Lecture 19

Properties of Power Spectral Density

1. \( S_x(f) \) is real and even.

2. (for a discrete time process) \[ \int_{-\infty}^{\infty} S_x(f) \, df = R_x(0) = \mathbb{E}[X^2] \]

3. \( S_x(f) \) is the average power density, i.e., the power of \( X(t) \) in the frequency band \([f_1, f_2]\)

   \[ \int_{f_1}^{f_2} S_x(f) \, df + \int_{-\infty}^{f_1} S_x(f) \, df = 2 \int_{f_1}^{f_2} S_x(f) \, df. \]

4. \( S_x(f) \geq 0 \) (from 3)

5. In general, \( S(f) \) is even, real, and non-negative with \( \int_{-\infty}^{\infty} S(f) < \infty \) then it is a power spectral density of some random process \( \{X(t)\} \)

Example

1. \( R_x(\tau) = e^{-a|\tau|}, \quad a > 0 \)

2. \( R_x(\tau) = \frac{\cos(\omega \tau)}{\omega} \)

3. (Stationary Gauss-Markov Process with \( \lambda = \frac{1}{2} \))

\[ S_x(f) = \frac{\lambda^2}{\lambda^2 + (\pi f)^2} \]

\[ S_x(f) = \frac{3}{5 - 4 \cos(2\pi f)} \]
If we let $B \to \infty$, then we obtain the white noise process

$$R_x(t) = \frac{N}{2} S_c(t)$$

Technically, the white noise process is not WSS, since $E(x(t)) = 0$ but $\mu_x(f) = \infty = \int_{-\infty}^{\infty} s_x(f) df$.

Moreover, $\{ \frac{dW(t)}{dt} \}_{t=0}^{\infty}$ can be viewed as having $s_x(f) = 1$.

**Response to LTI System to WSS Process Input**

- Consider a linear time-invariant (LTI) system with impulse response $h(t)$ and transfer function $H(f) = \mathcal{F}[h(t)]$.
- What happens if we drive the system with WSS process input $X(t)$?
- We wish to characterize the output (random process)

$$Y(t) = h(t) \ast X(t) = \int_{-\infty}^{\infty} X(\tau) h(t-\tau) d\tau$$

$$X(t) \rightarrow h(t) \rightarrow Y(t) = X(t) \ast h(t)$$
It can be shown that $x(t)$ and $y(t)$ are jointly WSS, namely,

- $x(t)$ and $y(t)$ are each WSS,
- Their cross correlation function $R_{xy}(t_1, t_2) = E[x(t_1)y(t_2)]$ is time invariant, i.e., $R_{xy}(t_1, t_2) = R_{xy}(t_1 + t_2, t_2)$ vs. $R_{xx}(t_1, t_2) = R_{xx}(t_1 + t_2, t_2)$

The cross correlation function of jointly WSS $(x(t), y(t))$

First, we relate

$$R_{xy}(t_1, t_2) = R_{xy}(t_1 - t_2, 0)$$

$$= R_{xy}(\tau)$$

where $\tau = t_2 - t_1$.

Unlike $R_{xx}(\tau)$, $R_{xy}(\tau)$ is not even symmetric.

But $R_{xy}(\tau)$ is skew-symmetric, i.e., $R_{xy}(\tau) = -R_{xy}(\tau)$.

It's Fourier Transform

$$S_{xy}(f) = \mathcal{F}[R_{xy}(\tau)]$$

is called cross power spectral density.

Example: let $y(t) = x(t) + z(t)$ where $x(t)$ and $z(t)$ are zero mean, uncorrelated WSS process. Then:

- $y(t)$ is zero mean and
- $R_y(t_1, t_2) = E[(x(t_1) + z(t_1))(x(t_2) + z(t_2))]$
  
  $$= E[x(t_1)x(t_2)] + E[z(t_1)z(t_2)]$$

  $$= R_x(t_1, t_2) + R_z(t_1, t_2)$$
  
  is time invariant.

