3.1 DEFINITION

It is often convenient to represent the outcome of a random experiment by a number. A random variable (r.v.) is such a representation. To be more precise, let \((\Omega, \mathcal{F}, P)\) be a probability space. Then a random variable \(X : \Omega \to \mathbb{R}\) is a mapping of the outcome.

\[ \Omega \xrightarrow{\omega} X(\omega) \]

**Figure 3.1.** Random variable as a mapping.

**Example 3.1.** Let the random variable \(X\) be the number of heads in \(n\) coin flips. The sample space is \(\Omega = \{H, T\}^n\), the possible outcomes of \(n\) coin flips; then

\[ X \in \{0, 1, 2, \ldots, n\} \]

**Example 3.2.** Consider packet arrival times \(t_1, t_2, \ldots\) in the interval \((0, T]\). The sample space \(\Omega\) consists of the empty string (no packet) and all finite length strings of the form \((t_1, t_2, \ldots, t_n)\) such that \(0 < t_1 \leq t_2 \leq \cdots \leq t_n \leq T\). Define the random variable \(X\) to be the length of the string; then \(X \in \{0, 1, 2, 3, \ldots\}\).

**Example 3.3.** Consider the voltage across a capacitor. The sample space \(\Omega = \mathbb{R}\). Define the random variables

\[
X(\omega) = \omega,
\]

\[
Y(\omega) = \begin{cases} 
+1, & \omega \geq 0, \\
-1, & \text{otherwise}.
\end{cases}
\]
Example 3.4. Let \((\Omega, \mathcal{F}, P)\) be a probability space. For a given event \(A \in \mathcal{F}\), define the *indicator random variable*

\[
X(\omega) = \begin{cases} 
1, & \omega \in A, \\
0, & \text{otherwise}. 
\end{cases}
\]

We use the notation \(1_A(\omega)\) or \(\chi_A(\omega)\) to denote the indicator random variable for \(A\).

Throughout the course, we use uppercase letters, say, \(X, Y, Z, \Phi, \Theta\), to denote random variables, and lowercase letters to denote the *values* taken by the random variables. Thus, \(X(\omega) = x\) means that the random variable \(X\) takes on the value \(x\) when the outcome is \(\omega\).

As a representation of a random experiment in the probability space \((\Omega, \mathcal{F}, P)\), the random variable \(X\) can be viewed as an outcome of a random experiment on its own. The sample space is \(\mathbb{R}\) and the set of events is the Borel \(\sigma\)-algebra \(\mathcal{B}\). An event \(A \in \mathcal{B}\) occurs if \(X \in A\) and its probability is

\[
P(\{\omega \in \Omega : X(\omega) \in A\}),
\]

which is determined by the probability measure \(P\) of the underlying random experiment and the *inverse image* of \(A\) under the mapping \(X : \Omega \rightarrow \mathbb{R}\). Thus, \((\Omega, \mathcal{F}, P)\) induces a probability space \((\mathbb{R}, \mathcal{B}, P_X)\), where

\[
P_X(A) = P(\{\omega \in \Omega : X(\omega) \in A\}), \quad A \in \mathcal{B}.
\]

An implicit assumption here is that for every \(A \in \mathcal{B}\), the inverse image \(\{\omega \in \Omega : X(\omega) \in A\}\) is an event in \(\mathcal{F}\). A mapping \(X(\omega)\) satisfying this condition is called *measurable* (with respect to \(\mathcal{F}\)) and we will always assume that a given mapping is measurable.

Since we typically deal with multiple random variables on the same probability space, we will use the notation \(P\{X \in A\}\) instead of the more formal notation \(P_X(A)\) or \(P(\{\omega \in \Omega : X(\omega) \in A\})\).

The inverse image of \(A\) under \(X(\omega)\), i.e., \(\{\omega : X(\omega) \in A\}\)
3.2 Cumulative Distribution Function

To determine $P\{X \in A\}$ for any Borel set $A$, i.e., any set generated by open intervals via countable unions, intersections, and complements, it suffices to specify $P\{X \in (a, b]\}$ or $P\{X \in (a, b]\}$ for all $-\infty < a < b < \infty$. Then the probability of any other Borel set can be determined by the axioms of probability. Equivalently, it suffices to specify the cumulative distribution function (cdf) of the random variable $X$:

$$F_X(x) = P\{X \leq x\} = P\{X \in (-\infty, x]\}, \quad x \in \mathbb{R}.$$

The cdf of a random variable satisfies the following properties.

