Solutions to Practice Final Examination (Winter 2015)

1. Drawing balls without replacement (20 pts). Suppose that we have an urn containing one red ball and \( n - 1 \) white balls. Each time we draw a ball at random from the urn without replacement (so after the \( n \)-th drawing, there is no ball left in the urn). For \( i = 1, 2, \ldots, n \), let

\[
X_i = \begin{cases} 
1 & \text{if the } i\text{-th ball is red,} \\
0 & \text{otherwise.}
\end{cases}
\]

(a) Find \( E[X_i] \), \( i = 1, 2, \ldots, n \).

(b) Find \( \text{Var}(X_i) \) and \( \text{Cov}(X_i, X_j) \), \( i, j = 1, 2, \ldots, n \).

Solution:

(a) Since the balls are drawn randomly,

\[
P\{X_1 = 1\} = \frac{1}{n},
\]

\[
P\{X_1 = 0\} = \frac{n - 1}{n}.
\]

For \( i = 2, \ldots, n \), \( X_i = 0 \) if the red ball is drawn before. Therefore, we have

\[
P\{X_i = 1 \mid \sum_{j=1}^{i-1} X_j = 1\} = 0,
\]

and hence

\[
P\{X_i = 1\} = P\{X_1 = \ldots = X_{i-1} = 0\} \cdot P\{X_i = 1 \mid X_1 = \ldots = X_{i-1} = 0\}
\]

\[
= \frac{n - 1}{n} \cdot \frac{n - 2}{n - 1} \cdot \frac{n - i + 1}{n - i + 2} \cdot \frac{1}{n - i + 1}
\]

\[
= \frac{1}{n}.
\]

Therefore for \( i = 1, 2, \ldots, n \),

\[
X_i = \begin{cases} 
1 & \text{w. p. } \frac{1}{n}, \\
0 & \text{w. p. } \frac{n - 1}{n}.
\end{cases}
\]

We have

\[
E[X_i] = 1 \times \frac{1}{n} + 0 \times \frac{n - 1}{n} = \frac{1}{n}.
\]
(b)

\[ \text{Var}(X_i) = E[X_i^2] - E[X_i]^2 = \frac{1}{n} - \frac{1}{n^2} = \frac{n - 1}{n^2}, \]

\[ \text{Cov}(X_i, X_j) = E[X_iX_j] - E[X_i]E[X_j] = 0 - \left( \frac{1}{n} \right)^2 = -\frac{1}{n^2}, \quad i \neq j. \]

\[ E[X_iX_j] = 0 \text{ since for } i \neq j \text{ at most one of } X_i \text{ and } X_j \text{ can be one.} \]

2. **Correlation coefficients (30 pts).** Let \( X_1, X_2, X_3 \) be three identically distributed, but not necessarily independent random variables with zero mean and unit variance. Let \( \rho_{ij} \) be the correlation coefficient between \( X_i \) and \( X_j \) for \( i \neq j \in \{1, 2, 3\} \).

(a) Is it possible to have \( \rho_{12} = \rho_{13} = \rho_{23} = 1? \) If so, construct a random triple with such correlation coefficients. If not, justify why not.

(b) Is it possible to have \( \rho_{12} = \rho_{13} = \rho_{23} = -1? \) If so, construct a random triple with such correlation coefficients. If not, justify why not.

(c) Is it possible to have \( \rho_{12} = \rho_{13} = \rho_{23} = -1/2? \) If so, construct a random triple with such correlation coefficients. If not, justify why not.

**Solution:**

(a) Yes. Consider \( X_1 = X_2 = X_3 = X \), where \( X \) is a random variable with finite and non-zero variance. Then

\[ \rho_{ij} = \frac{\text{Cov}(X_i, X_j)}{\sqrt{\text{Var}(X_i)\text{Var}(X_j)}} = \frac{E[X_iX_j] - E[X_i]E[X_j]}{\sqrt{\text{Var}(X_i)\text{Var}(X_j)}} = \frac{E[X_i^2] - E[X_i]^2}{\text{Var}(X_i)} = 1. \]

(b) No. Let \( \sigma^2 \) be the variance of the three identically distributed random variables. If \( \rho_{12} = \rho_{13} = \rho_{23} = -1 \), then the covariance matrix will be

\[ \sigma^2 \begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix}, \]

which is positive semidefinite if and only if \(-\frac{1}{2} \leq \rho \leq 1\) (to see this, compute the determinant).

