5.11 Product of two Gaussian broadcast channels. Consider two degraded DM-BCs $p(y_{11} | x_1) p(y_{21} | y_{11})$ and $p(y_{12} | x_2) p(y_{22} | y_{12})$. The product of these two degraded DM-BCs depicted in Figure 1 is a DM-BC with $X = (X_1, X_2)$, $Y_1 = (Y_{11}, Y_{12})$, $Y_2 = (Y_{21}, Y_{22})$, and $p(y_1, y_2 | x) = p(y_{11} | x_1) p(y_{21} | y_{11}) p(y_{12} | x_2) p(y_{22} | y_{12})$. Show that the private-message capacity region of the product DM-BC is the set of rate pairs $(R_1, R_2)$ such that

$$R_1 \leq I(X_1; Y_{11}) + I(X_2; Y_{12}),$$
$$R_2 \leq I(U_1; Y_{21}) + I(U_2; Y_{22})$$

for some pmf $p(u_1, x_1)p(u_2, x_2)$. Thus, the capacity region is the Minkowski sum of the capacity regions of the two component DM-BCs.

![Figure 1: Product of two degraded broadcast channels.](image1)

5.14 Reversely degraded broadcast channels with common message. Consider two reversedly degraded DM-BCs $p(y_{11} | x_1) p(y_{21} | y_{11})$ and $p(y_{12} | x_2) p(y_{22} | y_{12})$. The product of these two degraded DM-BCs depicted in Figure 2 is a DM-BC with $X = (X_1, X_2)$, $Y_1 = (Y_{11}, Y_{12})$, $Y_2 = (Y_{21}, Y_{22})$, and $p(y_1, y_2 | x) = p(y_{11} | x_1) p(y_{21} | y_{11}) p(y_{22} | x_2) p(y_{12} | y_{12})$. A common message $M_0 \in [1 : 2^{nR_0}]$ is to be communicated to both receivers. Show that the common-message capacity is

$$C_0 = \max_{p(x_1), p(x_2)} \min \{I(X_1; Y_{11}) + I(X_2; Y_{12}), I(X_1; Y_{21}) + I(X_2; Y_{22})\}.$$  

![Figure 2: Product of reversely degraded broadcast channels.](image2)

5.15 Duality between Gaussian broadcast and multiple access channels. Consider the following Gaussian BC and Gaussian MAC:

- Gaussian BC: $Y_1 = g_1X + Z_1$ and $Y_2 = g_2X + Z_2$, where $Z_1 \sim N(0, 1)$ and $Z_2 \sim N(0, 1)$. Assume average power constraint $P$ on $X$. 

• Gaussian MAC: \( Y = g_1X_1 + g_2X_2 + Z \), where \( Z \sim N(0, 1) \). Assume the average sum-power constraint
\[
\sum_{i=1}^{n} (x_{1i}^2(m_1) + x_{2i}^2(m_2)) \leq nP, \quad (m_1, m_2) \in [1 : 2^nR_1] \times [1 : 2^nR_2].
\]

(a) Characterize the (private-message) capacity regions of these two channels in terms of \( P, g_1, g_2 \), and power allocation parameter \( \alpha \in [0, 1] \).

(b) Show that the two capacity regions are equal.

(c) Show that every point \((R_1, R_2)\) on the boundary of the capacity region of the above Gaussian MAC is achievable using random coding and successive cancellation decoding. That is, time sharing is not needed in this case.

(d) Argue that the sequence of codes that achieves the rate pairs \((R_1, R_2)\) on the boundary of the Gaussian MAC capacity region can be used to achieve the same point on the capacity region of the above Gaussian BC.

5.18 MAC with degraded message sets. Consider a DM-MAC \( p(y|x_1, x_2) \) with message pair \((M_0, M_1)\) uniformly distributed over \([1 : 2^nR_0] \times [1 : 2^nR_1]\). Sender 1 encodes \((M_0, M_1)\), while sender 2 encodes only \( M_0 \). The receiver wishes to recover both messages. The probability of error, achievability, and the capacity region are defined as for the DM-MAC with private messages.

