Stationary Processes

9.1 STRICT-SENSE STATIONARITY

Stationarity refers to time invariance of some, or all, of the statistics of a random process, such as mean, autocorrelation, and n-th order distribution. We define two types of stationarity: strict-sense stationarity (SSS) and wide-sense stationarity (WSS).

A random process \{X(t)\} is said to be SSS (or just stationary) if all its finite-order distributions are time invariant, i.e., the joint cdfs (pdfs, pmfs) of

\[ X(t_1), X(t_2), \ldots, X(t_k) \]

and

\[ X(t_1 + \tau), X(t_2 + \tau), \ldots, X(t_k + \tau) \]

are the same for every \( k \), every \( t_1, t_2, \ldots, t_k \), and every time shift \( \tau \). So for a SSS process, the first-order distribution is independent of \( t \), and the second-order distribution — the distribution of any two samples \( X(t_1) \) and \( X(t_2) \) — depends only on \( \tau = t_2 - t_1 \). To see this, note that from the definition of stationarity, for any \( t \), the joint distribution of \( X(t_1) \) and \( X(t_2) \) is the same as the joint distribution of \( X(t) = X(t_1 + (t - t_1)) \) and \( X(t_2 + (t - t_1)) = X(t + (t_2 - t_1)) \).

Example 9.1. IID processes are SSS.

Example 9.2. The random walk is not SSS. In fact, no independent increment process is SSS.

Example 9.3. The Gauss–Markov process defined in Section 8.2.5 is not SSS. However, if we set \( X_1 \) to the steady-state distribution of \( X_n \), it becomes SSS (see Problem 9.6).

9.2 WIDE-SENSE STATIONARITY

A random process \{X(t)\} is said to be wide-sense stationary (WSS) if its mean and autocorrelation functions are time invariant, i.e.,

- \( E[X(t)] = \mu \) is independent of \( t \)
- \( R_X(t_1, t_2) \) is a function only of the time difference \( t_2 - t_1 \).
As a technical condition, we also assume that
\[ E[X(t)^2] < \infty. \]

Since \( R_X(t_1, t_2) = R_X(t_2, t_1) \), for any wide-sense stationary process \( \{X(t)\} \), the autocorrelation function \( R_X(t_1, t_2) \) is a function only of \( |t_2 - t_1| \). Clearly SSS implies WSS. The converse is not necessarily true.

**Example 9.4.** Let
\[ X(t) = \begin{cases} 
+\sin t & \text{w.p. } 1/4, \\
-\sin t & \text{w.p. } 1/4, \\
+\cos t & \text{w.p. } 1/4, \\
-\cos t & \text{w.p. } 1/4.
\end{cases} \]

Note that \( E[X(t)] = 0 \) and \( R_X(t_1, t_2) = (1/2) \cos(t_2 - t_1) \); thus \( X(t) \) is WSS. But \( X(0) \) and \( X(\pi/4) \) have different pmfs (i.e., the first-order pmf is not time-invariant) and the process is not SSS.

For a Gaussian random process, WSS implies SSS, since every finite-order distribution of the process is completely specified by its mean and autocorrelation functions. The random walk is not WSS, since \( R_X(n_1, n_2) = \min\{n_1, n_2\} \) is not time-invariant. In fact, no independent increment process can be WSS.

### 9.3 Autocorrelation Function of WSS Processes

Let \( \{X(t)\} \) be a WSS process. We relabel \( R_X(t_1, t_2) \) as \( R_X(\tau) \), where \( \tau = t_1 - t_2 \). The autocorrelation function \( R_X(\tau) \) satisfies the following properties.

1. \( R_X(\tau) \) is even, i.e., \( R_X(\tau) = R_X(-\tau) \) for every \( \tau \).
2. \( |R_X(\tau)| \leq R_X(0) = E[X^2(t)] \), the "average power" of \( X(t) \).
3. If \( R_X(T) = R_X(0) \) for some \( T \neq 0 \), then \( R_X(\tau) \) is periodic with period \( T \) and so is \( X(t) \) (with probability 1).

**Example 9.5.** Let \( X(t) = \alpha \cos(\omega t + \Theta) \) be periodic with random phase. Then
\[ R_X(\tau) = \frac{\alpha^2}{2} \cos \omega \tau, \]
which is also periodic.

