Exercise Set #6

1. **Markov chain.** Assume that the continuous random variables $X_1$ and $X_3$ are independent given $X_2$. Show that $f(x_1, x_2, x_3) = f(x_1)f(x_2|x_1)f(x_3|x_2) = f(x_3)f(x_2|x_3)f(x_1|x_2)$.

2. **Covariance matrices.** Which of the following matrices can be a covariance matrix? Justify your answer either by constructing a random vector $X$, as a function of the i.i.d zero mean unit variance random variables $Z_1, Z_2$, and $Z_3$, with the given covariance matrix, or by establishing a contradiction.

   (a) $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$  
   (b) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$  
   (c) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$  
   (d) $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 3 \end{bmatrix}$

3. The **correlation matrix $C$** for a random vector $X$ is the matrix whose entries are $c_{ij} = \mathbb{E}(X_iX_j)$. Show that it has the same properties as the covariance matrix, i.e., that it is real, symmetric, and positive semidefinite definite.

4. **Gaussian random vector.** Given a Gaussian random vector $X \sim \mathcal{N}(\mu, \Sigma)$, where $\mu = (1 \ 5 \ 2)^T$ and

   \[ \Sigma = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix} \]

   (a) Find the pdfs of
   i. $X_1$,
   ii. $X_2 + X_3$,
   iii. $2X_1 + X_2 + X_3$,
   iv. $X_3$ given $(X_1, X_2)$, and
   v. $(X_2, X_3)$ given $X_1$.

   (b) What is $P\{2X_1 + X_2 - X_3 < 0\}$? Express your answer using the $Q$ function.

   (c) Find the joint pdf on $Y = AX$, where

   \[ A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \]

5. **Gaussian Markov chain.** Let $X, Y,$ and $Z$ be jointly Gaussian random variables with zero mean and unit variance, i.e., $\mathbb{E}(X) = \mathbb{E}(Y) = \mathbb{E}(Z) = 0$ and $\mathbb{E}(X^2) = \mathbb{E}(Y^2) = \mathbb{E}(Z^2) = 1$. Let $\rho_{X,Y}$ denote the correlation coefficient between $X$ and $Y$, and let $\rho_{Y,Z}$ denote the correlation coefficient between $Y$ and $Z$. Suppose that $X$ and $Z$ are conditionally independent given $Y$.

   (a) Find $\rho_{X,Z}$ in terms of $\rho_{X,Y}$ and $\rho_{Y,Z}$.

   (b) Find the MMSE estimate of $Z$ given $(X,Y)$ and the corresponding MSE.
6. **Sufficient statistic.** The bias of a coin is a random variable $P \sim U[0, 1]$. Let $Z_1, Z_2, \ldots, Z_{10}$ be the outcomes of 10 coin flips. Thus $Z_i \sim \text{Bern}(P)$ and $Z_1, Z_2, \ldots, Z_{10}$ are conditionally independent given $P$. If $X$ is the total number of heads, then $X|\{P = p\} \sim \text{Binom}(10, p)$. Assuming that the total number of heads is 9, show that 

$$f_{P|Z_1, Z_2, \ldots, Z_{10}}(p|z_1, z_2, \ldots, z_{10}) = f_{P|X}(p|9)$$

is independent of the order of the outcomes.

7. **Packet switching.** Let $N$ be the number of packets per unit time arriving at a network switch. Each packet is routed to output port 1 with probability $p$ and to output port 2 with probability $1 - p$, independent of $N$ and of other packets. Let $X$ be the number of packets per unit time routed to output port 1. Thus 

$$X = \begin{cases} 
0 & N = 0 \\
\sum_{i=1}^{N} Z_i & N > 0 
\end{cases} \text{ where } Z_i = \begin{cases} 
1 \text{ packet } i \text{ routed to Port 1} \\
0 \text{ packet } i \text{ routed to Port 2},
\end{cases}$$

and $Z_1, Z_2, \ldots, Z_N$ are conditionally independent given $N$. Suppose that $N \sim \text{Poisson}(\lambda)$, i.e., has Poisson pmf with parameter $\lambda$. 

(a) Find the mean and variance of $X$.

(b) Find the pmf of $X$ and the pmf of $N - X$.

8. **Nonlinear estimator.** Consider a channel with the observation $Y = XZ$, where the signal $X$ and the noise $Z$ are uncorrelated Gaussian random variables. Let $E[X] = 1$, $E[Z] = 2$, $\sigma_X^2 = 5$, and $\sigma_Z^2 = 8$.

(a) Using the fact that $E(W^3) = \mu + 3\mu\sigma^2$ and $E(W^4) = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$ for $W \sim \mathcal{N}(\mu, \sigma^2)$, find the mean and covariance matrix of $[X \ Y \ Y^2]^T$.

(b) Find the MMSE linear estimate of $X$ given $Y$ and the corresponding MSE.

(c) Find the MMSE linear estimate of $X$ given $Y^2$ and the corresponding MSE.

(d) Find the MMSE linear estimate of $X$ given $Y$ and $Y^2$ and the corresponding MSE.

(e) Compare your answers in parts (b) through (d). Is the MMSE estimate of $X$ given $Y$ (namely, $E(X|Y)$) linear?

9. **Prediction.** Let $X$ be a random process with zero mean and covariance matrix

$$\Sigma_X = \begin{bmatrix}
1 & \alpha & \alpha^2 & \cdots & \alpha^{n-1} \\
\alpha & 1 & \alpha & \cdots & \alpha^{n-2} \\
\alpha^2 & \alpha & 1 & \cdots & \alpha^{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\alpha^{n-1} & \alpha^{n-2} & \alpha^{n-3} & \cdots & 1
\end{bmatrix}$$

for $|\alpha| < 1$. $X_1, X_2, \ldots, X_{n-1}$ are observed, find the best linear MSE estimate (predictor) of $X_n$. Compute its MSE.