(c) Yes. Let \( n = 3 \) in Problem 1 and consider the resulting identically distributed random variables \( X_1, X_2, X_3 \). For \( i \neq j \), we have

\[ \rho_{ij} = \frac{\text{Cov}(X_i, X_j)}{\sqrt{\text{Var}(X_i)\text{Var}(X_j)}} = \frac{-\frac{1}{n}}{\frac{2}{n^2}} = -\frac{1}{2}. \]
Alternatively, we can construct random variables $X_1, X_2, \text{and } X_3$ from three i.i.d. random variables $Z_1, Z_2, Z_3$ with mean $\mu_Z = 0$ and variance $\sigma^2_Z = 1$ as follows:

$$X_1 = \frac{1}{\sqrt{2}} (-Z_1 + Z_2),$$
$$X_2 = \frac{1}{\sqrt{2}} (Z_1 - Z_3),$$
$$X_3 = \frac{1}{\sqrt{2}} (-Z_2 + Z_3).$$

We have

$$\text{Var}(X_i) = \frac{1}{2} + \frac{1}{2} = 1,$$

and for $i \neq j$

$$\text{Cov}(X_i, X_j) = -\frac{1}{2}.$$ 

3. Sampled random walk (30 pts). Let $\{X_n\}$ be the (standard) symmetric random walk, i.e.,

$$X_0 = 0,$$
$$X_n = \sum_{i=1}^{n} Z_i, \quad n = 1, 2, \ldots,$$

where $Z_1, Z_2, \ldots$ are i.i.d. with $P\{Z_1 = -1\} = P\{Z_1 = 1\} = 1/2$. Let $\{Y_n\}$ be a sampled version of $\{X_n\}$ defined by

$$Y_n = X_{2n}, \quad n = 0, 1, 2, \ldots.$$

(a) Is $\{Y_n\}$ independent increment? Justify your answer.
(b) Is $\{Y_n\}$ Markov? Justify your answer.
(c) Find $E[Y_3 \mid Y_2]$.

Solution:

(a) Yes. $Y_{n1}, Y_{n2} - Y_{n1}, \ldots, Y_{nk} - Y_{nk-1}$ are independent for all $n_1 < n_2 < \ldots < n_k$ and all $k$, since they are sums of disjoint sets of $Z_i$’s and $Z_i$’s are independent.

(b) Yes. Since $\{Y_n\}$ is independent increment, it is Markov.

(c) We have $Y_3 = Y_2 + Z_5 + Z_6$. Therefore, given $Y_2 = y_2$

$$Y_3 = \begin{cases} 
  y_2 + 2 & \text{if } Z_5 = Z_6 = 1, \\
  y_2 & \text{if } Z_5 = 1, Z_6 = -1 \text{ or } Z_5 = -1, Z_6 = 1, \\
  y_2 - 2 & \text{if } Z_5 = Z_6 = -1.
\end{cases}$$

Hence

$$Y_3 = \begin{cases} 
  y_2 + 2 & \text{w. p. } \frac{1}{4}, \\
  y_2 & \text{w. p. } \frac{1}{2}, \\
  y_2 - 2 & \text{w. p. } \frac{1}{4}.
\end{cases}$$
So we have
\[ E[Y_3 \mid Y_2 = y_2] = \frac{1}{4}(y_2 + 2) + \frac{1}{2}y_2 + \frac{1}{4}(y_2 - 2) = y_2, \]
and
\[ E[Y_3 \mid Y_2] = Y_2. \]

4. Random binary modulation (20 pts). Let \( \{X_n\} \) be a zero-mean wide-sense stationary random process with autocorrelation function \( R_X(n) \), and \( Z_1, Z_2, \ldots \) be i.i.d. Bern(p) random variables, i.e.,
\[ Z_i = \begin{cases} 1, & \text{with probability } p, \\ 0, & \text{with probability } 1 - p. \end{cases} \]

Let
\[ Y_n = X_n \cdot Z_n, \quad n = 1, 2, \ldots. \]

(a) Find the mean and the autocorrelation function of \( \{Y_n\} \) in terms of \( R_X(n) \) and \( p \).

(b) Is \( \{Y_n\} \) jointly wide-sense stationary with \( \{X_n\} \)?

Solution:

(a) \[ E[Y_n] = E[X_n] E[Z_n] = 0, \quad n = 1, 2, \ldots. \]
\[
R_Y(m + n, m) = E[Y_{m+n}Y_m] = E[X_{m+n}X_mZ_{m+n}Z_m] = E[X_{m+n}X_m]E[Z_{m+n}Z_m] \quad \text{Since } Z_i\'s \text{ are independent of } \{X_n\} \\
= R_X(n)E[Z_{m+n}Z_m] \quad \text{if } n = 0, \\
= \begin{cases} R_X(0)E[Z_m^2] & \text{if } n = 0, \\ R_X(n)(E[Z_m])^2 & \text{otherwise}. \end{cases}
\]

(b) From part (a), we know that the mean and the autocorrelation function of \( \{Y_n\} \) are time invariant, so \( \{Y_n\} \) is WSS.

\[ R_{YX}(m + n, m) = E[Y_{m+n}X_m] = E[X_{m+n}Z_{m+n}X_m] = E[X_{m+n}X_mE[Z_{m+n}] = pR_X(n). \]

The cross correlation is also time invariant, so \( \{X_n\} \) and \( \{Y_n\} \) are jointly WSS.

