Practice Final Examination (Winter 2015)

There are 6 problems, each problem with multiple parts, each part worth 10 points. Your answer should be as clear and readable as possible. Justify any claim that you make.

1. **Drawing balls without replacement (20 pts).** Suppose that we have an urn containing one red ball and \( n - 1 \) white balls. Each time we draw a ball at random from the urn without replacement (so after the \( n \)-th drawing, there is no ball left in the urn). For \( i = 1, 2, \ldots, n \), let

\[
X_i = \begin{cases} 
1 & \text{if the } i\text{-th ball is red}, \\
0 & \text{otherwise}.
\end{cases}
\]

(a) Find \( E[X_i] \), \( i = 1, 2, \ldots, n \).
(b) Find \( \text{Var}(X_i) \) and \( \text{Cov}(X_i, X_j) \), \( i, j = 1, 2, \ldots, n \).

2. **Correlation coefficients (30 pts).** Let \( X_1, X_2, X_3 \) be three identically distributed, but not necessarily independent random variables with zero mean and unit variance. Let \( \rho_{ij} \) be the correlation coefficient between \( X_i \) and \( X_j \) for \( i \neq j \in \{1, 2, 3\} \).

(a) Is it possible to have \( \rho_{12} = \rho_{13} = \rho_{23} = 1 \)? If so, construct a random triple with such correlation coefficients. If not, justify why not.
(b) Is it possible to have \( \rho_{12} = \rho_{13} = \rho_{23} = -1 \)? If so, construct a random triple with such correlation coefficients. If not, justify why not.
(c) Is it possible to have \( \rho_{12} = \rho_{13} = \rho_{23} = -1/2 \)? If so, construct a random triple with such correlation coefficients. If not, justify why not.

3. **Sampled random walk (30 pts).** Let \( \{X_n\} \) be the (standard) symmetric random walk, i.e.,

\[
X_0 = 0, \\
X_n = \sum_{i=1}^{n} Z_i, \quad n = 1, 2, \ldots,
\]

where \( Z_1, Z_2, \ldots \) are i.i.d. with \( P\{Z_1 = -1\} = P\{Z_1 = 1\} = 1/2 \). Let \( \{Y_n\} \) be a sampled version of \( \{X_n\} \) defined by

\[
Y_n = X_{2n}, \quad n = 0, 1, 2, \ldots.
\]

(a) Is \( \{Y_n\} \) independent increment? Justify your answer.
(b) Is \( \{Y_n\} \) Markov? Justify your answer.
(c) Find \( E[Y_3 \mid Y_2] \).
4. Random binary modulation (20 pts). Let \( \{X_n\} \) be a zero-mean wide-sense stationary random process with autocorrelation function \( R_X(n) \), and \( Z_1, Z_2, \ldots \) be i.i.d. Bern(p) random variables, i.e.,
\[
Z_i = \begin{cases} 
1, & \text{with probability } p, \\
0, & \text{with probability } 1 - p.
\end{cases}
\]
Assume that \( \{X_n\} \) and \( \{Z_n\} \) are independent. Let
\[
Y_n = X_n \cdot Z_n, \quad n = 1, 2, \ldots.
\]
(a) Find the mean and the autocorrelation function of \( \{Y_n\} \) in terms of \( R_X(n) \) and \( p \).
(b) Is \( \{Y_n\} \) jointly wide-sense stationary with \( \{X_n\} \)?

5. Wiener process (30 pts). Recall the following definition of the (standard) Wiener process:

- \( W(0) = 0 \),
- \( \{W(t)\} \) is independent increment with \( W(t) - W(s) \sim N(0, t - s) \) for all \( t > s \),
- \( \mathbb{P}\{\omega : W(\omega, t) \text{ is continuous in } t\} = 1 \).

Let \( W_1(t) \) and \( W_2(t) \) be independent Wiener processes.
(a) Find the mean and the variance of
\[
X(t) = \frac{1}{\sqrt{2}} (W_1(t) + W_2(t)).
\]
Is \( \{X(t)\} \) a Wiener process? Justify your answer.
(b) Find the mean and the variance of
\[
Y(t) = \frac{1}{\sqrt{2}} (W_1(t) - W_2(t)).
\]
Is \( \{Y(t)\} \) a Wiener process? Justify your answer.
(c) Find \( \mathbb{E}[X(t)Y(s)] \).

6. Derivatives of stochastic processes (60 points). Let \( \{X(t)\} \) be a wide-sense stationary random process with mean zero and autocorrelation function \( R(\tau) = e^{-|\tau|} \). Recall that a random process \( \{Y(t)\} \) is continuous in mean square if \( \mathbb{E}[(Y(t + \epsilon) - Y(t))^2] \to 0 \) as \( \epsilon \to 0 \).
(a) Find the mean and the variance of \( X(t) \).
(b) Is \( X(t) \) continuous in mean square? Justify your answer.
(c) Now let
\[
Z_\epsilon(t) = \frac{X(t + \epsilon) - X(t)}{\epsilon}
\]
be an \( \epsilon \)-approximation of the derivative \( \dot{X}(t) \). Find the mean and the variance of \( Z_\epsilon(t) \).
(d) Find the linear MMSE estimate of \( Z_\epsilon(t) \) given \( (X(t), X(t + \epsilon)) \) and the associated MSE.
(e) Find the linear MMSE estimate of \( Z_\epsilon(t) \) given \( X(t) \) and the associated MSE.
(f) Find the limiting mean and variance of \( Z_\epsilon(t) \) as \( \epsilon \to 0 \).
There are 4 problems, each problem with multiple parts, each part worth 10 points. Your answer should be as clear and readable as possible. Justify any claim that you make.