The above properties of \( R_X(\tau) \) are necessary but not sufficient for a function to qualify as an autocorrelation function for a WSS process. The necessary and sufficient condition for a function \( R(\tau) \) to be an autocorrelation function for a WSS process is that it
be even and nonnegative definite, that is, for any \( n \), any \( t_1, t_2, \ldots, t_n \) and any real vector \( \mathbf{a} = (a_1, \ldots, a_n) \),
\[
\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j R(t_i - t_j) \geq 0.
\]

To see why this is necessary, recall that the correlation matrix for a random vector must be nonnegative definite, so if we take a set of \( n \) samples from the WSS random process, their correlation matrix must be nonnegative definite. The condition is sufficient since such an \( R(\tau) \) can specify a zero-mean stationary Gaussian random process. The nonnegative definite condition may be difficult to verify directly. It turns out, however, to be equivalent to the condition that the Fourier transform of \( R_X(\tau) \), which is called the power spectral density \( S_X(f) \), is nonnegative for all frequencies \( f \).

**Example 9.6.** Consider the following functions.

(a) \( e^{-\alpha \tau} \)

(b) \( e^{-|\tau|} \)

(c) \( \text{sinc} \ \tau \)

(d) \( 2^{-|n|} \)

The diagrams illustrate the functions for different values of \( \tau \) and \( n \).
Here, the functions in (a), (c), and (g) are not autocorrelation functions, and the other functions are autocorrelation functions of some WSS processes.

If $R_X(\tau)$ drops quickly with $\tau$, this means that samples become uncorrelated quickly as we increase $\tau$. Conversely, if $R_X(\tau)$ drops slowly with $\tau$, samples are highly correlated. Figure 9.1 illustrates autocorrelation functions in these two cases. Hence, $R_X(\tau)$ is a measure of the rate of change of $X(t)$ with time $t$. It turns out that this is not just an intuitive interpretation—as will be proved in Section 9.5—the Fourier transform of $R_X(\tau)$ (the power spectral density) is in fact the average power density of $X(t)$ over frequency.

![Autocorrelation functions](image)

**Figure 9.1.** Autocorrelation functions when (a) correlation is low and (b) correlation is high.

### 9.4 POWER SPECTRAL DENSITY

The **power spectral density** (psd) of a WSS random process $\{X(t)\}$ is the Fourier transform of $R_X(\tau)$:

$$S_X(f) = \mathcal{F}[R_X(\tau)] = \int_{-\infty}^{\infty} R_X(\tau)e^{-i2\pi f \tau} d\tau.$$
For a discrete-time process \( \{ X_n \} \), the power spectral density is the discrete-time Fourier transform of the sequence \( R_X(n) \):

\[
S_X(f) = \sum_{n=-\infty}^{\infty} R_X(n) e^{-i2\pi nf}, \quad |f| < \frac{1}{2}.
\]

By taking the inverse Fourier transform, \( R_X(\tau) \) (or \( R_X(n) \)) can be recovered from \( S_X(f) \) as

\[
R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) e^{i2\pi \tau f} df,
\]

\[
R_X(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_X(f) e^{i2\pi nf} df.
\]

The power spectral density \( S_X(f) \) satisfies the following properties.

1. \( S_X(f) \) is real and even.
2. \( \int_{-\infty}^{\infty} S_X(f) df = R_X(0) = E[X^2(t)] \), that is, the area under \( S_X(f) \) is the average power.
3. \( S_X(f) \) is the average power density, i.e., the average power of \( X(t) \) in the frequency band \([f_1, f_2]\) is

\[
\int_{-f_1}^{f_1} S_X(f) df + \int_{f_1}^{f_2} S_X(f) df + \int_{f_2}^{f_1} S_X(f) df = 2 \int_{f_1}^{f_2} S_X(f) df.
\]

4. \( S_X(f) \geq 0 \).

The first two properties follow from the definition of power spectral density as the Fourier transform of a real and even function \( R_X(\tau) \). The third and fourth properties will be proved later in Section 9.5. In general, a function \( S(f) \) is a psd if and only if it is real, even, nonnegative, and \( \int_{-\infty}^{\infty} S(f) df < \infty \).

**Example 9.7.** We consider a few pairs of autocorrelation functions and power spectral densities.

(a)

\[
R_X(\tau) = e^{-2a|\tau|}
\]

\[
S_X(f) = \frac{a}{a^2 + (\pi f)^2}
\]
Example 9.8 (Discrete-time white noise process). Let \( X_1, X_2, \ldots \) be zero mean, uncorrelated, with common variance \( N \), for example, i.i.d. \( \text{N}(0, N) \). The autocorrelation function and the power spectral density are plotted in Figure 9.2.