5. Wiener process (30 pts). Recall the following definition of the (standard) Wiener process:

- \( W(0) = 0 \),
- \( \{W(t)\} \) is independent increment with \( W(t) - W(s) \sim N(0, t - s) \) for all \( t > s \),
- \( P\{\omega : W(\omega, t) \text{ is continuous in } t\} = 1 \).

Let \( W_1(t) \) and \( W_2(t) \) be independent Wiener processes.
(a) Find the mean and the variance of

\[ X(t) = \frac{1}{\sqrt{2}} (W_1(t) + W_2(t)). \]

Is \( \{X(t)\} \) a Wiener process? Justify your answer.

(b) Find the mean and the variance of

\[ Y(t) = \frac{1}{\sqrt{2}} (W_1(t) - W_2(t)). \]

Is \( \{Y(t)\} \) a Wiener process? Justify your answer.

(c) Find \( E[X(t)Y(s)] \).

**Solution:**

(a) We have

\[ E[X(t)] = \frac{1}{\sqrt{2}} (E[W_1(t)] + E[W_2(t)]) = 0. \]

Therefore,

\[ \text{Var}(X(t)) = E[X(t)^2] = \frac{1}{2} (E[W_1(t)^2] + E[W_2(t)^2] + 2E[W_1(t)W_2(t)]) = \frac{1}{2} (t + t + 0) = t. \]

To see whether or not \( \{X(t)\} \) is a Wiener process, consider \( X(t) - X(s), t > s \).

\[ E[X(t) - X(s)] = 0. \]

\[ X(t) - X(s) = \frac{1}{\sqrt{2}} (W_1(t) - W_1(s)) + \frac{1}{\sqrt{2}} (W_2(t) - W_2(s)), \]

Therefore, \( X(t) - X(s) \) is Gaussian (since it is a linear combination of two independent Gaussian random variables) and

\[ \text{Var}(X(t) - X(s)) = \frac{1}{2} (\text{Var}(W_1(t) - W_1(s)) + \text{Var}(W_2(t) - W_2(s))) = t - s. \]

Hence, \( X(t) - X(s) \sim N(0, t - s) \) for all \( t > s \).

Since \( \{W_1(t)\} \) and \( \{W_2(t)\} \) are independent increment and independent from each other, the increments of \( \{X(t)\} \) are uncorrelated and hence independent (because they are Gaussian). Additionally, we have \( X(0) = 0 \), therefore, \( \{X(t)\} \) is a Wiener process.

(b) \[ E[Y(t)] = \frac{1}{\sqrt{2}} (E[W_1(t)] - E[W_2(t)]) = 0. \]

Therefore,

\[ \text{Var}(Y(t)) = E[Y(t)^2] = \frac{1}{2} (E[W_1(t)^2] + E[W_2(t)^2] - 2E[W_1(t)W_2(t)]) = \frac{1}{2} (t + t - 0) = t. \]

To see whether or not \( \{Y(t)\} \) is a Wiener process, consider \( Y(t) - Y(s), t > s \).

\[ E[Y(t) - Y(s)] = 0. \]
\[ Y(t) - Y(s) = \frac{1}{\sqrt{2}} (W_1(t) - W_1(s)) - \frac{1}{\sqrt{2}} (W_2(t) - W_2(s)), \]

Therefore, \( Y(t) - Y(s) \) is Gaussian (since it is a linear combination of two independent Gaussian random variables) and

\[ \text{Var}(Y(t) - Y(s)) = \frac{1}{2} (\text{Var}(W_1(t) - W_1(s)) + \text{Var}(W_2(t) - W_2(s))) = t - s. \]

Hence, \( Y(t) - Y(s) \sim N(0, t - s) \) for all \( t > s \).

Since \( \{W_1(t)\} \) and \( \{W_2(t)\} \) are independent increment and independent from each other, the increments of \( \{Y(t)\} \) are uncorrelated and hence independent (because they are Gaussian). Additionally, we have \( Y(0) = 0 \), therefore, \( \{Y(t)\} \) is a Wiener process.

(c) Assume \( t > s \), then we have

\[ E[W_1(t)W_1(s)] = E[(W_1(s) + W_1(t) - W_1(s))W_1(s)] = E[W_1(s)^2] = s. \]

Similarly,

\[ E[W_2(t)W_2(s)] = s. \]

Therefore,

\[ E[X(t)Y(s)] = \frac{1}{2} E[(W_1(t)+W_2(t))(W_1(s)-W_2(s))] = \frac{1}{2} (E[W_1(t)W_1(s)] - E[W_2(t)W_2(s)]) = 0. \]

6. Derivatives of stochastic processes (60 points). Let \( \{X(t)\} \) be a wide-sense stationary random process with mean zero and autocorrelation function \( R(\tau) = e^{-|\tau|} \). Recall that a random process \( \{Y(t)\} \) is continuous in mean square if \( E[(Y(t + \epsilon) - Y(t))^2] \to 0 \) as \( \epsilon \to 0 \).

