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Stationary random processes

- Stationarity refers to time invariance of some, or all, of the statistics of a random process, such as mean, autocorrelation, and n-th order distribution.

- We define two types of stationarity: strict sense stationarity (SSS) and wide sense stationarity (WSS).

- A random process \( \{X(t)\} \) is said to be SSS if all its finite order distributions are time-invariant, i.e., the joint distribution of \( X(t_1), X(t_2), \ldots, X(t_n) \) is equal to that of \( X(t_1+t), X(t_2+t), \ldots, X(t_n+t) \) for every \( t < t_2 < \ldots < t_n \), every \( T \), and every \( n \).

- For a strict sense stationary \( \{X(t)\} \), the first-order distribution is independent of \( t \), and the second order distribution of \( X(t_1) \) and \( X(t_2) \) depends only on \( T= t_2 - t_1 \).

- IID processes are SSS.

- Random walk, Poisson processes and the Brownian motion are NOT SSS.

- The Gauss-Markov process is NOT SSS. But if we choose the distribution of \( X_0 \) to be the steady state distribution, then we can make it SSS.

- A random process \( \{X(t)\} \) is said to be WSS if its mean and autocorrelation functions are time-invariant, i.e.,
  \[
  \begin{align*}
  \{ E[X(t)] \} &= \mu, \text{ independent of } t. \\
  R_X(t_1, t_2) &= \text{a function only of the time difference } t_2 - t_1.
  \end{align*}
  \]

  For technical reasons, we also define \( E[(X(t))^2] < \infty \).

  Since \( R_X(t_1, t_2) = R_X(t_2, t_1) \), it is a function of \( |t_2 - t_1| \).

  Also, \( R_X(t_1, t_1) = R_X(0, t_1) \). Hence, we write \( R_X(t) \) to mean \( R_X(0, T) = R_X(t, 0) \).

- Clearly, SSS \( \Rightarrow \) WSS. (assuming \( E[(X(t))^2] < \infty \))

- The converse is not true. (Think of your own example!)

- If \( \{X(t)\} \) is Gaussian and WSS, then it is also SSS.

- Random walk, Poisson processes and the Brownian motion are NOT WSS. For example, if \( \{X(t)\} \) is the random walk, then \( R_X(n, m) = \min\{n, m\} \).

  (Another check: If \( \{X(t)\} \) is WSS, then \( E[X(t)^2] = E[X(0)^2] \) is constant.)
Autocorrelation function of WSS processes

1. \( R_x(t) = E[x(t)x(t)] \) is even, i.e., \( R_x(t) = R_x(-t) \) for all \( t \). (Why? \( R_x(0,t) = R_x(t,0) = R_x(-t,0) \))

2. \( |R_x(t)| \leq R_x(0) = E[x^2(t)] \) (the average power of \( X(t) \))

   Why? \( (R_x(t))^2 = \left( E[x(t)x(t+\tau)] \right)^2 \leq E[x^2(t)]E[x^2(t+\tau)] = \left( E[x^2(t)] \right)^2 \) by CS ineq.

3. If \( R_x(t) = R_x(0) \) for some \( T \neq 0 \), then \( R_x(t) \) is periodic with period \( T \) and so is \( X(t) \) (with probability 1), i.e.,

   \[ R_x(t) = R_x(t+T) \] and \( P(X(t) = X(t+T)) \) for every \( t \) = 1.

Example: Let \( X(t) = \alpha \cos(\omega t + \Theta) \) where \( \Theta \sim U(0,2\pi) \).

Then, \( R_x(t) = E[x(t)x(t)] = \alpha^2 E[\cos(\Theta)\cos(\omega t + \Theta)] = \frac{\alpha^2}{2}\cos(\omega t) \).

Here, \( X(t) \) is periodic with period \( T = \frac{2\pi}{\omega} \) and so is \( R_x(t) \).

To prove property, note that for every \( \tau \)

\[
(R_x(t) - R_x(t+T))^2 = \left( E[x(t)(x(t+\tau) - x(t+\tau+T))] \right)^2 \leq E[x^2(t)] E[(x(t+\tau) - x(t+\tau+T))^2]
\]

\[
= R_x(0)(2R_x(0) - 2R_x(\tau)) = 0 \Rightarrow R_x(t) = R_x(t+T)
\]

\( \{X(t)\} \) is said to be cyclostationary with cycle \( T \) if \( X(t_1), X(t_2), \ldots, X(t_n) \) has the same distribution as \( X(t_1+kT), X(t_2+kT), \ldots, X(t_n+kT) \) for every \( t_1, t_2, \ldots, t_n \), every \( k = 1, 2, \ldots \) and every \( n \).