1. **Additive exponential noise channel (60 pts)**. A device has two equally likely states $S = 0$ and $S = 1$. When it is inactive ($S = 0$), it transmits $X = 0$. When it is active ($S = 1$), it transmits $X \sim \text{Exp}(1)$. Now suppose the signal is observed through the additive exponential noise channel with output

$$Y = X + Z,$$

where $Z \sim \text{Exp}(2)$ is independent of $(X, S)$. One wishes to decide whether the device is active or not.

(a) Find $f_{Y|S}(y|0)$.
(b) Find $f_{Y|S}(y|1)$.
(c) Find $f_Y(y)$.
(d) Find $p_{S|Y}(0|y)$ and $p_{S|Y}(1|y)$.
(e) Find the decision rule $d(y)$ that minimizes the probability of error

$$P(S \neq d(Y)).$$

(f) Find the corresponding probability of error.

(Hint: Recall that $Z \sim \text{Exp}(\lambda)$ means that its pdf is $f_Z(z) = \lambda e^{-\lambda z}, z \geq 0$.)

2. **Brownian bridge (40 pts)**. Let $\{W(t)\}_{t=0}^{\infty}$ be the standard Brownian motion (Wiener process). Recall that the process is independent increment and $W(t) - W(s), 0 \leq s < t$, has the pmf

$$p_{W(t) - W(s)}(n) = e^{-\lambda(t-s)} \frac{\lambda(t-s)^n}{n!}, \quad n = 0, 1, \ldots.$$ 

In the following, we investigate several properties of the process conditioned on $\{W(1) = 0\}$.

(a) Find the conditional distribution of $W(1/2)$ given $W(1) = 0$.
(b) Find $E[W(t) \mid W(1) = 0]$ for $t \in [0, 1]$.
(c) Find $E[(W(t))^2 \mid W(1) = 0]$ for $t \in [0, 1]$.
(d) Find $E[W(t_1)W(t_2) \mid W(1) = 0]$ for $t_1, t_2 \in [0, 1]$.

3. **Convergence of random processes (30 pts)**. Let $\{N(t)\}_{t=0}^{\infty}$ be a Poisson process with rate $\lambda$. Recall that the process is independent increment and $N(t) - N(s), 0 \leq s < t$, has the pmf

$$p_{N(t) - N(s)}(n) = e^{-\lambda(t-s)} \frac{(\lambda(t-s))^n}{n!}, \quad n = 0, 1, \ldots.$$
Define
\[ M(t) = \frac{N(t)}{t}, \quad t > 0. \]

(a) Find the mean and autocorrelation function of \( \{M(t)\}_{t>0} \).

(b) Does \( \{M(t)\}_{t>0} \) converge in mean square as \( t \to \infty \), that is,
\[ \lim_{t \to \infty} E[(M(t) - M)^2] = 0 \]
for some random variable (or constant) \( M \)? If so, what is the limit?

Now consider
\[ L(t) = \frac{1}{t} \int_0^t N(s) \frac{ds}{s}, \quad t > 0. \]

(c) Does \( \{L(t)\}_{t>0} \) converge in mean square as \( t \to \infty \)? If so, what is the limit?

(Hint: \( \int 1/x \, dx = \ln x + C \), \( \int \ln x \, dx = x \ln x - x + C \), and \( \lim_{x \to 0} x \ln x = 0 \).)

4. **Random binary waveform (40 pts).** Let \( \{N(t)\}_{t=0}^{\infty} \) be a Poisson process with rate \( \lambda \), and \( Z \) be independent of \( \{N(t)\} \) with \( P(Z=1) = P(Z=-1) = 1/2 \). Define
\[ X(t) = Z \cdot (-1)^{N(t)}, \quad t \geq 0. \]

(a) Find the mean and autocorrelation function of \( \{X(t)\}_{t=0}^{\infty} \).

(b) Is \( \{X(t)\}_{t=0}^{\infty} \) wide-sense stationary?

(c) Find the first-order pmf \( p_{X(t)}(x) = P(X(t) = x) \).

(d) Find the second-order pmf \( p_{X(t_1),X(t_2)}(x_1,x_2) = P(X(t_1) = x_1, X(t_2) = x_2) \).

(Hint: \( \sum_{k \text{ even}} x^k/k! = (e^x + e^{-x})/2 \) and \( \sum_{k \text{ odd}} x^k/k! = (e^x - e^{-x})/2 \).)