**Figure 9.2.** Autocorrelation function and power spectral density of the discrete-time white noise process.

Example 9.9 (Band-limited white noise process). Let \( \{ X(t) \} \) be a WSS zero-mean process with autocorrelation function and power spectral density plotted in Figure reffig:band-
limited-white. Note that for any \( t \), the samples \( X \left( t \pm \frac{n}{2B} \right) \) for \( n = 0, 1, 2, \ldots \) are uncorrelated.

\[
S_X(f) = \frac{N}{2} \quad \text{for all } f,
\]
\[
R_X(\tau) = NB \text{sinc} 2B\tau.
\]

Figure 9.3. Autocorrelation function and power spectral density of the band-limited white noise process.

**Example 9.10 (White noise process).** If we let \( B \to \infty \) in the previous example, we obtain a white noise process, which has

\[
S_X(f) = \frac{N}{2} \quad \text{for all } f,
\]
\[
R_X(\tau) = \frac{N}{2} \delta(\tau).
\]

If, in addition, \( \{X(t)\} \) is Gaussian, then we obtain the famous white Gaussian noise (WGN) process. A sample path of the white noise process is depicted in Figure 9.4. For a white noise process, all samples are uncorrelated. The process is not physically realizable, since it has infinite power. However, it plays a similar role in random processes to the role of a point mass in physics and delta function in EE. Thermal noise and shot noise are well modeled as white Gaussian noise, since they have very flat psd over very wide band (GHz).

Figure 9.4. A sample function of the white noise process.
9.5 WSS PROCESSES AND LTI SYSTEMS

Consider a linear time invariant (LTI) system with impulse response \( h(t) \) and transfer function \( H(f) = \mathcal{F}[h(t)] \). Suppose that the input to the system is a WSS process \( \{X(t) : t \in \mathbb{R}\} \), as depicted in Figure 9.5. We wish to characterize its output

\[
Y(t) = X(t) \ast h(t) = \int_{-\infty}^{\infty} X(\tau) h(t - \tau) \, d\tau.
\]

![Diagram](image)

**Figure 9.5.** An LTI system driven by a WSS process input.

Assuming that the system is in **steady state**, it turns out (not surprisingly) that the output is also a WSS process. In fact, \( X(t) \) and \( Y(t) \) are jointly WSS, namely,

- \( X(t) \) and \( Y(t) \) are WSS, and
- their crosscorrelation function

\[
R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)]
\]

is time invariant, \( R_{XY}(t_1, t_2) \) is a function of \( \tau = t_1 - t_2 \).

Henceforth, we relabel \( R_{XY}(t_1, t_2) \) as \( R_{XY}(\tau) \), where \( \tau = t_1 - t_2 \). Note that unlike \( R_X(\tau) \), \( R_{XY}(\tau) \) is not necessarily even, but satisfies

\[
R_{XY}(\tau) = R_{YX}(-\tau).
\]

**Example 9.11.** Let \( \Theta \sim \text{Unif}[0, 2\pi] \). Consider two processes

\[
X(t) = \alpha \cos(\omega t + \Theta), \quad Y(t) = \alpha \sin(\omega t + \Theta).
\]

These processes are jointly WSS, since each is WSS (in fact SSS) and

\[
R_{XY}(t_1, t_2) = E[\alpha^2 \cos(\omega t_1 + \Theta) \sin(\omega t_2 + \Theta)]
\]

\[
= \frac{\alpha^2}{4\pi} \int_{0}^{2\pi} \sin(\omega(t_1 + t_2) + 2\theta) - \sin(\omega(t_1 - t_2)) \, d\theta
\]

\[
= -\frac{\alpha^2}{2} \sin(\omega(t_1 - t_2)),
\]

which is a function only of \( t_1 - t_2 \).
We define the cross power spectral density for jointly WSS processes \( \{X(t)\} \) and \( \{Y(t)\} \) as
\[
S_{XY}(f) = \mathcal{F}[R_{XY}(\tau)].
\]

**Example 9.12.** Let \( Y(t) = X(t) + Z(t) \), where \( \{X(t)\} \) and \( \{Z(t)\} \) are zero-mean uncorrelated WSS processes with power spectral densities \( S_X(f) \) and \( S_Z(f) \). Then, \( \{X(t)\} \) and \( \{Y(t)\} \) are jointly WSS. First, we show that \( \{Y(t)\} \) is WSS, since it is zero mean and
\[
R_Y(t_1, t_2) = \mathbb{E}[(X(t_1) + Z(t_1))(X(t_2) + Z(t_2))]
= \mathbb{E}[X(t_1)X(t_2)] + \mathbb{E}[Z(t_1)Z(t_2)]
= R_X(\tau) + R_Z(\tau),
\]
where \((a)\) follows since \( \{X(t)\} \) and \( \{X(t)\} \) and \( \{Z(t)\} \) are zero mean and uncorrelated.

