Final Examination
(Total: 260 points)

There are 6 problems, each problem with multiple parts. Your answer should be as clear and readable as possible. Please justify any claim that you make.

   
   (a) Find the joint pdf $f_{X,Z}(x, z)$ of $X$ and $Z$.
   (b) Find the joint pdf $f_{Z,W}(z, w)$ of $Z$ and $W$.
   (c) Find $E[Z|X]$.
   (d) Find $E[X|Z]$.

2. MMSE estimation (30 pts). Let $X \sim \text{Exp}(1)$ and $Y = \min\{X, 1\}$.
   
   (a) Find $E[Y]$.
   (b) Find the estimate $\hat{X} = g(Y)$ of $X$ given $Y$ that minimizes the mean square error $E[(X - \hat{X})^2] = E[(X - g(Y))^2]$, and plot $g(y)$ as a function of $y$.
   (c) Find the mean square error of the estimate found in part (b).

3. Is the grass always greener on the other side? (30 pts). Let $X$ and $Y$ be two i.i.d. continuous nonnegative random variables with invertible common cdf $F$, i.e.,

   $$P\{X \leq x\} = P\{Y \leq x\} = F(x).$$

   (a) Find $P\{X > Y\}$ and $P\{X < Y\}$.

   Suppose now that we observe the value of $X$ and make a decision on whether $X$ is larger or smaller than $Y$.

   (b) Find the optimal decision rule $d(x)$ that minimizes the error probability. Your answer should be in terms of the common cdf $F$.
   (c) Find the probability of error for the decision rule found in part (b).
4. **Sampled Wiener process (60 pts).** Let \( \{W(t), t \geq 0\} \) be the standard Brownian motion. For \( n = 1, 2, \ldots, \) let
\[
X_n = n \cdot W \left( \frac{1}{n} \right).
\]
(a) Find the mean and autocorrelation functions of \( \{X_n\} \).
(b) Is \( \{X_n\} \) WSS? Justify your answer.
(c) Is \( \{X_n\} \) Markov? Justify your answer.
(d) Is \( \{X_n\} \) independent increment? Justify your answer.
(e) Is \( \{X_n\} \) Gaussian? Justify your answer.
(f) For \( n = 1, 2, \ldots, \) let \( S_n = X_n/n \). Find the limit
\[
\lim_{n \to \infty} S_n
\]
in probability.

5. **Poisson process (40 pts).** Let \( \{N(t), t \geq 0\} \) be a Poisson process with arrival rate \( \lambda > 0 \). Let \( s \leq t \).
(a) Find the conditional pmf of \( N(t) \) given \( N(s) \).
(b) Find \( \mathbb{E}[N(t)|N(s)] \) and its pmf.
(c) Find the conditional pmf of \( N(s) \) given \( N(t) \).
(d) Find \( \mathbb{E}[N(s)|N(t)] \) and its pmf.

6. **Hidden Markov process (60 pts).** Let \( X_0 \sim N(0, \sigma^2) \) and \( X_n = \frac{1}{2} X_{n-1} + Z_n \) for \( n \geq 1 \), where \( Z_1, Z_2, \ldots \) are i.i.d. \( N(0, 1) \), independent of \( X_0 \). Let \( Y_n = X_n + V_n \), where \( V_n \) are i.i.d. \( \sim N(0, 1) \), independent of \( \{X_n\} \).
(a) Find the variance \( \sigma^2 \) such that \( \{X_n\} \) and \( \{Y_n\} \) are jointly WSS.
Under the value of \( \sigma^2 \) found in part (a), answer the following.
(b) Find \( R_Y(n) \).
(c) Find \( R_{XY}(n) \).
(d) Find the MMSE estimate of \( X_n \) given \( Y_n \).
(e) Find the MMSE estimate of \( X_n \) given \( (Y_n, Y_{n-1}) \).
(f) Find the MMSE estimate of \( X_n \) given \( (Y_n, Y_{n+1}) \).