(a) Find the mean and the variance of \( X(t) \).

(b) Is \( X(t) \) continuous in mean square? Justify your answer.

(c) Now let

\[ Z_\epsilon(t) = \frac{X(t + \epsilon) - X(t)}{\epsilon} \]

be an \( \epsilon \)-approximation of the derivative \( \dot{X}(t) \). Find the mean and the variance of \( Z_\epsilon(t) \).

(d) Find the linear MMSE estimate of \( Z_\epsilon(t) \) given \( (X(t), X(t + \epsilon)) \) and the associated MSE.

(e) Find the linear MMSE estimate of \( Z_\epsilon(t) \) given \( X(t) \) and the associated MSE.

(f) Find the limiting mean and variance of \( Z_\epsilon(t) \) as \( \epsilon \to 0 \).

Solution:

(a)

\[ E[X(t)] = 0. \]

\[ \text{Var}(X(t)) = E[X(t)^2] = R_X(0) = 1. \]

(b) We have

\[ E[(X(t+\epsilon)-X(t))^2] = E[X(t+\epsilon)^2] + E[X(t)^2] - 2E[X(t+\epsilon)X(t)] = 2R_X(0) - 2R_X(\epsilon) = 2 - 2e^{-|\epsilon|}. \]

\[ E[(X(t + \epsilon) - X(t))^2] \to 0 \text{ as } \epsilon \to 0, \text{ therefore, } X(t) \text{ is continuous in mean square.} \]
(c) We have
\[ E[Z_{\epsilon}(t)] = \frac{E[X(t + \epsilon)] - E[X(t)]}{\epsilon} = 0, \]
and
\[ \text{Var}(Z_{\epsilon}(t)) = E[Z_{\epsilon}(t)^2] = \frac{1}{\epsilon^2} \left( E[X(t + \epsilon)^2] + E[X(t)^2] - 2E[X(t + \epsilon)X(t)] \right) = \frac{2 - 2e^{-|\epsilon|}}{\epsilon^2}. \]

(d) \( Z_{\epsilon}(t) \) is a linear function of \( X(t + \epsilon) \) and \( X(t) \) and hence the linear MMSE estimate of \( Z_{\epsilon}(t) \) given \( (X(t), X(t + \epsilon)) \) is \( \frac{X(t + \epsilon) - X(t)}{\epsilon} \) and the associated MSE is zero. This can also be verified using the vector case linear MMSE formulas.

(e) We have
\[ \text{Cov}(Z_{\epsilon}(t), X(t)) = E[Z_{\epsilon}(t)X(t)] = \frac{1}{\epsilon} \left( E[X(t + \epsilon)X(t)] - E[X(t)^2] \right) = \frac{1}{\epsilon} \left( R_X(\epsilon) - R_X(0) \right) = \frac{1}{\epsilon} (e^{-|\epsilon|} - 1). \]

Therefore, the linear MMSE estimate is
\[ Z_{\epsilon}^*(t) = \frac{\text{Cov}(Z_{\epsilon}(t), X(t))}{\text{Var}(X(t))} (X(t) - E[X(t)]) + E[Z_{\epsilon}(t)] = \frac{e^{-|\epsilon|} - 1}{\epsilon} X(t), \]
and the corresponding MSE is
\[ \text{MSE} = \text{Var}(Z_{\epsilon}(t)) - \frac{\text{Cov}(Z_{\epsilon}(t), X(t))^2}{\text{Var}(X(t))} = \frac{2 - 2e^{-|\epsilon|}}{\epsilon^2} - \frac{(e^{-|\epsilon|} - 1)^2}{\epsilon^2} = 1 - e^{-2|\epsilon|}. \]

(f) We have
\[ \lim_{\epsilon \to 0} E[Z_{\epsilon}(t)] = 0, \]
and since the function is even,
\[ \lim_{\epsilon \to 0} \text{Var}(Z_{\epsilon}(t)) = \lim_{\epsilon \to 0^+} \text{Var}(Z_{\epsilon}(t)) = \lim_{\epsilon \to 0^+} \frac{2 - 2e^{-\epsilon}}{\epsilon^2} = \lim_{\epsilon \to 0^+} \frac{2e^{-\epsilon}}{2\epsilon} = \infty. \]
1. Additive exponential noise channel (60 pts). A device has two equally likely states $S = 0$ and $S = 1$. When it is inactive ($S = 0$), it transmits $X = 0$. When it is active ($S = 1$), it transmits $X \sim \text{Exp}(1)$. Now suppose the signal is observed through the additive exponential noise channel with output

$$Y = X + Z,$$

where $Z \sim \text{Exp}(2)$ is independent of $(X, S)$. One wishes to decide whether the device is active or not.