Taking the Fourier transform of both sides,
\[
S_Y(f) = S_X(f) + S_Z(f).
\]

To show that \( Y(t) \) and \( X(t) \) are jointly WSS, consider
\[
R_{XY}(t_1, t_2) = \mathbb{E}[X(t_1)(X(t_2) + Z(t_2))]
= \mathbb{E}[X(t_1)X(t_2)] + \mathbb{E}[X(t_1)Z(t_2)]
= R_X(t_1, t_2),
\]
which is time invariant. Relabeling \( R_X(t_1, t_2) = R_X(\tau) \) and taking the Fourier transform,
\[
S_{XY}(f) = S_X(f).
\]

Let \( \{X(t): t \in \mathbb{R}\} \) be a WSS process input to a LTI system with impulse response \( h(t) \) and transfer function \( H(f) \). If the system is stable, i.e.,
\[
\left| \int_{-\infty}^{\infty} h(t) \, dt \right| = |H(0)| < \infty,
\]
then the input \( X(t) \) and the output \( Y(t) \) are jointly WSS with the following properties:

1. \( \mathbb{E}[Y(t)] = H(0) \mathbb{E}[X(t)] \).
2. \( R_{YX}(\tau) = h(\tau) * R_X(\tau) \).
3. \( R_Y(\tau) = h(\tau) * R_X(\tau) * h(-\tau) \).
4. \( S_{YX}(f) = H(f)S_X(f) \).
5. \( S_Y(f) = |H(f)|^2S_X(f) \).

These properties are summarized in Figure 9.6.
To show the first property, consider

\[
E[Y(t)] = E \left[ \int_{-\infty}^{\infty} X(\tau) h(t - \tau) \, d\tau \right] \\
= \int_{-\infty}^{\infty} E[X(\tau)] h(t - \tau) \, d\tau \\
= E[X(t)] \int_{-\infty}^{\infty} h(t - \tau) \, d\tau \\
= E[X(t)] H(0).
\]

For the second property,

\[
R_{YX}(\tau) = E[Y(t + \tau)X(t)] \\
= E \left[ \int_{-\infty}^{\infty} h(s)X(t + \tau - s)X(t) \, ds \right] \\
= \int_{-\infty}^{\infty} h(s)R_X(\tau - s) \, ds \\
= h(\tau) \ast R_X(\tau).
\]

For the third property, consider

\[
R_{Y}(\tau) = E[Y(t + \tau)Y(t)] \\
= E \left[ \int_{-\infty}^{\infty} h(s)X(t + \tau - s) \, ds \right] \\
= \int_{-\infty}^{\infty} h(s)R_{YX}(\tau + s) \, ds \\
= R_{YX}(\tau) \ast h(-\tau).
\]

In the above, we have shown that \(E[Y(t + \tau)X(t)]\) and \(E[Y(t + \tau)Y(t)]\) are functions of \(\tau\) (not of \(t\)), confirming that \(\{X(t)\}\) and \(\{Y(t)\}\) are jointly WSS. The fourth and fifth properties follow by taking the Fourier transforms of \(R_{YX}(\tau)\) and \(R_{Y}(\tau)\), respectively.
We can use the above properties to prove that $S_X(f)$ is indeed the power spectral density of $X(t)$, that is, $S_X(f), df$ is the average power contained in the frequency band $[f, f + df]$. Consider an ideal band-pass filter shown in Figure 9.7 with output

$$Y(t) = h(t) * X(t)$$

driven by a WSS process $\{X(t)\}$. Then the average power of $X(t)$ in the band $[f_1, f_2]$ is

$$E[Y^2(t)] = \int_{-\infty}^{\infty} S_Y(f) \, df = \int_{-\infty}^{\infty} |H(f)|^2 S_X(f) \, df = \int_{-f_2}^{-f_1} S_X(f) \, df + \int_{f_1}^{f_2} S_X(f) \, df = 2 \int_{f_1}^{f_2} S_X(f) \, df.$$

This argument also shows that $S_X(f) \geq 0$ for all $f$.

