
(a) Find the joint pdf $f_{X,Z}(x, z)$ of $X$ and $Z$.
(b) Find the joint pdf $f_{Z,W}(z, w)$ of $Z$ and $W$.
(c) Find $E[Z|X]$.
(d) Find $E[X|Z]$.

**Solution:**

(a) For $z < x$, we have

$$F_{Z|X}(z|x) = P\{Z \leq z \mid X = x\} = 0.$$ 

For $0 \leq x \leq z$,

$$F_{Z|X}(z|x) = P\{Z \leq z \mid X = x\}$$
$$= P\{X + Y \leq z \mid X = x\}$$
$$= P\{Y \leq z - x \mid X = x\}$$
$$\overset{(a)}{=} P\{Y \leq z - x\}$$
$$= 1 - e^{-(z-x)},$$

where (a) follows from the independence of $X$ and $Y$. We therefore have

$$f_{Z|X}(z|x) = \begin{cases} e^{-(z-x)}, & \text{if } 0 \leq x \leq z \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$f_{X,Z}(x, z) = \begin{cases} e^{-z}, & \text{if } 0 \leq x \leq z \\ 0, & \text{otherwise.} \end{cases}$$
(b) From the previous part, we have, for $0 \leq x \leq z$,
\[
f_{X|Z}(x|z) = \frac{f_{X,Z}(x,z)}{f_Z(z)} = \frac{\int_0^z f_{X,Z}(x,z)dx}{f_Z(z)} = \frac{1}{z}.
\]
Thus for $z \geq 0$, $X \mid \{Z = z\} \sim \text{Unif}[0, z]$. We have $W = X - Y = 2X - Z$. Therefore,
\[
F_{W|Z}(w|z) = P\{W \leq w \mid Z = z\} = P\{2X - Z \leq w \mid Z = z\} = P\{X \leq \frac{z + w}{2} \mid Z = z\} = \begin{cases} 0, & \text{if } w < -z \\ \frac{z + w}{2z}, & \text{if } -z \leq w \leq z \\ 1, & \text{if } w > z. \end{cases}
\]
Thus,
\[
f_{W|Z}(w|z) = \begin{cases} \frac{1}{2z}, & \text{if } |w| \leq z \\ 0, & \text{otherwise}, \end{cases}
\]
which leads us to conclude that
\[
f_{Z,W}(z, w) = f_{W|Z}(w|z)f_Z(z) = \begin{cases} \frac{1}{2}e^{-z}, & \text{if } |w| \leq z \\ 0, & \text{otherwise}. \end{cases}
\]
(c) We have
\[
E[Z|X] = E[X + Y \mid X] = X + E[Y|X] = X + E[Y] = X + 1,
\]
where $E[Y|X] = E[Y]$ since $X$ and $Y$ are independent.
(d) From part (b), we have $X \mid \{Z = z\} \sim \text{Unif}[0, z]$. Therefore,
\[
E[X|Z] = \frac{Z}{2}.
\]
2. **MMSE estimation (30 pts).** Let $X \sim \text{Exp}(1)$ and $Y = \min\{X, 1\}$. 

2
(a) Find $E[Y]$.

(b) Find the estimate $\hat{X} = g(Y)$ of $X$ given $Y$ that minimizes the mean square error $E[(X - \hat{X})^2] = E[(X - g(Y))^2]$, and plot $g(y)$ as a function of $y$.

(c) Find the mean square error of the estimate found in part (b).

**Solution:**

(a) We have

$$E[Y] = E[\min\{X, 1\}]$$
$$= \int_0^\infty \min\{x, 1\} e^{-x}dx$$
$$= \int_0^1 xe^{-x}dx + \int_1^\infty e^{-x}dx$$
$$= -xe^{-x} - e^{-x}\bigg|_0^1 + e^{-1}$$
$$= 1 - e^{-1}.$$

(b) We have $g(y) = E[X \mid Y = y]$. For $y < 1$,

$$E[X \mid Y = y] = E[X \mid X = y] = y.$$

For $y = 1$, we have

$$E[X \mid Y = y] = E[X \mid X \geq 1]$$
$$\overset{(a)}{=} E[X] + 1$$
$$= 2,$$

where $(a)$ follows from the memorylessness property of the exponential distribution. Thus,

$$g(y) = \begin{cases} y, & 0 \leq y < 1 \\ 2, & y = 1. \end{cases}$$

The plot of $g(y)$ vs $y$ is shown in Fig. 1.