(a) Find $f_{Y|S}(y|0)$.
(b) Find $f_{Y|S}(y|1)$.
(c) Find $f_Y(y)$.
(d) Find $p_{S|Y}(0|y)$ and $p_{S|Y}(1|y)$.
(e) Find the decision rule $d(y)$ that minimizes the probability of error $P(S \neq d(Y))$.
(f) Find the corresponding probability of error.

(Hint: Recall that $Z \sim \text{Exp}(\lambda)$ means that its pdf is $f_Z(z) = \lambda e^{-\lambda z}$, $z \geq 0$.)

Solution:

(a) Given $S = 0$, $X = 0$ and thus $Y = Z \sim \text{Exp}(2)$. Hence,

$$f_{Y|S}(y|0) = \begin{cases} 2e^{-2y}, & y \geq 0 \\ 0, & y < 0. \end{cases}$$

(b) Given $S = 1$, $X \sim \text{Exp}(1)$ and $Y$ is the sum of two independent exponential random variables. Hence,

$$f_{Y|S}(y|1) = f_{X|S}(y) \ast f_Z(y) = e^{-y} \mathbb{1}_{y \geq 0} * 2e^{-2y} \mathbb{1}_{y \geq 0}$$

$$= \int_{-\infty}^{\infty} 2e^{-2t} \mathbb{1}_{t \geq 0} e^{-(y-t)} \mathbb{1}_{y-t \geq 0} dt$$

$$= \int_0^y 2e^{-2t+t-y} dt$$

$$= \begin{cases} 2e^{-y}(1-e^{-y}), & y \geq 0 \\ 0, & \text{otherwise}. \end{cases}$$

(c) We have

$$f_Y(y) = f_{Y|S}(y|0)p(S = 0) + f_{Y|S}(y|1)p(S = 1)$$

$$= \frac{1}{2} \left(2e^{-2y} + 2e^{-y} - 2e^{-2y}\right)$$

$$= e^{-y}, \quad y \geq 0.$$

Thus, $Y \sim \text{Exp}(1)$. 

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(d) We have

\[ p_{S|Y}(0|y) = \frac{f_{Y|S}(y|0)P(S = 0)}{f_Y(y)} = \frac{e^{-2y}}{e^{-y}} = e^{-y}. \]

We similarly have

\[ p_{S|Y}(1|y) = \frac{f_{Y|S}(y|1)P(S = 1)}{f_Y(y)} = \frac{e^{y}(1 - e^{-y})}{e^{-y}} = 1 - e^{-y}. \]

(Alternatively, \( p_{S|Y}(1|y) = 1 - p_{S|Y}(0|y) \).)

(e) We have

\[ d(y) = \arg\max_{s \in \{0, 1\}} p_{S|Y}(s|y) = \begin{cases} 0, & e^{-y} > 1 - e^{-y} \\ 1, & \text{otherwise.} \end{cases} \]

The condition \( e^{-y} > 1 - e^{-y} \) is equivalent to \( y < \ln 2 \), and hence

\[ d(y) = \begin{cases} 0, & 0 \leq y < \ln 2 \\ 1, & y \geq \ln 2. \end{cases} \]

(f) We have

\[
\begin{align*}
P(d(Y) \neq S) &= P(d(Y) \neq S, S = 0) + P(d(Y) \neq S, S = 1) \\
&= P(Y \geq \ln 2|S = 0)P(S = 0) + P(Y < \ln 2|S = 1)P(S = 1) \\
&= \frac{1}{2} \left( \int_{\ln 2}^{\infty} f_{Y|S}(y|0)dy + \int_{0}^{\ln 2} f_{Y|S}(y|1)dy \right) \\
&= \frac{1}{2} \left( \int_{\ln 2}^{\infty} 2e^{-2y}dy + \int_{0}^{\ln 2} 2e^{-y}(1 - e^{-y})dy \right) \\
&= \frac{1}{2} \left( e^{-2\ln 2} + 2(1 - e^{-\ln 2}) - (1 - e^{-2\ln 2}) \right) \\
&= \frac{1}{2} \left( \frac{1}{4} + 2 - 1 - 1 + \frac{1}{4} \right) \\
&= \frac{1}{4}.
\end{align*}
\]
2. Brownian bridge (40 pts). Let \( \{W(t)\}_{t=0}^\infty \) be the standard Brownian motion (Wiener process). Recall that the process is independent-increment with \( W(0) = 0 \) and
\[
W(t) - W(s) \sim \mathcal{N}(0, t - s), \quad 0 \leq s < t.
\]
In the following, we investigate several properties of the process conditioned on \( \{W(1) = 0\} \).