![Figure 9.7. The transfer function of an ideal band-pass filter over $[f_1, f_2]$.](image)

**Example 9.13 (KT/C noise).** The noise in a resistor $R$ (in ohms) due to thermal noise is modeled as a WGN voltage source $V(t)$ in series with $R$; see Figure 9.8. The power spectral density of $V(t)$ is $S_V(f) = 2kT_R V^2/\text{Hz}$ for all $f$, where $k$ is the Boltzmann constant and $T$ is the temperature in degrees K. We find the average output noise power for an RC circuit shown in Figure 9.9. First, note that the transfer function for the circuit is

$$H(f) = \frac{1}{1 + i2\pi fRC},$$

which implies that

$$|H(f)|^2 = \frac{1}{1 + (2\pi fRC)^2}.$$
Now we write the output power spectral density in terms of the input power spectral density as

\[ S_{V_s}(f) = S_V(f) |H(f)|^2 = 2kTR \frac{1}{1 + (2\pi f RC)^2}. \]

Thus the average output power is

\[
E[V_o^2(t)] = \int_{-\infty}^{\infty} S_{V_s}(f) df \\
= \frac{2kTR}{2\pi RC} \int_{-\infty}^{\infty} \frac{1}{1 + (2\pi f RC)^2} d(2\pi f RC) \\
= \frac{kT}{\pi C} \int_{-\infty}^{\infty} \frac{1}{1 + x^2} dx \\
= \frac{kT}{\pi C} \arctan x \bigg|_{-\infty}^{\infty} \\
= \frac{kT}{C},
\]

which is independent of \( R \).
9.6 LINEAR ESTIMATION OF RANDOM PROCESSES

Let \( \{X(t)\} \) and \( \{Y(t)\} \) be zero mean jointly WSS processes with known autocorrelation and crosscorrelation functions \( R_X(\tau) \), \( R_Y(\tau) \), and \( R_{XY}(\tau) \). We observe the random process \( Y(s) \) for \( t - a \leq s \leq t + b (\ -a \leq b) \) and wish to find the linear MMSE estimate of the signal \( X(t) \), i.e., the estimate of the form

\[
\hat{X}(t) = \int_{-b}^{a} h(\tau) Y(t - \tau) \, d\tau
\]

that minimizes the mean square error

\[
E[(X(t) - \hat{X}(t))^2].
\]

By the orthogonality principle, the linear MMSE estimate must satisfy

\[
(X(t) - \hat{X}(t)) \perp Y(t - \tau), \quad -b \leq \tau \leq a,
\]

or equivalently,

\[
E[(X(t) - \hat{X}(t))Y(t - \tau)] = 0, \quad -b \leq \tau \leq a.
\]

Thus, for \( -b \leq \tau \leq a \), the optimal estimation filter \( h(\tau) \) must satisfy

\[
R_{XY}(\tau) = E[X(t)Y(t - \tau)] = E[\hat{X}(t)Y(t - \tau)]
\]

\[
= E\left[ \int_{-b}^{a} h(s)Y(t - s)Y(t - \tau) \, ds \right]
\]

\[
= \int_{-b}^{a} h(\alpha)R_Y(\tau - s) \, ds.
\]

To find \( h(s) \), we need to solve an infinite set of integral equations. Solving these equations analytically is not possible in general. However, it can be done for two important special cases:

- **Infinite smoothing**: \( a, b \to \infty \).
- **Filtering**: \( a \to \infty \) and \( b = 0 \)

We discuss only the first case. The second case leads to the Wiener-Hopf equations from which the famous Wiener filter is derived.

When \( a, b \to \infty \), the integral equations for the linear MMSE estimate become

\[
R_{XY}(\tau) = \int_{-\infty}^{\infty} h(s)R_Y(\tau - s) \, ds, \quad -\infty < \tau < +\infty.
\]

In other words,

\[
R_{XY}(\tau) = h(\tau) * R_Y(\tau).
\]

By taking Fourier transforms, we have

\[
S_{XY}(f) = H(f)S_Y(f),
\]

where \( S_{XY}(f) \) is the cross-spectrum of \( X(t) \) and \( Y(t) \), and \( H(f) \) is the Fourier transform of the optimal estimation filter \( h(s) \).
which implies that the optimal infinite smoothing filter is

\[ H(f) = \frac{S_{XY}(f)}{S_{Y}(f)} \]

Observe the similarity between the optimal filter and the LMMSE estimate

\[ \hat{X} = \Sigma_{Y}^{-1} \Sigma_{X} \Sigma_{Y}^{-1} (Y - E[Y]) + E[X] \]

for the vector case discussed in Section 6.6.