(c) For $0 \leq y < 1$, $\text{Var}(X \mid Y = y) = 0$. For $y = 1$,

$$\text{Var}(X \mid Y = y) = \text{Var}(X \mid X \geq 1)$$
$$\overset{(a)}{=} \text{Var}(X)$$
$$= 1,$$

where the step $(a)$ follows from the memoryless property. We therefore have

$$\text{MSE} = E[\text{Var}(X|Y)]$$
$$= \text{Var}(X \mid Y = 1)P\{Y = 1\}$$
$$= e^{-1}.$$
3. Is the grass always greener on the other side? (30 pts). Let $X$ and $Y$ be two i.i.d. continuous nonnegative random variables with invertible common cdf $F$, i.e.,

$$P\{X \leq x\} = P\{Y \leq x\} = F(x).$$

(a) Find $P\{X > Y\}$ and $P\{X < Y\}$.

Suppose now that we observe the value of $X$ and make a decision on whether $X$ is larger or smaller than $Y$.

(b) Find the optimal decision rule $d(x)$ that minimizes the error probability. Your answer should be in terms of the common cdf $F$.

(c) Find the probability of error for the decision rule found in part (b).

Solution:

(a) By symmetry, $P\{X > Y\} = P\{X < Y\} = 1/2$. Alternatively, let $f$ be the common pdf of $X$ and $Y$. Then

$$P\{X > Y\} = \int_0^\infty P\{X > Y \mid Y = y\}f(y)dy$$

$$= \int_0^\infty P\{X > y\}f(y)dy$$

$$= \int_0^\infty (1 - F(y))f(y)dy$$

$$= 1 - \int_0^\infty f(y)F(y)dy.$$

Here, (a) follows from the independence of $X$ and $Y$. We now have, integrating
by parts,

\[
I := \int_{0}^{\infty} f(y) F(y) \, dy \\
= \left[ F(y)^2 \right]_{0}^{\infty} - \int_{0}^{\infty} F(y) f(y) \, dy \\
= \lim_{y \to \infty} F(y)^2 - I \\
= 1 - I,
\]

whence \( I = 1/2 \). Thus,

\[
P\{X > Y\} = \frac{1}{2}.
\]

By interchanging the roles of \( X \) and \( Y \), we conclude that

\[
P\{X < Y\} = \frac{1}{2}.
\]

Note: We can also compute \( I \) by noting that

\[
f(y) F(y) = \frac{1}{2} \frac{d}{dy} F(y)^2.
\]

(b) Let us define a random variable \( Z \) as

\[
Z = \begin{cases} 
1, & \text{if } X > Y \\
0, & \text{if } X \leq Y.
\end{cases}
\]

Then, we have to find a decision rule \( d(\cdot) \), such that \( P\{d(X) \neq Z\} \) is minimized. We know that this should be the MAP decision rule. We have

\[
p_{Z|X}(1|x) = P\{Z = 1 \mid X = x\} \\
= P\{Y < X \mid X = x\} \\
= P\{Y < x \mid X = x\} \\
= F(x).
\]

Therefore, \( p_{Z|X}(0|x) = 1 - F(x) \), i.e., we should choose \( d(x) = 1 \) if \( F(x) > 1 - F(x) \), i.e., if \( x > F^{-1}(1/2) \) (which is the median of \( X \) and is unique since \( F \) is invertible). Thus the optimal decision rule is given by

\[
d(x) = \begin{cases} 
1, & \text{if } x > F^{-1}(1/2) \\
0, & \text{if } x \leq F^{-1}(1/2).
\end{cases}
\]

In other words, we predict that \( X \) is larger than \( Y \) if the observed value of \( X \) is larger than the median.
(c) We have

\[ P \{d(X) \neq Z\} = P \{X > Y, X \leq F^{-1}(1/2)\} + P \{X < Y, X > F^{-1}(1/2)\} \]

\[ = P\{Y < X \leq F^{-1}(1/2)\} + P\{F^{-1}(1/2) < X < Y\} \]

\[ = \int_0^{F^{-1}(1/2)} \int_0^x f(x)f(y)dydx + \int_{F^{-1}(1/2)}^\infty \int_{F^{-1}(1/2)}^\infty f(x)f(y)dydx \]

\[ = \int_0^{F^{-1}(1/2)} f(x)F(x)dx + \int_{F^{-1}(1/2)}^\infty f(x)(1 - F(x))dx \]

\[ = \frac{1}{2} + \int_0^{F^{-1}(1/2)} f(x)F(x)dx - \int_{F^{-1}(1/2)}^\infty f(x)F(x)dx \]

\[ = \frac{1}{2} + \frac{1}{2} \left( F \left( F^{-1}(1/2) \right) \right)^2 - \frac{1}{2} \left( 1 - \left( F \left( F^{-1}(1/2) \right) \right) \right)^2 \]

\[ = \frac{1}{2} + \frac{1}{8} - \frac{3}{8} \]

\[ = \frac{1}{4}. \]

Here, (a) follows from the observation made at the end of part (a).

