2.4 Inequalities. Label each of the following statements with $=,$ $\leq,$ or $\geq$. Justify each answer.

(a) $h(X + Y)$ vs. $h(X)$, if $X$ and $Y$ are independent continuous random variables.
(b) $h(X + aY)$ vs. $h(X + Y)$, if $Y \sim N(0, 1)$ is independent of $X$ and $a \geq 1$.
(c) $I(aX + Y; bX)$ vs. $I(X + Y/a; X)$, if $a, b \neq 0$ and $Y \sim N(0, 1)$ is independent of $X$.

3.9 Capacity with input cost. Consider the DMC $p(y|x)$ with cost constraint $B$.

(a) Using the operational definition of the capacity–cost function $C(B)$, show that it is nondecreasing and concave for $B \geq 0$.
(b) Show that the information capacity–cost function $C(B)$ is nondecreasing, concave, and continuous for $B \geq 0$.

3.12 Output scaling. Show that the capacity of the Gaussian channel $Y = gX + Z$ remains the same if we scale the output by a nonzero constant $a$.

3.14 List codes. A $(2^nR, 2^nL, n)$ list code for a DMC $p(y|x)$ with capacity $C$ consists of an encoder that assigns a codeword $x^n(m)$ to each message $m \in [1 : 2^nR]$ and a decoder that upon receiving $y^n$ tries to find the list of messages $\mathcal{L}(y^n) \subseteq [1 : 2^nR]$ of size $|\mathcal{L}| \leq 2^nL$ that contains the transmitted message. An error occurs if the list does not contain the transmitted message $M$, i.e., $P_e^{(n)} = P\{M \notin \mathcal{L}(y^n)\}$.

A rate–list exponent pair $(R, L)$ is said to be achievable if there exists a sequence of $(2^nR, 2^nL, n)$ list codes with $P_e^{(n)} \to 0$ as $n \to \infty$.

(a) Using random coding and joint typicality decoding, show that any $(R, L)$ is achievable, provided $R < C + L$.
(b) Show that for every sequence of $(2^nR, 2^nL, n)$ list codes with $P_e^{(n)} \to 0$ as $n \to \infty$, we must have $R \leq C + L$. (Hint: You will need to develop a modified Fano’s inequality.)

10.1 Conditional lossless source coding. Consider the lossless source coding setup depicted in Figure 1. Let $(X, Y)$ be a 2-DMS. The source sequence $X^n$ is to be sent losslessly to a decoder with side information $Y^n$ available at both the encoder and the decoder. Thus, a $(2^nR, n)$ code is defined by an encoder $m(x^n, y^n)$ and a decoder $\hat{x}^n(m, y^n)$, and the probability of error is defined as $P_e^{(n)} = P\{\hat{x}^n \neq X^n\}$.

![Figure 1: Source coding with side information.](image_url)

(a) Find the optimal rate $R^*$.
(b) Prove achievability using $|T_e^{(n)}(X|y^n)| \leq 2^n(H(X|Y) + \delta(c))$ for $y^n \in T_e^{(n)}(Y)$.
(c) Prove the converse using Fano’s inequality.
10.6 Lossless source coding with degraded source sets. Consider the distributed lossless source coding setup depicted in Figure 2. Let \((X, Y)\) be a 2-DMS. Sender 1 encodes \((X^n, Y^n)\) into an index \(M_1\), while sender 2 encodes only \(Y^n\) into an index \(M_2\). Find the optimal rate region (a) when the decoder wishes to recover both \(X\) and \(Y\) losslessly, and (b) when the decoder wishes to recover only \(Y\).

Figure 2: Distributed lossless source coding with degraded source sets.

10.8 Lossless source coding with two decoders and side information. Consider the lossless source coding setup for a 3-DMS \((X, Y_1, Y_2)\) depicted in Figure 3. Source \(X\) is encoded into an index \(M\), which is broadcast to two decoders. Decoder \(j = 1, 2\) wishes to recover \(X\) losslessly from the index \(M\) and side information \(Y_j\). The probability of error is defined as \(P\{\hat{X}_i^n \neq X^n \text{ or } \hat{X}_2^n \neq X^n\}\). Find the optimal lossless source coding rate.

Figure 3: Lossless source coding with multiple side information.