Properties of power spectral density

1. \( S_x(f) \) is real and even.

2. \( \int_{-\infty}^{\infty} S_x(f) df = R_x(0) \) is the average power of the process.
   (For a discrete time process, \( \sum_{k} S_x(k) = R_x(0) = \text{Var}(x) \).

3. \( S_x(f) \) is the average power density, i.e., the average power of \( X(t) \)
in the frequency band \( [f_1, f_2] \) is

\[
\int_{-f_2}^{-f_1} S_x(f) df + \int_{f_1}^{f_2} S_x(f) df = 2 \int_{f_1}^{f_2} S_x(f) df.
\]

4. \( S_x(f) \geq 0 \) (from (3)).

5. In general, \( S(f) \) is even, real, and nonnegative with \( \int_{-\infty}^{\infty} S_x(f) df < \infty \),
   then it is a power spectral density of some random process \( \{X(t)\} \).

Examples

1. \( R_x(t) = e^{-2|t|} \), \( a > 0 \)

2. \( R_x(t) = \frac{a}{2} \cos(wt) \)

3. \( R_x(n) = 2^n \)
   (Stationary Gaussian Markov with \( \alpha = 1/2 \))

4. \( R_x(n) = \delta(n - 0) \) for \( N \), otherwise.
   (Discrete time white Gaussian noise)
If we let $B \to \infty$, then we obtain the white noise process:

\[ R_x(t) = \frac{N}{2} \delta(t) \]

Technically, the white noise process is not WSS since $E[X(t)] = 0 = \int_{-\infty}^{\infty} S_x(f) df$ but we use it as a mathematically convenient model for real-world noise processes. Moreover, $\int_{-\infty}^{\infty} \frac{dW(t)}{dt} dt \bigg|_{t=-\infty}^{t=\infty}$ can be viewed as having $S_x(f) = 1$.

Response of LTI system to WSS process input:

- Consider a linear time-invariant (LTI) system with impulse response $h(t)$ and transfer function $H(f) = \mathcal{F}[h(t)]$.
- What happens if we drive this system with WSS process input $X(t)$?
- We wish to characterize the output (random process)

\[ Y(t) = X(t) * h(t) = \int_{-\infty}^{\infty} X(t) h(t-T) dt. \]

\[ X(t) \rightarrow h(t) \rightarrow Y(t) = h(t) * X(t). \]

- It can be shown that $X(t)$ and $Y(t)$ are jointly WSS, namely,
  - $X(t)$ and $Y(t)$ are each WSS.
  - Their cross correlation function $R_{xy}(t_1, t_2) = E[X(t_1)Y(t_2)]$ is time-invariant, i.e., $R_{xy}(t_1, t_2) = R_{xy}(t_1 + s, t_2 + s)$ for all $s$. 
Cross correlation function of jointly WSS \((x(t), y(t))\)

Firstly we redefine
\[ R_{xy}(t_1, t_2) = R_{xy}(t_1 - t_2, 0) = R_{xy}(t), \]  
where \( t = t_1 - t_2 \).

Unlike \( R_x(t) \), \( R_{xy}(t) \) is NOT even (symmetric).

But \( R_{xy}(t) \) is skew-symmetric, i.e., \( R_{xy}(t) = -R_{xy}(-t) \).

Its Fourier transform \( S_{xy}(f) = \mathcal{F}[R_{xy}(t)] \) is called the cross power spectral density.

Example: Let \( y(t) = x(t) + z(t) \), where \( x(t) \) and \( z(t) \) are zero-mean uncorrelated WSS processes. Then \( y(t) \) is zero-mean and

\[
\begin{align*}
R_y(t_1, t_2) &= E[(x(t_1) + z(t_1))(x(t_2) + z(t_2))] \\
&= E[x(t_1)x(t_2)] + E[z(t_1)z(t_2)] \\
&= R_x(t_1, t_2) + R_z(t_1, t_2),
\end{align*}
\]

is time-invariant. Hence, \( y(t) \) is WSS with \( R_y(t) = R_x(t) + R_z(t) \) and \( S_y(f) = S_x(f) + S_z(f) \). Now consider

\[
E[x(t_1)y(t_2)] = E[(x(t_1)x(t_2) + z(t_1)z(t_2))],
\]

which is time-invariant. Hence, \((x(t), y(t))\) is jointly WSS with \( R_{xy}(t) = R_{x(t)} \) and \( S_{xy}(f) = S_x(f) \).

Output process of the LTI system

If the system is stable, i.e., \( \int_{-\infty}^{\infty} |h(t)| dt < \infty \), then \((x(t), y(t))\) is jointly WSS with

1. \( E[y(t)] = H(0)E[x(t)] \)
2. \( R_{xy}(t) = E[y(t) x(t^\prime)] = H(t) * R_x(t^\prime) \)
3. \( R_y(t) = h(t) * R_x(t) * h(-t) \).

\[
\begin{align*}
R_x(t) &\xrightarrow{h(t)} R_{xy}(t) \xrightarrow{h(-t)} R_y(t) \\
S_{xy}(f) &\xrightarrow{H(f)} S_y(f) \xrightarrow{H(-f)} S_y(f)
\end{align*}
\]
Proof: (1) Consider \( E[Y(t)] = E \left( \int_{-\infty}^{\infty} x(t) h(t-t) dt \right) \)
\[
= \int_{-\infty}^{\infty} E[x(t)] h(t-t) dt = E[x(t)] \int_{-\infty}^{\infty} h(t) dt = E[x(t)] H(0).
\]

(2) Consider \( R_x(t) = E[Y(t)X(t)] \)
\[
= E \left( \int_{-\infty}^{\infty} h(s) X(t+s) ds X(t) \right) = \int_{-\infty}^{\infty} h(s) E[X(t+s)X(t)] ds = \int_{-\infty}^{\infty} h(s) R_x(t-s) ds = h(t) \star R_x(t).
\]

(3) Consider \( R_y(t) = E[Y(t)Y(t)] \)
\[
= E \left[ Y(t+\tau) \int_{-\infty}^{\infty} h(s) X(t-s) ds \right] = \int_{-\infty}^{\infty} h(s) E[Y(t+\tau)X(t-s)] ds = \int_{-\infty}^{\infty} h(s) R_y(t+\tau) ds = \tau = -\tau
\]
\[
= \int_{-\infty}^{\infty} h(t) R_y(t-\tau) dt = h(-\tau) \star R_y(\tau) = h(\tau) \star h(t) \star R_y(t).
\]

(4) + (4) follow from the properties of Fourier transform. 

\( S_x(f) \) is power spectral density.

Let \( h(t) \) (or \( H(f) \)) be a bandpass filter on \([f_1, f_2]\), i.e.,

Drive this system with WSS \( x(t) \) with power spectral density \( S_x(f) \).

Then the output \( Y(t) \) has power spectral density
\[
S_y(f) = |H(f)|^2 S_x(f)
\]
\[
= \left\{ \begin{array}{ll}
S_x(f) & \text{if } |f| \in [f_1, f_2] \\
0 & \text{otherwise}
\end{array} \right.
\]

Furthermore, the output average power is
\[
\int_{-\infty}^{\infty} S_y(f) df = 2 \int_{f_1}^{f_2} S_x(f) df.
\]