4. *Sampled Wiener process (60 pts)*. Let \( \{W(t), t \geq 0\} \) be the standard Brownian motion. For \( n = 1, 2, \ldots \), let

\[ X_n = n \cdot W \left( \frac{1}{n} \right). \]

(a) Find the mean and autocorrelation functions of \( \{X_n\} \).

(b) Is \( \{X_n\} \) WSS? Justify your answer.

(c) Is \( \{X_n\} \) Markov? Justify your answer.

(d) Is \( \{X_n\} \) independent increment? Justify your answer.

(e) Is \( \{X_n\} \) Gaussian? Justify your answer.

(f) For \( n = 1, 2, \ldots \), let \( S_n = X_n/n \). Find the limit

\[ \lim_{n \to \infty} S_n \]

in probability.

**Solution:**

(a) We have

\[ E[X_n] = nE[W(1/n)] = 0. \]
For \( m, n \in \mathbb{N} \) and \( m \geq n \), we have
\[
E[X_m X_n] = mnE[W(1/m)W(1/n)]
\]
\[
= mn \cdot \min\{1/m, 1/n\}
\]
\[
= mn \cdot \frac{1}{m}
\]
\[
= n.
\]
Thus in general,
\[
E[X_m X_n] = \min\{m, n\}.
\]

(b) No. Since the autocorrelation function is not time-invariant, \( \{X_n\} \) is not WSS.

(c) Yes. Clearly, \( \{X_n\} \) is a Gaussian process (see the solution to part (e)) with mean and autocorrelation functions as found in part (a). Therefore, for integers \( m_1 < m_2 \leq m_3 < m_4 \), we have
\[
E[(X_{m_2} - X_{m_1})(X_{m_4} - X_{m_3})] = E[X_{m_2}X_{m_4}] + E[X_{m_1}X_{m_3}] - E[X_{m_2}X_{m_3}] - E[X_{m_1}X_{m_4}]
\]
\[
= \min\{m_2, m_4\} + \min\{m_1, m_3\} - \min\{m_2, m_3\} - \min\{m_1, m_4\}
\]
\[
= m_2 + m_1 - m_2 - m_1
\]
\[
= 0
\]
\[
= E[X_{m_2} - X_{m_1}]E[X_{m_4} - X_{m_3}].
\]

Therefore, since \( (X_{m_2} - X_{m_1}) \) and \( (X_{m_4} - X_{m_3}) \) are jointly Gaussian and uncorrelated, they are independent. Now, for positive integers \( n_1 < n_2 < \cdots < n_k \) for some \( k \), \( (X_{n_1}, X_{n_2} - X_{n_1}, \ldots, X_{n_k} - X_{n_{k-1}}) \), being a linear transformation of a Gaussian random vector, is itself Gaussian. Moreover, from what we just showed, \( (X_{n_1}, X_{n_2} - X_{n_1}, \ldots, X_{n_k} - X_{n_{k-1}}) \) are pairwise independent. Therefore, they are all independent, which implies that \( \{X_n\} \) is independent-increment. This implies Markovity.

(d) Yes. See the solution to part (c).

(e) Yes. For integers \( n_1, n_2, \ldots, n_k \) for any \( k \), we have
\[
\begin{bmatrix}
X_{n_1} \\
X_{n_2} \\
\vdots \\
X_{n_k}
\end{bmatrix}
= 
\begin{bmatrix}
n_1 & 0 & \cdots & 0 \\
0 & n_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & 0 \\
0 & 0 & \cdots & n_k
\end{bmatrix}
\begin{bmatrix}
W(1/n_1) \\
W(1/n_2) \\
\vdots \\
W(1/n_k)
\end{bmatrix}
\]
Thus, \( [X_{n_1}, \cdots, X_{n_k}]^T \), being a linear transformation of a Gaussian random vector, is itself a Gaussian random vector. Therefore, \( \{X_n\} \) is Gaussian.
Recall that $X_n \sim N(0, n)$, which implies that $X_n/\sqrt{n} \sim N(0, 1)$. Therefore, for any fixed $\epsilon > 0$, we have

$$P\{|S_n| > \epsilon\} = P\{|X_n| > n\epsilon\} = P\left\{\frac{|X_n|}{\sqrt{n}} > \epsilon\sqrt{n}\right\} = 2Q(\epsilon\sqrt{n}) \to 0,$$

as $n \to \infty$. Therefore, $\lim_{n \to \infty} S_n = 0$ in probability. Alternatively, note that $W(0) = 0$ and $W(t)$ is continuous with probability 1. Therefore

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} W\left(\frac{1}{n}\right) = W(0) = 0.$$

5. **Poisson process (40 pts).** Let $\{N(t), t \geq 0\}$ be a Poisson process with arrival rate $\lambda > 0$. Let $s \leq t$.