1. $F_X(x)$ is nonnegative, i.e.,
   $$F_X(x) \geq 0, \quad x \in \mathbb{R}.$$

2. $F_X(x)$ is monotonically nondecreasing, i.e.,
   $$F_X(a) \leq F_X(b), \quad a < b.$$

3. Limits.
   $$\lim_{x \to -\infty} F_X(x) = 0 \quad \text{and} \quad \lim_{x \to +\infty} F_X(x) = 1.$$

4. $F_X(x)$ is right continuous, i.e.,
   $$F_X(a^+) := \lim_{x \to a^+} F_X(x) = F_X(a).$$

5. Probability of a singleton.
   $$P\{X = a\} = F_X(a) - F_X(a^-),$$
   where $F_X(a^-) := \lim_{x \to a^-} F_X(x)$.

Throughout, we use the notation $X \sim F(x)$ means that the random variable $X$ has the cdf $F(x)$.

![Figure 3.2. An illustration of a cumulative distribution function (cdf).](image-url)
3.3 PROBABILITY MASS FUNCTION (PMF)

A random variable $X$ is said to be discrete if $F_X(x)$ consists only of steps over a countable set $\mathcal{X}$ as illustrated in Figure 3.3.

![Figure 3.3. The cdf of a discrete random variable.](image)

A discrete random variable $X$ can be completely specified by its probability mass function (pmf)

$$p_X(x) = P\{X = x\}, \quad x \in \mathcal{X}.$$  

The set $\mathcal{X}$ is often referred to as the alphabet of $X$. Clearly, $p_X(x) \geq 0$, $\sum_{x \in \mathcal{X}} p_X(x) = 1$, and

$$P(X \in A) = \sum_{x \in A \cap \mathcal{X}} p_X(x).$$

Throughout, we use the notation $X \sim p(x)$ to mean that $X$ is a discrete random variable $X$ with pmf $p(x)$.

We review a few famous discrete random variables.

**Bernoulli.** $X \sim \text{Bern}(p)$, $p \in [0, 1]$, has the pmf

$$p_X(1) = p \quad \text{and} \quad p_X(0) = 1 - p.$$  

This is the indicator of observing a head from flipping a coin with bias $p$.

**Geometric.** $X \sim \text{Geom}(p)$, $p \in [0, 1]$, has the pmf

$$p_X(k) = p(1 - p)^{k-1}, \quad k = 1, 2, 3, \ldots.$$  

This is the number of independent coin flips of bias $p$ until the first head.

**Binomial.** $X \sim \text{Binom}(n, p)$, $p \in [0, 1]$, $n = 1, 2, \ldots$, has the pmf

$$p_X(k) = \binom{n}{k} p^k (1 - p)^{n-k}, \quad k = 0, 1, \ldots, n.$$  

This is the number of heads in $n$ independent coin flips of bias $p$.

**Poisson.** $X \sim \text{Poisson}(\lambda)$, $\lambda > 0$, has the pmf

$$p_X(k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 0, 1, 2, \ldots.$$
This is often used to characterize the number of random arrivals in a unit time interval, the number of random points in a unit area, and so on.

Let $X \sim \text{Binom}(n, \lambda/n)$. Then, its pmf for a fixed $k$ is

$$p_X(k) = \binom{n}{k} \left( \frac{\lambda}{n} \right)^k \left( 1 - \frac{\lambda}{n} \right)^{n-k}$$

$$= \frac{n(n-1) \cdots (n-k+1)}{n^k} \frac{\lambda^k}{k!} \left( 1 - \frac{\lambda}{n} \right)^n \left( 1 - \frac{\lambda}{n} \right)^{-k},$$

which converges to the Poisson($\lambda$) pmf $(\lambda^k/k!)e^{-\lambda}$ as $n \to \infty$. Thus, Poisson($\lambda$) is the limit of Binom($n, \lambda/n$).