The LMMSE filter achieves the MSE

\[
E[(X(t) - \hat{X}(t))^2] = E[(X(t) - \hat{X}(t))X(t)] - E[(X(t) - \hat{X}(t)]\hat{X}(t)]
\]

\[(a) \rightarrow E[(X(t) - \hat{X}(t))X(t)]
\]

\[ = E[(X(t)^2) - E[\hat{X}(t)X(t)],
\]

where \((a)\) follows by orthogonality. To evaluate the second term, consider

\[ R_{X\hat{X}}(\tau) = E[X(t + \tau)\hat{X}(t)]
\]

\[ = E \left[ X(t + \tau) \int_{-\infty}^{\infty} h(s)Y(t - s) \, ds \right]
\]

\[ = \int_{-\infty}^{\infty} h(s)R_{XY}(\tau + s) \, ds
\]

\[ = R_{XY}(\tau) \ast h(-\tau).
\]

Therefore,

\[ E[\hat{X}(t)X(t)] = R_{X\hat{X}}(0)
\]

\[ = \int_{-\infty}^{\infty} H(-f)S_{XY}(f) \, df
\]

\[ = \int_{-\infty}^{\infty} \frac{|S_{XY}(f)|^2}{S_{Y}(f)} \, df
\]

and the minimum MSE is

\[
E[(X(t) - \hat{X}(t))^2] = E[(X(t)^2) - E[\hat{X}(t)X(t)]
\]

\[ = \int_{-\infty}^{\infty} S_{X}(f) \, df - \int_{-\infty}^{\infty} \frac{|S_{XY}(f)|^2}{S_{Y}(f)} \, df
\]

\[ = \int_{-\infty}^{\infty} \left( S_{X}(f) - \frac{|S_{XY}(f)|^2}{S_{Y}(f)} \right) \, df.
\]

This MSE can be compared to

\[ \sigma_{X}^2 - \Sigma_{Y}^{-1} \Sigma_{Y}^{-1} \Sigma_{Y}
\]

for the vector case.
Example 9.14 (Additive white noise channel). Let \( \{X(t)\} \) and \( \{Z(t)\} \) be zero-mean uncorrelated WSS processes with

\[
S_X(f) = \begin{cases} 
\frac{P}{2} & |f| \leq B, \\
0 & \text{otherwise}, 
\end{cases}
\]

and

\[
S_Z(f) = \frac{N}{2} \quad \text{for all } f,
\]

as shown in Figure 9.10. In other words, the signal \( \{X(t)\} \) is band-limited white noise and the noise \( \{Z(t)\} \) is white. We find the optimal infinite smoothing filter for estimating \( X(t) \) given

\[
Y(\tau) = X(\tau) + Z(\tau), \quad -\infty < \tau < +\infty.
\]

The transfer function of the optimal filter is

\[
H(f) = \frac{S_{XY}(f)}{S_Y(f)} = \frac{S_X(f)}{S_X(f) + S_Z(f)} = \begin{cases} 
\frac{P}{P + N} & |f| \leq B, \\
0 & \text{otherwise}, 
\end{cases}
\]

as shown in Figure 9.11. The MSE of the optimal filter is

\[
\int_{-\infty}^{\infty} S_X(f) - \left[ \frac{S_{XY}(f)}{S_Y(f)} \right]^2 \, df = \int_{-B}^{B} \frac{P}{2} - \frac{(P/2)^2}{P/2 + N/2} \, df = PB - \frac{P^2/4}{(P + N)/2} 2B = \frac{NPB}{N + P}.
\]
Figure 9.11. The transfer function $H(f)$ of the optimal infinite smoothing filter.

PROBLEMS

9.1. Moving average process. Let $Z_0, Z_1, Z_2, \ldots$ be i.i.d. $\sim N(0, 1)$.
(a) Let $X_n = \frac{1}{2}Z_{n-1} + Z_n$ for $n \geq 1$. Find the mean and autocorrelation function of $X_n$.
(b) Is $\{X_n\}$ wide-sense stationary?
(c) Is $\{X_n\}$ Gaussian?
(d) Is $\{X_n\}$ strict-sense stationary?
(e) Find $E(X_3|X_1, X_2)$.
(f) Find $E(X_3|X_2)$.
(g) Is $\{X_n\}$ Markov?
(h) Is $\{X_n\}$ independent increment?
(i) Let $Y_n = Z_{n-1} + Z_n$ for $n \geq 1$. Find the mean and autocorrelation functions of $\{Y_n\}$.
(j) Is $\{Y_n\}$ wide-sense stationary?
(k) Is $\{Y_n\}$ Gaussian?
(l) Is $\{Y_n\}$ strict-sense stationary?
(m) Find $E(Y_3|Y_1, Y_2)$.
(n) Find $E(Y_3|Y_2)$.
(o) Is $\{Y_n\}$ Markov?
(p) Is $\{Y_n\}$ independent increment?

