}(0) = P_{x}(0) P_{y|x}(0|0) + P_{x}(1) P_{y|x}(0|1) \]

\[ = (1 - \alpha)(1 - P) + \alpha P \]

\[ P_{y}(1) = P_{x}(0) P_{y|x}(1|0) + P_{x}(1) P_{y|x}(1|1) \]

\[ = 1 - P_{y}(0) = (1 - \alpha) P + \alpha (1 - P) \]

**Second:**

\[ P_{x|y}(0|0) = \frac{P_{x}(0) P_{y|x}(0|0)}{P_{y}(0)} = \frac{P_{x}(0)}{P_{y}(0)} = \frac{(1 - \alpha)(1 - P)}{(1 - \alpha)(1 - P) + \alpha P} \]

\[ P_{x|y}(1|1) = \frac{\alpha (1 - P)}{(1 - \alpha)(1 - P) + \alpha (1 - P)} \]

Find \( p(\{ x \neq y \}) \)

\[ p(\{ x \neq y \}) = p(\{ x \neq y, x = 1 \}) + p(\{ x \neq y, x = 0 \}) \]

\[ = p(\{ x = 0 \} | x = 1 \}) P_{x}(1) + p(\{ x \neq y, x = 0 \}) P_{x}(0) \]

\[ = p(\{ y = 1 \} | x = 0 \}) P_{x}(1) + p(\{ y = 1 \} | x = 0 \}) P_{x}(0) \]

\[ = pa \]

\[ \Rightarrow p(x \neq y) = P \]

\[ x \text{ and } y \text{ are independent when } P = \frac{1}{2} \]

When \( P = \frac{1}{2} \), \( P_{x}(0) = \frac{1}{2} = P_{x}(1) \)

But since \( P = \frac{1}{2} \)

\[ P_{y|x}(0|0) = \frac{1}{2} = P_{y|x}(0|1) \]

\[ P_{y|x}(1|0) = P_{y|x}(1|1) = P_{y|x}(1|1) \]

**Continuous random pair \((x, y)\)**

The probability of any random pair \((x, y)\) is fully specified by their joint cdf

\[ F_{xy}(x, y) = p(\{ x \leq x, y \leq y \}) \quad x, y \in \mathbb{R} \]
Properties of joint cdf

1) \( F_{xy}(x,y) \geq 0 \)

2) If \( x_1 \leq x_2 \) and \( y_1 \leq y_2 \) then \( F_{xy}(x_1,y_1) \leq F_{xy}(x_2,y_2) \)

3) \( \lim_{(x,y)\to(\infty,\infty)} F_{xy}(x,y) = 1 \)

4) \( \lim_{x \to -\infty} F_{xy}(x,y) = \lim_{y \to -\infty} F_{xy}(x,y) = 0 \)

5) \( \lim_{y \to +\infty} F_{xy}(x,y) = F_x(x) \), \( \lim_{x \to +\infty} F_{xy}(x,y) = F_y(y) \)

6) The probability of any event can be determined by \( F_{xy}(x,y) \)

   for example:

\[
P\left\{ a < X \leq b \ , \ c < Y \leq d \right\} = F_{xy}(b, d) - F_{xy}(a, d) - F_{xy}(b, c) + F_{xy}(a, c)
\]

7) \( X \) and \( Y \) are said to be independent if

\[ F_{xy}(x,y) = F_x(x) \cdot F_y(y) \quad \forall x, y \in \mathbb{R} \]

We say that \( X \) and \( Y \) are jointly continuous if

\( F_{x,y}(x,y) \) is continuous in \((x,y)\)

Then,

1) \( X \) is continuous and \( Y \) is continuous

2) There exist a function \( f_{xy}(x,y) \) such that

\[ F_{xy}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{xy}(u,v) \, du \, dv \]

3) If \( f_{xy}(x,y) \) is differentiable at \((x,y)\)