Exercise Set #7

1. **Minimum waiting time.** Let $X_1, X_2, \ldots$ be i.i.d. exponentially distributed random variables with parameter $\lambda$, i.e., $f_{X_i}(x) = \lambda e^{-\lambda x}$, for $x \geq 0$.

   (a) Does $Y_n = \min\{X_1, X_2, \ldots, X_n\}$ converge in probability as $n$ approaches infinity?

   (b) If it converges, what is the limit?

   (c) What about $Z_n = nY_n$?

2. **Roundoff errors.** The sum of a list of 100 real numbers is to be computed. Suppose that these numbers are rounded off to the nearest integer so that each number has an error that is uniformly distributed in the interval $(-0.5, 0.5)$. Use the central limit theorem to estimate the probability that the total error in the sum of the 100 numbers exceeds 6.

3. The signal received over a wireless communication channel can be represented by two sums

   \[ X_{1n} = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} Z_j \cos \Theta_j, \quad \text{and} \]

   \[ X_{2n} = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} Z_j \sin \Theta_j, \]

   where $Z_1, Z_2, \ldots$ are i.i.d. with mean $\mu$ and variance $\sigma^2$ and $\Theta_1, \Theta_2, \ldots$ are i.i.d. $\text{U}[0, 2\pi]$ independent of $Z_1, Z_2, \ldots$. Find the distribution of $\begin{bmatrix} X_{1n} \\ X_{2n} \end{bmatrix}$ as $n$ approaches $\infty$.

4. **Polya’s urn.** An urn initially has one red ball and one white ball. Let $X_1$ denote the name of the first ball drawn from the urn. Replace that ball and one like it. Let $X_2$ denote the name of the next ball drawn. Replace it and one like it. Continue, drawing and replacing.

   (a) Argue that the probability of drawing $k$ reds followed by $n-k$ whites is

   \[ \frac{1}{2} \cdot \frac{2}{3} \cdots \frac{k}{k+1} \cdot \frac{1}{(k+2)} \cdots \frac{(n-k)}{(n+1)} = \frac{k!(n-k)!}{(n+1)!} = \frac{1}{(n+1)} \cdot \frac{1}{n+1}. \]

   (b) Let $P_n$ be the proportion of red balls in the urn after the $n$-th drawing. Argue that $P_n = \frac{k}{n+2}$, for $k = 1, 2, \ldots, n+1$. Thus all proportions are equally probable. This shows that $P_n$ tends to a uniformly distributed random variable in distribution, i.e.,

   \[ \lim_{n \to \infty} P\{P_n \leq t\} \longrightarrow t, \quad 0 \leq t \leq 1. \]

   (c) What can you say about the behavior of the proportion $P_n$ if you started initially with one red ball in the urn and two white balls? Specifically, what is the limiting distribution of $P_n$? Can you show $P\{P_n = \frac{k}{n+3}\} = \frac{k}{n+3}$, for $k = 1, 2, \ldots, n+1$?
5. **Symmetric random walk.** Let $X_n$ be a random walk defined by

\[
X_0 = 0, \\
X_n = \sum_{i=1}^{n} Z_i,
\]

where $Z_1, Z_2, \ldots$ are i.i.d. with $P\{Z_1 = -1\} = P\{Z_1 = 1\} = \frac{1}{2}$.

(a) Find $P\{X_{10} = 10\}$.
(b) Approximate $P\{-10 \leq X_{100} \leq 10\}$ using the central limit theorem.
(c) Find $P\{X_n = k\}$.

6. **Absolute-value random walk.** Consider the symmetric random walk $X_n$ in the previous problem. Define the absolute value random process $Y_n = |X_n|$. 

(a) Find $P\{Y_n = k\}$.
(b) Find $P\{\max_{1 \leq i < 20} Y_i = 10 \mid Y_{20} = 0\}$.

7. **Discrete-time Wiener process.** Let $Z_n$, $n \geq 0$, be a discrete time white Gaussian noise process, i.e., $Z_1, Z_2, \ldots$ are i.i.d. $N(0, 1)$. Define the process $X_n$, $n \geq 1$, such that $X_0 = 0$, and $X_n = X_{n-1} + Z_n$, for $n \geq 1$.

(a) Is $X_n$ an independent increment process? Justify your answer.
(b) Is $X_n$ a Gaussian process? Justify your answer.
(c) Find the mean and autocorrelation functions of $X_n$.
(d) Specify the first-order pdf of $X_n$.
(e) Specify the joint pdf of $X_3, X_5$, and $X_8$.
(f) Find $E(X_{20} \mid X_1, X_2, \ldots, X_{10})$.
(g) Given $X_1 = 4$, $X_2 = 2$, and $0 \leq X_3 \leq 4$, find the minimum MSE estimate of $X_4$.

8. **A random process.** Let $X_n = Z_{n-1} + Z_n$ for $n \geq 1$, where $Z_0, Z_1, Z_2, \ldots$ are i.i.d. $\sim N(0, 1)$.

(a) Find the mean and autocorrelation functions of $\{X_n\}$.
(b) Is $\{X_n\}$ Gaussian? Justify your answer.
(c) Find $E(X_3 \mid X_1, X_2)$.
(d) Find $E(X_3 \mid X_2)$.
(e) Is $\{X_n\}$ Markov? Justify your answer.
(f) Is $\{X_n\}$ independent increment? Justify your answer.

9. **Moving average process.** Let $X_n = \frac{1}{2}Z_{n-1} + Z_n$ for $n \geq 1$, where $Z_0, Z_1, Z_2, \ldots$ are i.i.d. $\sim N(0, 1)$. Find the mean and autocorrelation function of $X_n$.

10. **Autoregressive process.** Let $X_0 = 0$ and $X_n = \frac{1}{2}X_{n-1} + Z_n$ for $n \geq 1$, where $Z_1, Z_2, \ldots$ are i.i.d. $\sim N(0, 1)$. Find the mean and autocorrelation function of $X_n$. 

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11. Random binary waveform. In a digital communication channel the symbol “1” is represented by the fixed duration rectangular pulse
\[ g(t) = \begin{cases} 1 & \text{for } 0 \leq t < 1 \\ 0 & \text{otherwise,} \end{cases} \]
and the symbol “0” is represented by \(-g(t)\). The data transmitted over the channel is represented by the random process
\[ X(t) = \sum_{k=0}^{\infty} A_k g(t - k), \quad \text{for } t \geq 0, \]
where \(A_0, A_1, \ldots\) are i.i.d random variables with
\[ A_i = \begin{cases} +1 & \text{w.p. } \frac{1}{2} \\ -1 & \text{w.p. } \frac{1}{2}. \end{cases} \]
(a) Find its first and second order pmfs.
(b) Find the mean and the autocorrelation function of the process \(X(t)\).