(a) Find the conditional pmf of $N(t)$ given $N(s)$.

(b) Find $E[N(t)|N(s)]$ and its pmf.

(c) Find the conditional pmf of $N(s)$ given $N(t)$.

(d) Find $E[N(s)|N(t)]$ and its pmf.

**Solution:**

(a) Assume $0 \leq n_s \leq n_t$. By the independent increment property of the Poisson process, we would get

$$P\{N(t) = n_t|N(s) = n_s\} = P\{N(t) - N(s) = n_t - n_s|N(s) = n_s\}
\hspace{1cm} = P\{N(t) - N(s) = n_t - n_s\}
\hspace{1cm} = e^{-\lambda(t-s)} \frac{\lambda(t-s)^{n_t-n_s}}{(n_t-n_s)!}$$

for $n_s = 0, 1, \ldots$ and $n_t = n_s, n_s + 1, \ldots$. Thus,

$$N(t)|\{N(s) = n_s\} \sim n_s + \text{Poisson}(\lambda(t-s)).$$

(b) From part (a), it immediately follows that

$$E[N(t)|N(s)] = N(s) + \lambda(t-s).$$

Therefore, the pmf of $E[N(t)|N(s)]$ is

$$p_{E[N(t)|N(s)]}(x) = \begin{cases} e^{-\lambda s} (\frac{\lambda s}{k})^k / k! & \text{if } x = k + \lambda(t-s), \quad k = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases}$$
(c) From part (a), the joint pmf of \((N(t), N(s))\) for \(0 \leq n_s \leq n_t\), is
\[
P\{N(t) = n_t, N(s) = n_s\} = P\{N(s) = n_s\}P\{N(t) = n_t|N(s) = n_s\}
= e^{-\lambda s}(\lambda s)^{n_s}/n_s! e^{-\lambda(t-s)}(\lambda(t-s))^{n_t-n_s}/(n_t-n_s)!
= e^{-\lambda t}n_t^{n_t}(t-s)^{n_t-n_s}/n_s!(n_t-n_s)!
\]
Therefore, the conditional pmf of \(N(s)|\{N(t) = n_t\}\) is for \(n_t \geq n_s \geq 0\)
\[
P\{N(s) = n_s|N(t) = n_t\} = P\{N(s) = n_s, N(t) = n_t\}/P\{N(t) = n_t\}
= \left(\frac{e^{-\lambda t}n_t^{n_t}(t-s)^{n_t-n_s}}{n_s!(n_t-n_s)!}\right) \left(\frac{e^{-\lambda t}(\lambda t)^{n_t}}{n_t!}\right)^{-1}
= \left(\frac{n_t}{n_s}\right)\left(\frac{s}{t}\right)^{n_s} \left(1 - \frac{s}{t}\right)^{n_t-n_s}.
\]
Hence,
\[N(s)|\{N(t) = n_t\} \sim \text{Binom}\left(n_t, \frac{s}{t}\right).\]
(d) From part (c), it immediately follows that
\[E[N(s)|N(t)] = \frac{s}{t}N(t),\]
and its pmf is
\[p_{E[N(s)|N(t)]}(x) = \begin{cases} e^{-\lambda t}(\lambda t)^k/k! & \text{if } x = \frac{k}{t}, \quad k = 0, 1, 
\vdots \\
0 & \text{otherwise} \end{cases} \]

6. **Hidden Markov process (60 pts).** Let \(X_0 \sim N(0, \sigma^2)\) and \(X_n = \frac{1}{2}X_{n-1} + Z_n\) for \(n \geq 1\), where \(Z_1, Z_2, \ldots\) are i.i.d. \(N(0, 1)\), independent of \(X_0\). Let \(Y_n = X_n + V_n\), where \(V_n\) are i.i.d. \(\sim N(0, 1)\), independent of \(\{X_n\}\).

(a) Find the variance \(\sigma^2\) such that \(\{X_n\}\) and \(\{Y_n\}\) are jointly WSS. Under the value of \(\sigma^2\) found in part (a), answer the following.