**Example 3.5.** In a popular lottery called “Powerball,” the winning combination of numbers is selected uniformly at random among 292,201,338 possible combinations. Suppose that $\alpha n$ tickets are sold. What is the probability that there is no winner?

Since the number $N$ of winners is a Binom($\alpha n, 1/n$) random variable with $n = 2.92 \times 10^8$ very large, we can use the Poisson approximation to obtain

$$P\{N = 0\} = \left( 1 - \frac{1}{n} \right)^{\alpha n} \to e^{-\alpha}, \quad \text{as } n \to \infty.$$ 

If $\alpha = 1$, there is no winner with probability of 37%. If $\alpha = 2$, this probability decreases to 14%. Thus, even with 600 million tickets sold, there is a significant chance that the lottery has no winner and rolls over to the next week (with a bigger jackpot).

### 3.4 Probability Density Function

A random variable is said to be *continuous* if its cdf is continuous as illustrated in Figure 3.4.

If $F_X(x)$ is continuous and differentiable (except possibly over a countable set), then $X$ can be completely specified by its *probability density function* (pdf) $f_X(x)$ such that

$$F_X(x) = \int_{-\infty}^{x} f_X(u) \, du.$$
Random Variables

If \( F_X(x) \) is differentiable everywhere, then by the definition of derivative

\[
f_X(x) = \frac{dF_X(x)}{dx} = \lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{P(x < X \leq x + \Delta x)}{\Delta x}.
\]

The pdf of a random variable satisfies the following properties.

1. \( f_X(x) \) is nonnegative, i.e.,
   \[ f_X(x) \geq 0, \quad x \in \mathbb{R}. \]

2. **Normalization.** \( \int_{-\infty}^{\infty} f_X(x) \, dx = 1. \)

3. For any event \( A \subset \mathbb{R} \),
   \[ P(X \in A) = \int_{x \in A} f_X(x) \, dx. \]

In particular,

\[ P(a < X \leq b) = P(a < X < b) = P(a \leq X < b) = P(a \leq X \leq b) = \int_{a}^{b} f_X(x) \, dx. \]

Note that \( f_X(x) \) should *not* be interpreted as the probability that \( X = x \). In fact, \( f_X(x) \) can be greater than 1. It is \( f_X(x) \Delta x \) that can be interpreted as the approximation of the probability \( P(x < X \leq x + \Delta x) \) for \( \Delta x \) sufficiently small.

Throughout, we use the notation \( X \sim f(x) \) to mean that \( X \) is a continuous random variable with pdf \( f(x) \).

We review a few famous continuous random variables.

**Uniform.** \( X \sim \text{Unif}[a, b], \, a < b \), has the pdf

\[
f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b], \\ 0 & \text{otherwise.} \end{cases}
\]

This is often used to model quantization noise.

**Exponential.** \( X \sim \text{Exp}(\lambda), \, \lambda > 0 \), has the pdf

\[
f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases}
\]

This is often used to model the service time in a queue or the time between two random arrivals. An exponential random variable satisfies the *memoryless property*

\[
P(X > x + t \mid X > t) = \frac{P(X > x + t)}{P(X > t)} = P(X > x), \quad t, x > 0.
\]
Example 3.6. Suppose that for every $t > 0$, the number of packet arrivals during time interval $(0, t]$ is a Poisson$(\lambda t)$ random variable, i.e.,

$$p_N(n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \quad n = 0, 1, 2, \ldots.$$  

Let $X$ be the time until the first packet arrival. Then the event \{\(X > t\)\} is equivalent to the event \{\(N = 0\)\} and thus

$$F_X(t) = 1 - P\{X > t\} = 1 - P\{N = 0\} = 1 - e^{-\lambda t}.$$  

Hence, $f_X(t) = \lambda e^{-\lambda t}$ and $X \sim \text{Exp}(\lambda)$.