(a) Show that the capacity region is the set of rate pairs \((R_0, R_1)\) such that
\[
R_1 \leq I(X_1; Y | X_2),
R_0 + R_1 \leq I(X_1, X_2; Y)
\]
for some pmf \( p(x_1, x_2) \).

(b) Characterize the capacity region of the Gaussian MAC with noise power 1, channel gains \( g_1 \) and \( g_2 \), and average power constraint \( P \) on each of \( X_1 \) and \( X_2 \).

6.7 Successive cancellation decoding for the Gaussian IC. In class, we found that for the DM-MAC, successive cancellation decoding with time sharing achieves the same inner bound as simultaneous decoding. In this problem, we show that this is not the case for the interference channel.

Consider the Gaussian IC with SNRs \( S_1 \) and \( S_2 \) and INRs \( I_1 \) and \( I_2 \).

(a) Write down the rate region achieved by successive cancellation decoding with Gaussian codes and no power control.

(b) Under what conditions is this region equal to the simultaneous-nonunique-decoding inner bound?

(c) How much worse can successive cancellation decoding be than simultaneous nonunique decoding?

6.8 Handoff. Consider two symmetric Gaussian ICs, one with SNR \( S \) and INR \( I > S \), and the other with SNR \( I \) and INR \( S \). Thus, the second Gaussian IC is equivalent to the setting where the messages are sent to the other receivers in the first Gaussian IC. Which channel has a larger capacity region?

7.6 No state information. Show that the capacity of the DMC with DM state \( p(y|x, s) \) \( p(s) \) when no state information is available at either the encoder or decoder is
\[
C = \max_{p(x)} I(X; Y),
\]
where \( p(y|x) = \sum_s p(s)p(y|x, s) \). Further show that any \((2^nR, n)\) code for the DMC \( p(y|x) \) achieves the same average probability of error when used over the DMC with DM state \( p(y|x, s)p(s) \), and vice versa.
7.8 **Strictly causal state information.** Consider the DMC with DM state \( p(y|x, s)p(s) \). Suppose that the state information is available strictly causally at the encoder, that is, the encoder is specified by \( x_i(m, s_{i-1}), i \in [1 : n] \). Establish the capacity (a) when the state information is not available at the decoder and (b) when the state information is also available at the decoder.

7.10 **Value of state information.** Consider the DMC with DM state \( p(y|x, s)p(s) \). Quantify how much state information can help by proving the following statements:

(a) \( C_{SI-D} - C \leq \max_{p(x)} H(S|Y) \).
(b) \( C_{SI-ED} - C_{SI-E} \leq C_{SI-ED} - C_{CSI-E} \leq \max_{p(x|s)} H(S|Y) \).

Thus, the state information at the decoder is worth at most \( H(S) \) bits. Show that the state information at the encoder can be much more valuable by providing an example for which \( C_{SI-E} - C > H(S) \).

7.15 **MMSE estimation via writing on dirty paper.** Consider the additive noise channel with output (observation)

\[ Y = X + S + Z, \]

where \( X \) is the transmitted signal and has mean \( \mu \) and variance \( P \), \( S \) is the state and has zero mean and variance \( Q \), and \( Z \) is the noise and has zero mean and variance \( N \). Assume that \( X, S, \) and \( Z \) are uncorrelated. The sender knows \( S \) and wishes to transmit a signal \( U \), but instead transmits \( X \) such that \( U = X + \alpha S \) for some constant \( \alpha \).

(a) Find the mean squared error (MSE) of the linear MMSE estimate of \( U \) given \( Y \) in terms only of \( \mu, \alpha, P, Q, \) and \( N \).
(b) Find the value of \( \alpha \) that minimizes the MSE in part (a).
(c) How does the minimum MSE obtained in part (b) compare to the MSE of the linear MMSE when there is no state at all, i.e., \( S = 0 \)? Interpret the result.