(a) Find the conditional distribution of \( W(1/2) \) given \( W(1) = 0 \).
(b) Find \( \mathbb{E}[W(t) \mid W(1) = 0] \) for \( t \in [0, 1] \).
(c) Find \( \mathbb{E}[(W(t))^2 \mid W(1) = 0] \) for \( t \in [0, 1] \).
(d) Find \( \mathbb{E}[W(t_1)W(t_2) \mid W(1) = 0] \) for \( t_1, t_2 \in [0, 1] \).

Solution:

(a) By the property of a Brownian motion,
\[
\begin{bmatrix} W(1/2) \\ W(1) \end{bmatrix} \sim \mathcal{N} \left( \begin{bmatrix} 0 \\ \mathcal{N}(1/2, 1/2) \end{bmatrix}, \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1 \end{bmatrix} \right).
\]
Therefore,
\[
\mathbb{E}[W(1/2) \mid W(1)] = \mathbb{E}[W(1/2)] + \frac{\text{Cov}(W(1/2), W(1))}{\text{Var}[W(1)]} \left( W(1) - \mathbb{E}[W(1)] \right)
\]
\[
= \frac{1}{2} W(1).
\]

Also,
\[
\text{Var}[W(1/2) \mid W(1)] = \text{Var}[W(1/2)] - \frac{\text{Cov}(W(1/2), W(1))^2}{\text{Var}[W(1)]}
\]
\[
= \frac{1}{2} - \frac{1}{4}
\]
\[
= \frac{1}{4}.
\]
Thus, \( W(1/2) \mid \{W(1) = 0\} \sim \mathcal{N}(0, 1/4) \).

(b) For \( t \in [0, 1] \),
\[
\begin{bmatrix} W(t) \\ W(1) \end{bmatrix} \sim \mathcal{N} \left( \begin{bmatrix} 0 \\ \mathcal{N}(t, t) \end{bmatrix}, \begin{bmatrix} t & t \\ t & 1 \end{bmatrix} \right).
\]
Therefore,
\[
\mathbb{E}[W(t) \mid W(1)] = \mathbb{E}[W(t)] + \frac{\text{Cov}(W(t), W(1))}{\text{Var}[W(1)]} \left( W(1) - \mathbb{E}[W(1)] \right)
\]
\[
= t W(1).
\]
Thus, \( \mathbb{E}[W(t) \mid W(1) = 0] = 0 \).
(c) 
\[
\text{Var}[W(t)|W(1)] = \text{Var}[W(t)] - \frac{\text{Cov}(W(t), W(1))^2}{\text{Var}[W(1)]}
\]
\[= t - t^2.\]

Thus, \(E[W(t)^2|W(1) = 0] = t(1 - t).\)

(d) For \(0 \leq t_1 \leq t_2 \leq 1,\)
\[
\begin{bmatrix}
W(t_1) \\
W(t_2) \\
W(1)
\end{bmatrix}
\sim \mathcal{N}
\left(0,
\begin{bmatrix}
t_1 & t_1 & t_1 \\
t_2 & t_2 & t_2 \\
1 & 1 & 1
\end{bmatrix}
\right).
\]

Hence,
\[
\text{Cov}
\left(
\begin{bmatrix}
W(t_1) \\
W(t_2)
\end{bmatrix}
\middle| W(1)
\right)
\]
\[= \text{Cov}
\left(
\begin{bmatrix}
W(t_1) \\
W(t_2)
\end{bmatrix}
\right) - \text{Cov}
\left(
\begin{bmatrix}
W(t_1) \\
W(t_2)
\end{bmatrix}, W(1)
\right) \text{Var}[W(1)]^{-1} \text{Cov}
\left(
\begin{bmatrix}
W(t_1) \\
W(t_2)
\end{bmatrix}, W(1)
\right)^T
\]
\[= \begin{bmatrix}
t_1 & t_1 & t_1 \\
t_2 & t_2 & t_2
\end{bmatrix} - \begin{bmatrix}
t_1 & t_1 & t_1 \\
t_2 & t_2 & t_2
\end{bmatrix}
\]
\[= \begin{bmatrix}
t_1(1 - t_1) & t_1(1 - t_2) \\
t_2(1 - t_1) & t_2(1 - t_2)
\end{bmatrix}.
\]

This shows that
\[
E[W(t_1)W(t_2)|W(1) = 0] = t_1(1 - t_2)
\]
\[= \text{min}(t_1, t_2) - t_1t_2.
\]

3. Convergence of random processes (30 pts). Let \(\{N(t)\}_{t=0}^{\infty}\) be a Poisson process with rate \(\lambda\). Recall that the process is independent increment and \(N(t) - N(s), 0 \leq s < t,\) has the pmf
\[
p_{N(t)-N(s)}(n) = \frac{e^{-\lambda(t-s)}(\lambda(t-s))^n}{n!}, \quad n = 0, 1, \ldots.
\]

Define
\[
M(t) = \frac{N(t)}{t}, \quad t > 0.
\]

(a) Find the mean and autocorrelation function of \(\{M(t)\}_{t>0}\).