Gaussian. $X \sim \text{N}(\mu, \sigma^2)$ has the pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$  

This characterizes many random phenomena such as thermal and shot noise, and is also called a normal random variable. The cdf of the standard normal random variable $\text{N}(0, 1)$ is

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du.$$  

Its complement is

$$Q(x) = 1 - \Phi(x) = P\{X > x\}.$$  

The numerical values of the $Q$ function is often used to compute probabilities of any Gaussian random variable $Y \sim \text{N}(\mu, \sigma^2)$ as

$$P\{Y > y\} = P\left\{X > \frac{y-\mu}{\sigma}\right\} = Q\left(\frac{y-\mu}{\sigma}\right). \hspace{1cm} (3.1)$$

### 3.5 Functions of a Random Variable

Let $X$ be a random variable and $g : \mathbb{R} \rightarrow \mathbb{R}$ be a given function. Then $Y = g(X)$ is a random variable and its probability distribution can be expressed through that of $X$. For example, if $X$ is discrete, then $Y$ is discrete and

$$p_Y(y) = P\{Y = y\} = P\{g(X) = y\} = \sum_{x : g(x) = y} p_X(x).$$
In general,

\[ F_Y(y) = P\{Y \leq y\} = P\{g(X) \leq y\}, \]

which can be further simplified in many cases.

**Example 3.7 (Linear function).** Let \( X \sim F_X(x) \) and \( Y = aX + b, a \neq 0 \). If \( a > 0 \), then

\[ F_Y(y) = P\{aX + b \leq y\} = P\{X \leq \frac{y-b}{a}\} = F_X\left(\frac{y-b}{a}\right). \]

Taking derivative with respect to \( y \), we have

\[ f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right) \]

We can similarly show that if \( a < 0 \), then

\[ F_Y(y) = 1 - F_X\left(\frac{y-b}{a}\right) \]

and

\[ f_Y(y) = -\frac{1}{a} f_X\left(\frac{y-b}{a}\right). \]
Combining both cases,
\[ f_Y(y) = \frac{1}{|a|} f_X \left( \frac{y - b}{a} \right) \]

As a special case, let \( X \sim N(\mu, \sigma^2) \), i.e.,
\[ f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \]

Again setting \( Y = aX + b \), we have
\[ f_Y(y) = \frac{1}{|a|} f_X \left( \frac{y - b}{a} \right) \]
\[ = \frac{1}{|a|} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-b-\mu)^2}{2a^2\sigma^2}} \]
\[ = \frac{1}{\sqrt{2\pi(a\sigma)^2}} e^{-\frac{(y-b-a\mu)^2}{2a^2\sigma^2}}. \]

Therefore, \( Y \sim N(a\mu + b, a^2\sigma^2) \). This result justifies the use of the Q function in (3.1) to compute probabilities for an arbitrary Gaussian random variable.

**Example 3.8 (Quadratic function).** Let \( X \sim F_X(x) \) and \( Y = X^2 \). If \( y < 0 \), then \( F_Y(y) = 0 \).

Otherwise,
\[ F_Y(y) = P \{-\sqrt{y} \leq X \leq \sqrt{y} \} = F_X(\sqrt{y}) - F_X((-\sqrt{y})^-) \]

If \( X \) is continuous with pdf \( f_X(x) \), then
\[ f_Y(y) = \frac{1}{2\sqrt{y}} (f_X(-\sqrt{y}) + f_X(\sqrt{y})). \]

![Figure 3.7. A quadratic function.](image-url)
The above two examples can be generalized as follows.

**Proposition 3.1.** Let \( X \sim f_X(x) \) and \( Y = g(X) \) be differentiable. Then

\[
f_Y(y) = \sum_{i=1}^{\infty} \frac{f_X(x_i)}{|g'(x_i)|},
\]

where \( x_1, x_2, \ldots \) are the solutions of the equation \( y = g(x) \) and \( g'(x_i) \) is the derivative of \( g \) evaluated at \( x_i \).

The distribution of \( Y \) can be written explicitly even when \( g \) is not differentiable.

**Example 3.9 (Limiter).** Let \( X \) be a r.v. with Laplacian pdf \( f_X(x) = \frac{1}{2} e^{-|x|} \), and let \( Y \) be defined by the function of \( X \) shown in Figure 3.8. Consider the following cases.