9.2. Random binary modulation. Let $\{X_n\}$ be a zero-mean wide-sense stationary random process with autocorrelation function $R_X(n)$, and $Z_1, Z_2, \ldots$ be i.i.d. Bern($p$) random variables, i.e.,

$$Z_j = \begin{cases} 
1, & \text{with probability } p, \\
0, & \text{with probability } 1 - p.
\end{cases}$$
Let
\[ Y_n = X_n \cdot Z_n, \quad n = 1, 2, \ldots. \]

(a) Find the mean and the autocorrelation function of \( \{Y_n\} \) in terms of \( R_X(n) \) and \( p \).
(b) Is \( \{Y_n\} \) jointly wide-sense stationary with \( \{X_n\} \)?

9.3. Random binary waveform (40 pts). Let \( \{N(t)\}_{t=0}^{\infty} \) be a Poisson process with rate \( \lambda \), and \( Z \) be independent of \( \{N(t)\} \) with \( P(Z = 1) = P(Z = -1) = 1/2 \). Define
\[ X(t) = Z \cdot (-1)^{N(t)}, \quad t \geq 0. \]

(a) Find the mean and autocorrelation function of \( \{X(t)\}_{t=0}^{\infty} \).
(b) Is \( \{X(t)\}_{t=0}^{\infty} \) wide-sense stationary?
(c) Find the first-order pmf \( p_{X(t)}(x) = P(X(t) = x) \).
(d) Find the second-order pmf \( p_{X(t_1),X(t_2)}(x_1, x_2) = P(X(t_1) = x_1, X(t_2) = x_2) \).
(Hint: \( \sum_{k \text{ even}} x^k / k! = (e^x + e^{-x})/2 \) and \( \sum_{k \text{ odd}} x^k / k! = (e^x - e^{-x})/2 \).)

9.4. QAM random process. Consider the random process
\[ X(t) = Z_1 \cos \omega t + Z_2 \sin \omega t, \quad -\infty < t < \infty, \]
where \( Z_1 \) and \( Z_2 \) are i.i.d. discrete random variables such that \( P_{Z_1}(+1) = P_{Z_1}(-1) = 1/2 \).
(a) Is \( X(t) \) wide-sense stationary? Justify your answer.
(b) Is \( X(t) \) strict-sense stationary? Justify your answer.

9.5. Mixture of two WSS processes. Let \( X(t) \) and \( Y(t) \) be two zero-mean WSS processes with autocorrelation functions \( R_X(\tau) \) and \( R_Y(\tau) \), respectively. Define the process
\[ Z(t) = \begin{cases} X(t), \quad &\text{with probability } \frac{1}{2} \\ Y(t), \quad &\text{with probability } \frac{1}{2}. \end{cases} \]
Find the mean and autocorrelation functions for \( Z(t) \). Is \( Z(t) \) a WSS process?

\[ X_0 \sim N(0, a) \]
\[ X_n = \frac{1}{2} X_{n-1} + Z_n, \quad n \geq 1, \]
where \( Z_1, Z_2, Z_3, \ldots \) are i.i.d. \( N(0, 1) \) independent of \( X_0 \).
(a) Find \( a \) such that \( X_n \) is stationary. Find the mean and autocorrelation functions of \( X_n \).
(b) (Difficult.) Consider the sample mean \( S_n = \frac{1}{n} \sum_{i=1}^{n} X_i, \ n \geq 1. \) Show that \( S_n \) converges to the process mean in probability even though the sequence \( X_n \) is not i.i.d. (A stationary process for which the sample mean converges to the process mean is called *mean ergodic*.)

9.7. *AM modulation.* Consider the AM modulated random process

\[
X(t) = A(t) \cos(2\pi t + \Theta),
\]

where the amplitude \( A(t) \) is a zero-mean WSS process with autocorrelation function \( R_A(\tau) = e^{-\frac{1}{2}|\tau|} \), the phase \( \Theta \) is a \text{Unif}[0, 2\pi) \) random variable, and \( A(t) \) and \( \Theta \) are independent. Is \( X(t) \) a WSS process?

9.8. *LTI system with WSS process input.* Let \( Y(t) = h(t) \ast X(t) \) and \( Z(t) = X(t) - Y(t) \) as shown in the Figure 9.12

(a) Find \( S_Y(f) \).

(b) Find \( E(Z^2(t)) \).