(b) Does \(\{M(t)\}_{t>0}\) converge in mean square as \(t \to \infty,\) that is,
\[
\lim_{t \to \infty} E[(M(t) - M)^2] = 0
\]

for some random variable (or constant) \(M\)? If so, what is the limit?
Now consider
\[ L(t) = \frac{1}{t} \int_0^t \frac{N(s)}{s} \, ds, \quad t > 0. \]

(c) Does \( \{L(t)\}_{t>0} \) converge in mean square as \( t \to \infty \)? If so, what is the limit?

(Hint: \( \int 1/x \, dx = \ln x + C \), \( \int \ln x \, dx = x \ln x - x + C \), and \( \lim_{x \to 0} x \ln x = 0 \).)

Solution:

(a) We have

\[
E[M(t)] = \frac{E[N(t)]}{t}
= \frac{\lambda t}{t}
= \lambda.
\]

Also, for \( \tau \geq 0 \), we have

\[
E[M(t)M(t+\tau)] = \frac{E[N(t)N(t+\tau)]}{t(t+\tau)}
= \frac{E[N(t)](N(t) + N(t+\tau) - N(t))}{t(t+\tau)}
= \frac{E[N(t)^2] + E[N(t+\tau) - N(t)]E[N(t)]}{t(t+\tau)}
\]

(by independent-increment property)

\[
= \frac{\lambda t + \lambda^2 t^2 + \lambda \tau \cdot \lambda t}{t(t+\tau)}
= \frac{\lambda + \lambda^2 (t + \tau)}{t + \tau}
= \lambda^2 + \frac{\lambda}{t + \tau}.
\]

Thus the autocorrelation function is given by

\[
R_M(s, t) = \lambda^2 + \frac{\lambda}{\max(s, t)}.
\]

(b) We have

\[
\text{Var}[M(t)] = E[M(t)^2] - \left( E[M(t)] \right)^2
= R_M(t, t) - \lambda^2
= \frac{\lambda}{t}.
\]
Thus if we let $M = \lambda$, we have
\[
\lim_{t \to \infty} E[(M(t) - M)^2] = \lim_{t \to \infty} \text{Var}[M(t)] \\
= \lim_{t \to \infty} \frac{\lambda}{t} \\
= 0.
\]

This shows that $M(t) \longrightarrow M$ in mean square.

(c) We have
\[
E[L(t)] = \frac{1}{t} \int_0^t E[M(s)]ds \\
= \lambda.
\]

Also,
\[
\text{Var}[L(t)] = E[L(t)^2] - \left( E[L(t)] \right)^2 \\
= \frac{1}{t^2} \int_0^t \int_0^t E[M(u)M(v)]dudv - \lambda^2 \\
= \frac{1}{t^2} \int_0^t \int_0^t \left( \lambda^2 + \frac{\lambda}{\max(u,v)} \right)dudv - \lambda^2 \\
= \frac{1}{t^2} \int_0^t \int_0^t \frac{\lambda}{\max(u,v)}dudv \\
= \frac{\lambda}{t^2} \int_0^t \left( \int_0^v \frac{1}{u}du + \int_v^t \frac{1}{u}du \right)dv \\
= \frac{\lambda}{t^2} \int_0^t \left( 1 + \ln \left( \frac{t}{v} \right) \right)dv \\
= \frac{\lambda}{t^2} \left( t + t \ln t - t \ln t + t \right) \\
= \frac{2\lambda}{t}.
\]

Thus if we let $L = \lambda$, we have
\[
\lim_{t \to \infty} E[(L(t) - L)^2] = \lim_{t \to \infty} \text{Var}[L(t)] \\
= \lim_{t \to \infty} \frac{2\lambda}{t} \\
= 0.
\]

This shows that $L(t) \longrightarrow L$ in mean square.

4. Random binary waveform (40 pts). Let $\{N(t)\}_{t=0}^{\infty}$ be a Poisson process with rate $\lambda$, and $Z$ be independent of $\{N(t)\}$ with $P(Z = 1) = P(Z = -1) = 1/2$. Define
\[
X(t) = Z \cdot (-1)^{N(t)}, \quad t \geq 0.
\]
(a) Find the mean and autocorrelation function of \( \{X(t)\}_{t=0}^{\infty} \).
(b) Is \( \{X(t)\}_{t=0}^{\infty} \) wide-sense stationary?
(c) Find the first-order pmf \( p_{X(t)}(x) = P(X(t) = x) \).
(d) Find the second-order pmf \( p_{X(t_1),X(t_2)}(x_1,x_2) = P(X(t_1) = x_1, X(t_2) = x_2) \).