Corollary: \( X(t), X(t+T), X(t+2T), \ldots \) is a discrete-time stationary process.

4. \( R_x(t) \) is positive definite (nonnegative definite), i.e., for any \( n \), any \( t_1, t_2, \ldots, t_n \), and \( a_1, a_2, \ldots, a_n \in \mathbb{R} \)

   \[
   (a_1, a_2, \ldots, a_n)^T \begin{bmatrix}
   R_x(t_1-t_1) & \cdots & R_x(t_1-t_n) \\
   \vdots & \ddots & \vdots \\
   R_x(t_n-t_1) & \cdots & R_x(t_n-t_n)
   \end{bmatrix}
   \begin{bmatrix}
   a_1 \\
   a_2 \\
   \vdots \\
   a_n
   \end{bmatrix} \geq 0.
   
   \]

   (Later, we argue that this property can be checked easily by considering the Fourier transform of \( R_x(t) \) is nonnegative.)
Which functions can be $R_{x}(t)$?

(1) \[ \text{NO} \] \hspace{1cm} \text{NOT even} \hspace{1cm} \text{sampling} \hspace{1cm} \text{stationary Gauss Markov} \hspace{1cm} \text{Even, periodic, and FT is nonnegative} \]

(2) \[ e^{-\alpha t} \] \hspace{1cm} \text{YES} \hspace{1cm} \text{not peaked at $t=0$} \hspace{1cm} \text{YES} \]

(3) \[ \text{NO} \] \hspace{1cm} \text{not nonnegative} \hspace{1cm} \text{YES} \]

(4) \[ \text{sin}(t) \] \hspace{1cm} \text{YES} \hspace{1cm} \text{YES for every $t$} \]

**Interpretation of autocorrelation function**

- Let $\{x(t)\}$ be WSS with zero mean. Then the fact that $R_{x}(t)$ drops rapidly in $t$ implies that the samples become uncorrelated quickly.
- Conversely, if $R_{x}(t)$ drops slowly, then the samples are highly correlated.
- So in some sense, $R_{x}(t)$ is the measure of the rate of change in $X(t)$ with time $t$ (in terms of average power) the frequency response of $X(t)$.
- This interpretation can be made precise if we consider its Fourier transform $S_{x}(f)$, which captures the amount of power contained in the frequency components of $X(t)$.

**Power spectral density**

- The power spectral density (psd) of a WSS process $\{x(t)\}$ is the Fourier transform of $R_{x}(t)$:
  \[ S_{x}(f) = \mathcal{F}[R_{x}(t)] = \int_{-\infty}^{\infty} R_{x}(t)e^{-j2\pi ft} \, dt \]
- If the process is discrete-time, then we use the discrete-time Fourier transform (DTFT) \[ S_{x}(f) = \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi fn}, \quad f \in [-\frac{1}{2}, \frac{1}{2}] \text{ (periodic with period 1)} \]
- $R_{x}(t)$ or $R_{x}(n)$ can be reversed from the psd:
  \[ R_{x}(t) = \mathcal{F}^{-1}[S_{x}(f)] = \int_{-\infty}^{\infty} S_{x}(f)e^{j2\pi ft} \, df \quad \text{and similarly} \]
  \[ R_{x}(n) = \sum_{f=-\frac{1}{2}}^{\frac{1}{2}} S_{x}(f)e^{j2\pi fn} \, df \]
Properties of psd

1. $S_x(f)$ is even and real (since $R_x(t)$ is even and real)

2. $\int_{-\infty}^{\infty} S_x(f) df = R_x(0) = \mathbb{E}[x^2(t)]$, the average power of $X(t)$.

3. $S_x(f)$ is the average power density:

$$\int_{f_1}^{f_2} S_x(f) df + \int_{-f_2}^{-f_1} S_x(f) df = 2 \int_{f_1}^{f_2} S_x(f) df$$

is the average power of $X(t)$ in the frequency band $[f_1, f_2]$.

(We will prove it later.)