- If \( y < -a \), clearly \( F_Y(y) = 0 \).
- If \( y = -a \),

\[
F_Y(-a) = F_X(-1) = \int_{-\infty}^{-1} \frac{1}{2} e^x \, dx = \frac{1}{2} e^{-1}.
\]

- If \( -a < y < a \),

\[
F_Y(y) = P\{Y \leq y\} = P\{aX \leq y\} = P\{X \leq \frac{y}{a}\} = F_X\left(\frac{y}{a}\right) = \frac{1}{2} e^{-1} + \int_{-1}^{y/a} \frac{1}{2} e^{-|x|} \, dx.
\]

![Figure 3.8. The limiter function.](image-url)
• If $y \geq a$, $F_Y(y) = 1$.

Combining these cases, the cdf of $Y$ is sketched in Figure 3.9.

![Figure 3.9. The cdf of the random variable $Y$.]

### 3.6 Generation of Random Variables

Suppose that we are given a uniform random variable $X \sim \text{Unif}[0, 1]$ and wish to generate a random variable $Y$ with prescribed cdf $F(y)$. If $F(y)$ is continuous and strictly increasing, set

$$Y = F^{-1}(X).$$

Then, since $X \sim \text{Unif}[0, 1]$ and $0 \leq F(y) \leq 1$,

$$F_Y(y) = P\{Y \leq y\}$$

$$= P\{F^{-1}(X) \leq y\}$$

$$= P\{X \leq F(y)\}$$

$$= F(y).$$

(3.2)

Thus, $Y$ has the desired cdf $F(y)$. For example, to generate $Y \sim \text{Exp}(\lambda)$ from $X \sim \text{Unif}[0, 1]$, we set

$$Y = -\frac{1}{\lambda} \ln(1 - X).$$
More generally, for an arbitrary cdf $F(y)$, we define
\[
F^{-1}(x) := \min\{y : x \leq F(y)\}, \quad x \in (0, 1).
\] (3.3)
Since $F(y)$ is right continuous, the above minimum is well-defined. Furthermore, since $F(y)$ is monotonically nondecreasing, $F^{-1}(x) \leq y$ iff $x \leq F(y)$. We now set $Y = F^{-1}(X)$ as before, but under this new definition of “inverse.” It follows immediately that the equality in (3.2) continues to hold and that $Y \sim F(y)$. For example, to generate $Y \sim \text{Bern}(p)$, we set
\[
Y = \begin{cases} 
0 & X \leq 1 - p, \\
1 & \text{otherwise}.
\end{cases}
\]
In conclusion, we can generate a random variable with any desired distribution from a $\text{Unif}[0, 1]$ random variable.

Conversely, a uniform random variable can be generated from any continuous random variable. Let $X$ be a continuous random variable with cdf $F(x)$ and $Y = F(X)$. Since $F(x) \in [0, 1]$, $F_Y(y) = P[Y \leq y] = 0$ for $y < 0$ and $F_Y(y) = 1$ for $y > 1$. For $y \in [0, 1]$, let $F^{-1}(y)$ be defined as in (3.3). Then
\[
F_Y(y) = P[Y \leq y] = P[F(X) \leq y] = P[X \leq F^{-1}(y)] = F(F^{-1}(y)) = y,
\] (3.4)
where the equality in (3.4) follows by the definition of $F^{-1}(y)$. Hence, $Y \sim U[0, 1]$. For example, let $X \sim \exp(\lambda)$ and
\[
Y = \begin{cases} 
1 - \exp(-\lambda X) & X \geq 0, \\
0 & \text{otherwise}.
\end{cases}
\]
Then $Y \sim \text{Unif}[0, 1]$. 

Figure 3.11. Generation of a Bern(p) random variable.
The exact generation of a uniform random variable, which requires an infinite number of bits to describe, is not possible in any digital computer. One can instead use the following approximation. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed (i.i.d.) Bern(1/2) random variables, and

$$Y = X_1 X_2 \cdots X_n$$

be a fraction in base 2 that lies between 0 and 1. Then $Y$ is a discrete random variable uniformly distributed over the set $\{k/2^n : k = 0, 1, \ldots, 2^n - 1\}$ and its cdf $F(y)$ converges to that of a Unif[0, 1] random variable for every $y$ as $n \to \infty$. Thus, by flipping many fair coin flips, one can simulate a uniform random variable.