$y(t)$ is WSS with $R_y(\tau) = R_x(\tau) + R_z(\tau)$

Now consider:

$$E[x(t_1)y(t_2)] = E[(x(t_1) + z(t_1))y(t_2)]$$

$$= E[x(t_1)x(t_2)]$$

$$= R_x(t_1, t_2)$$

which is time invariant, hence, $x(t)$, $y(t)$ is jointly WSS

with $R_{xy}(\tau) = R_x(\tau)$

$$S_{xy}(f) = S_x(f)$$
Output process of LTI system

If the system is stable, i.e.,
\[ \left| \sum_{-\infty}^{\infty} h(n) \, d\tau \right| = |H(\omega)| < \infty \]
then \((x(t), y(t))\) is jointly WSS with

1. \(E[y(t)] = H(\omega)E[x(t)]\)
2. \(R_y(\tau) = E[y(t) \, x(t - \tau)]\)
   \[ = h(\tau) \ast R_x(\tau) \]
3. \(R_y(\omega) = H(\omega) \ast R_x(\omega) + H(-\omega)\)
   \[ R_x(\omega) \rightarrow \underbrace{H(\omega)}_{R_y(\omega)} \rightarrow \underbrace{R_y(\omega)}_{H(-\omega)} \rightarrow R_y(\omega) \]
4. \(S_y(f) = H(f) \, S_x(f)\)
5. \(S_y(f) = |H(f)|^2 \, S_x(f)\)

\[ S_x(f) \rightarrow \underbrace{H(f)}_{S_y(f)} \rightarrow \underbrace{S_y(f)}_{H(f)} \]

Proof #1:
\[ E[y(t)] = E\left[ \sum_{-\infty}^{\infty} x(t) \, h(t, \tau) \, d\tau \right] \]
\[ = \int_{-\infty}^{\infty} E[x(t)] \, h(t, \tau) \, d\tau \]
\[ = E[x(t)] \int_{-\infty}^{\infty} h(t, \tau) \, d\tau \]
\[ = E[x(t)] \int_{-\infty}^{\infty} h(s) \, ds\]
\[ = E[x(t)] \int_{-\infty}^{\infty} h(s) \, ds = \frac{Y(\omega)}{H(\omega)} \]

Proof #2:
\[ R_y(\tau) = E\left[ y(t) \, x(t - \tau) \right] \]
\[ = E\left[ \int_{-\infty}^{\infty} h(s) \, x(t - \tau - s) \, x(t) \, ds \right] \]
\[ = \int_{-\infty}^{\infty} h(s) \, E[x(t - \tau - s) \, x(t)] \, ds \]
\[ = \int_{-\infty}^{\infty} h(s) \, R_x(\tau - s) \, ds \]
\[ = h(\tau) \ast R_x(\tau) \]
Proof #3:

\[ R_y(\tau) = \mathbb{E}[y(t+\tau)y(t)] = \mathbb{E}\left[ y(t+\tau) \int_{-\infty}^{\infty} h(s) x(t-s) \, ds \right] = \int_{-\infty}^{\infty} h(s) \mathbb{E}[y(t+\tau) x(t-s)] \, ds = \int_{-\infty}^{\infty} h(s) R_{yx}(s+\tau) \, ds = R_{xy}(\tau) * h(-\tau) \]

\[ R_x(\tau) \xrightarrow{h(\tau)} h_{\tau}(\tau) \xrightarrow{h(-\tau)} R_y(\tau) \]

Proof 4.5: Follow from the properties of Fourier Transform.

\( S_x(f) \) is power spectral density.

Let \( h(t) \) (or \( H(f) \)) be a band-pass filter on \([f_1, f_2]\), i.e.,

\[ f_1 \xrightarrow{h(t)} \xrightarrow{f} f_2 \]

Drive this system with WSS with power spectral density \( S_x(f) \)

Then, the output \( y(t) \) has power spectral density

\[ S_y(f) = |H(f)|^2 S_x(f) \]

\[ = \begin{cases} S_x(f) & \text{if } |f| \in [f_1, f_2] \\ 0 & \text{else} \end{cases} \]

Furthermore, the output average power's

\[ \int_{-\infty}^{\infty} S_y(f) \, df = 2 \int_{f_1}^{f_2} S_x(f) \, df \]

\[ \int \xrightarrow{\text{average power}} \int \]