(b) Find \(R_Y(n)\).
(c) Find \(R_{XY}(n)\).
(d) Find the MMSE estimate of \(X_n\) given \(Y_n\).
(e) Find the MMSE estimate of \(X_n\) given \((Y_n, Y_{n-1})\).
(f) Find the MMSE estimate of $X_n$ given $(Y_n, Y_{n+1})$.

**Solution:**

(a) If $\{X_n\}$ is WSS, then $\text{Var}(X_n) = \text{Var}(X_0) = \sigma^2$ for all $n \geq 0$. From the recursive relation, we would get

$$\text{Var}(X_n) = \frac{1}{4}\text{Var}(X_{n-1}) + \text{Var}(Z_n),$$

which implies $\sigma^2 = \frac{4}{3}$.

(b) First, note that for $n \geq 0$,

$$X_{m+n} = \frac{1}{2}X_{m+n-1} + Z_{m+n}$$

$$= \frac{1}{4}X_{m+n-2} + \frac{1}{2}Z_{m+n-1} + Z_{m+n}$$

$$= \ldots$$

$$= \frac{1}{2^n}X_m + \frac{1}{2^{n-1}}Z_{m+1} + \ldots + \frac{1}{2}Z_{m+n-1} + Z_{m+n}.$$ 

Hence, it follows that

$$R_X(n) = \mathbb{E}[X_{m+n}X_n] = 2^{-n}\mathbb{E}[X_m^2] = \frac{4}{3}2^{-|n|}.$$ 

Now we can find the autocorrelation function of $\{Y_n\}$ easily.

$$R_Y(n) = \mathbb{E}[Y_{m+n}Y_m]$$

$$= \mathbb{E}[(X_{m+n} + V_{m+n})(X_m + V_m)]$$

$$= \mathbb{E}[X_{m+n}X_m + X_{m+n}V_m + V_{m+n}X_m + V_{m+n}V_m]$$

$$= R_X(n) + \delta(n)$$

$$= \frac{4}{3}2^{-|n|} + \delta(n)$$

Here $\delta(n)$ denotes the Kronecker delta function, that is,

$$\delta(n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise}. \end{cases}$$

(c) The cross correlation function $R_{XY}(n)$ is

$$R_{XY}(n) = \mathbb{E}[X_{m+n}Y_m]$$

$$= \mathbb{E}[X_{m+n}X_m + X_{m+n}V_m]$$

$$= R_X(n) = \frac{4}{3}2^{-|n|}.$$
(d) Since $X_n$ and $Y_n$ are jointly Gaussian, we can find the conditional expectation $E[X_n|Y_n]$, which is the MMSE estimate of $X_n$ given $Y_n$, as follows:

\[
E[X_n|Y_n] = E[X_n] + \frac{\text{Cov}(X_n, Y_n)}{\text{Var}(Y_n)}(Y_n - E[Y_n])
\]

\[
= \frac{R_{XY}(0)}{R_Y(0)}Y_n
\]

\[
= \frac{4}{7}Y_n.
\]

(e) As in part (d), the MMSE estimate of $X_n$ given $(Y_n, Y_{n-1})$ is

\[
E[X_n|Y_n, Y_{n-1}] = E[X_n] + \Sigma_{X_n,(Y_n,Y_{n-1})}^{-1}\Sigma_{(Y_n,Y_{n-1})}^{-1} \left( \begin{bmatrix} Y_n \\ Y_{n-1} \end{bmatrix} - E \begin{bmatrix} Y_n \\ Y_{n-1} \end{bmatrix} \right)
\]

\[
= \begin{bmatrix} R_{XY}(0) & R_{XY}(1) \\ R_Y(1) & R_Y(0) \end{bmatrix}^{-1} \begin{bmatrix} Y_n \\ Y_{n-1} \end{bmatrix}
\]

\[
= \begin{bmatrix} 4/3 & 2/3 \\ 7/3 & 2/3 \end{bmatrix}^{-1} \begin{bmatrix} Y_n \\ Y_{n-1} \end{bmatrix}
\]

\[
= \begin{bmatrix} 8/15 & 2/15 \\ 2/15 & 7/15 \end{bmatrix} \begin{bmatrix} Y_n \\ Y_{n-1} \end{bmatrix}
\]

\[
= \frac{8}{15}Y_n + \frac{2}{15}Y_{n-1}.
\]

(f) Since $(X_n, Y_n)$ are jointly WSS, from part (e) it immediately follows that the conditional expectation $E[X_n|Y_n, Y_{n+1}]$ has the same form with $E[X_n|Y_n, Y_{n-1}]$:

\[
E[X_n|Y_n, Y_{n+1}] = \frac{8}{15}Y_n + \frac{2}{15}Y_{n+1}.
\]