Your answers should be in terms of \( S_X(f) \) and the transfer function \( H(f) = \mathcal{F}[h(t)] \).

9.9. *Echo filtering.* A signal \( X(t) \) and its echo arrive at the receiver as \( Y(t) = X(t) + X(t - \Delta) + Z(t) \). Here the signal \( X(t) \) is a zero-mean WSS process with power spectral density \( S_X(f) \) and the noise \( Z(t) \) is a zero-mean WSS with power spectral density \( S_Z(f) = \frac{N_0}{2} \), uncorrelated with \( X(t) \).

(a) Find \( S_Y(f) \) in terms of \( S_X(f), \Delta, \) and \( N_0 \).

(b) Find the best linear filter to estimate \( X(t) \) from \( \{Y(s)\}_{-\infty < s < \infty} \).

9.10. *Discrete-time LTI system with white noise input.* Let \( \{X_n: -\infty < n < \infty\} \) be a discrete-time white noise process, i.e., \( E(X_n) = 0, -\infty < n < \infty, \) and

\[
R_X(n) = \begin{cases} 
1 & n = 0, \\
0 & \text{otherwise.}
\end{cases}
\]
The process is filtered using a linear time invariant system with impulse response
\[ h(n) = \begin{cases} \alpha & n = 0, \\ \beta & n = 1, \\ 0 & \text{otherwise}. \end{cases} \]

Find \( \alpha \) and \( \beta \) such that the output process \( Y_n \) has
\[ R_Y(n) = \begin{cases} 2 & n = 0, \\ 1 & |n| = 1, \\ 0 & \text{otherwise}. \end{cases} \]

9.11. Finding time of flight. Finding the distance to an object is often done by sending a signal and measuring the time of flight, the time it takes for the signal to return (assuming speed of signal, e.g., light, is known). Let \( X(t) \) be the signal sent and \( Y(t) = X(t - \delta) + Z(t) \) be the signal received, where \( \delta \) is the unknown time of flight. Assume that \( X(t) \) and \( Z(t) \) (the sensor noise) are uncorrelated zero mean WSS processes. The estimated crosscorrelation function of \( Y(t) \) and \( X(t) \), \( R_{YX}(t) \) is shown in Figure 9.13. Find the time of flight \( \delta \).

9.12. Finding impulse response of LTI system. To find the impulse response \( h(t) \) of an LTI system (e.g., a concert hall), i.e., to identify the system, white noise \( X(t), -\infty < t < \infty \), is applied to its input and the output \( Y(t) \) is measured. Given the input and output sample functions, the crosscorrelation \( R_{YX}(\tau) \) is estimated. Show how \( R_{YX}(\tau) \) can be used to find \( h(t) \).

9.13. Generating a random process with a prescribed power spectral density. Let \( S(f) \geq 0 \), for \(-\infty < f < \infty \), be a real and even function such that
\[ \int_{-\infty}^{\infty} S(f) df = 1. \]

Define the random process
\[ X(t) = \cos(2\pi Ft + \Theta), \]
where \( F \sim S(f) \) and \( \Theta \sim U[-\pi, \pi] \) are independent. Find the power spectral density of \( X(t) \). Interpret the result.
9.14. Integrators. Let $Y(t)$ be a short-term integration of a WSS process $X(t)$:

$$Y(t) = \frac{1}{T} \int_{t-T}^{t} X(u) du.$$  

Find $S_Y(f)$ in terms of $S_X(f)$.

9.15. Derivatives of stochastic processes. Let $\{X(t)\}$ be a wide-sense stationary random process with mean zero and autocorrelation function $R(\tau) = e^{-|\tau|}$. Recall that a random process $\{Y(t)\}$ is continuous in mean square if $\mathbb{E}[(Y(t + \epsilon) - Y(t))^2] \to 0$ as $\epsilon \to 0$.

(a) Find the mean and the variance of $X(t)$.

(b) Is $X(t)$ continuous in mean square? Justify your answer.

(c) Now let

$$Z_\epsilon(t) = \frac{X(t + \epsilon) - X(t)}{\epsilon}$$

be an $\epsilon$-approximation of the derivative $\dot{X}(t)$. Find the mean and the variance of $Z_\epsilon(t)$.

(d) Find the linear MMSE estimate of $Z_\epsilon(t)$ given $(X(t), X(t + \epsilon))$ and the associated MSE.

(e) Find the linear MMSE estimate of $Z_\epsilon(t)$ given $X(t)$ and the associated MSE.

(f) Find the limiting mean and variance of $Z_\epsilon(t)$ as $\epsilon \to 0$. 