The fairness of coin flips is not essential to this procedure. Suppose that $Z_1$ and $Z_2$ are i.i.d. Bern($p$) random variable. The following procedure due to von Neumann can generate a single Bern(1/2) random variable, even when the bias $p$ is unknown. Let

$$X = \begin{cases} 0 & (Z_1, Z_2) = (0, 1), \\ 1 & (Z_1, Z_2) = (1, 0). \end{cases}$$

If $(Z_1, Z_2) = (0, 0)$ or $(1, 1)$, then the outcome is ignored. Clearly $p_X(0) = p_X(1) = 1/2$. By repeating the same procedure, one can generate a sequence of i.i.d. Bern(1/2) random variables from a sequence of i.i.d. Bern($p$) random variables.

**PROBLEMS**

3.1. **Probabilities from a cdf.** Let $X$ be a random variable with the cdf shown below.

![CDF Graph](image)

Find the probabilities of the following events.

(a) $\{X = 2\}$.
(b) $\{X < 2\}$.
(c) $\{X = 2\} \cup \{0.5 \leq X \leq 1.5\}$.
(d) $\{X = 2\} \cup \{0.5 \leq X \leq 3\}$.

3.2. **Gaussian probabilities.** Let $X \sim \text{N}(1000, 400)$. Express the following in terms of the Q function.
Random Variables

(a) $P(0 < X < 1020)$.
(b) $P(X < 1020 | X > 960)$.

3.3. **Laplacian.** Let $X \sim f(x) = \frac{1}{2} e^{-|x|}$.

(a) Sketch the cdf of $X$.
(b) Find $P(|X| \leq 2 \text{ or } X \geq 0)$.
(c) Find $P(|X| + |X - 3| \leq 3)$.
(d) Find $P(X \geq 0 | X \leq 1)$.

3.4. **Distance to the nearest star.** Let the random variable $N$ be the number of stars in a region of space of volume $V$. Assume that $N$ is a Poisson r.v. with pmf

$$p_N(n) = \frac{e^{-\rho V}(\rho V)^n}{n!}, \quad \text{for } n = 0, 1, 2, \ldots$$

where $\rho$ is the “density” of stars in space. We choose an arbitrary point in space and define the random variable $X$ to be the distance from the chosen point to the nearest star. Find the pdf of $X$ (in terms of $\rho$).

3.5. **Uniform arrival.** The arrival time of a professor to his office is uniformly distributed in the interval between 8 and 9 am. Find the probability that the professor will arrive during the next minute given that he has not arrived by 8:30. Repeat for 8:50.

3.6. **Lognormal distribution.** Let $X \sim N(0, \sigma^2)$. Find the pdf of $Y = e^X$ (known as the lognormal pdf).

3.7. **Random phase signal.** Let $Y(t) = \sin(\omega t + \Theta)$ be a sinusoidal signal with random phase $\Theta \sim U[-\pi, \pi]$. Find the pdf of the random variable $Y(t)$ (assume here that both $t$ and the radial frequency $\omega$ are constant). Comment on the dependence of the pdf of $Y(t)$ on time $t$.

3.8. **Quantizer.** Let $X \sim \exp(\lambda)$, i.e., an exponential random variable with parameter $\lambda$ and $Y = [X]$, i.e., $Y = k$ for $k \leq X < k + 1$, $k = 0, 1, 2, \ldots$. Find the pmf of $Y$. Define the quantization error $Z = X - Y$. Find the pdf of $Z$.

3.9. **Gambling.** Alice enters a casino with one unit of capital. She looks at her watch to generate a uniform random variable $U \sim \text{unif}[0, 1]$, then bets the amount $U$ on a fair coin flip. Her wealth is thus given by the r.v.

$$X = \begin{cases} 1 + U, & \text{with probability 1/2}, \\ 1 - U, & \text{with probability 1/2}. \end{cases}$$

Find the cdf of $X$.

3.10. **Nonlinear processing.** Let $X \sim \text{Unif}[-1, 1]$. Define the random variable

$$Y = \begin{cases} X^2 + 1, & \text{if } |X| \geq 0.5 \\ 0, & \text{otherwise}. \end{cases}$$
Find and sketch the cdf of $Y$. 