(Hint: \( \sum_{k \text{ even}} x^k/k! = (e^x + e^{-x})/2 \) and \( \sum_{k \text{ odd}} x^k/k! = (e^x - e^{-x})/2 \).)

Solution:

(a) Since \( Z \) is independent of the process \( N(t) \), we have

\[
E[X(t)] = E[Z] \cdot E[(-1)^{N(t)}] = 0,
\]

and, for \( \tau \geq 0 \),

\[
E[X(t)X(t+\tau)] = E[Z^2] \cdot E[(-1)^{N(t)+N(t+\tau)}] = E[(-1)^{N(t+\tau)-N(t)}] = \sum_{k=0}^{\infty} (-1)^k \frac{e^{-\lambda \tau} (\lambda \tau)^k}{k!} = \sum_{k=0}^{\infty} e^{-\lambda \tau} \left( -\lambda \tau \right)^k = e^{-2\lambda \tau}.
\]

Thus the autocorrelation function is given by

\[
R_X(s, t) = e^{-2\lambda |s-t|}.
\]

(b) Since \( E[X(t)] \) is constant and \( R_X(s, t) \) depends only on \( |s-t| \), \( X(t) \) is wide-sense stationary.

(c) We have

\[
P(X(t) = 1) = P(X(t) = 1|Z = 1)P(Z = 1) + P(X(t) = 1|Z = -1)P(Z = -1) = \frac{1}{2} \left( P((-1)^{N(t)} = 1) + P((-1)^{N(t)} = -1) \right) = \frac{1}{2} \left( P(N(t) = \text{even}) + P(N(t) = \text{odd}) \right) = \frac{1}{2}.
\]

Thus, \( p_{X(t)}(1) = p_{X(t)}(-1) = 1/2 \).
(d) Let $t_2 \geq t_1$. Since the process $N(t)$ is independent of $Z$ and $(N(t_2) - N(t_1))$ is independent of $N(t_1)$, we conclude that $N(t_2) - N(t_1)$ is independent of $(N(t_1), Z)$. Hence, $N(t_2) - N(t_1)$ is also independent of $X(t_1) = Z \cdot (-1)^{N(t_1)}$.

We have

$$P(X(t_2) = 1|X(t_1) = 1) = P\left(\frac{X(t_2)}{X(t_1)} = 1 \Big| X(t_1) = 1\right)$$

$$= P((-1)^{N(t_2) - N(t_1)} = 1|X(t_1) = 1)$$

$$= P((-1)^{N(t_2) - N(t_1)} = 1)$$

$$= P(N(t_2) - N(t_1) = \text{ even })$$

$$= \sum_{k \text{ even}} e^{-\lambda(t_2-t_1)} \frac{\left(\lambda(t_2 - t_1)\right)^k}{k!}$$

$$= e^{-\lambda(t_2-t_1)} \left(\frac{e^{\lambda(t_2-t_1)} + e^{-\lambda(t_2-t_1)}}{2}\right)$$

$$= 1 + e^{-2\lambda(t_2-t_1)} \frac{1}{2}. $$

Similarly,

$$P(X(t_2) = -1|X(t_1) = -1) = P(N(t_2) - N(t_1) = \text{ even })$$

$$= 1 + e^{-2\lambda(t_2-t_1)} \frac{1}{2}, $$

and

$$P(X(t_2) = -1|X(t_1) = 1) = P(X(t_2) = 1|X(t_1) = -1)$$

$$= P(N(t_2) - N(t_1) = \text{ odd })$$

$$= \sum_{k \text{ odd}} e^{-\lambda(t_2-t_1)} \frac{\left(\lambda(t_2 - t_1)\right)^k}{k!}$$

$$= e^{-\lambda(t_2-t_1)} \left(\frac{e^{\lambda(t_2-t_1)} - e^{-\lambda(t_2-t_1)}}{2}\right)$$

$$= 1 - e^{-2\lambda(t_2-t_1)} \frac{1}{2}. $$

Thus,

$$P_{X(t_1),X(t_2)}(x_1, x_2) = \begin{cases} 
\frac{1+e^{-2\lambda(t_2-t_1)}}{4}, & (x_1, x_2) = (1, 1) \text{ or } (-1, -1) \\
\frac{1-e^{-2\lambda(t_2-t_1)}}{4}, & (x_1, x_2) = (1, -1) \text{ or } (-1, 1).
\end{cases}$$

Finally, if we remove the restriction $t_2 \geq t_1$, the above becomes

$$P_{X(t_1),X(t_2)}(x_1, x_2) = \begin{cases} 
\frac{1+e^{-2\lambda|t_2-t_1|}}{4}, & (x_1, x_2) = (1, 1) \text{ or } (-1, -1) \\
\frac{1-e^{-2\lambda|t_2-t_1|}}{4}, & (x_1, x_2) = (1, -1) \text{ or } (-1, 1).
\end{cases}$$