3.19 **Bounds on the quadratic rate–distortion function.** Let $X$ be an arbitrary memoryless (stationary) source with variance $P$, and let $d(x, \hat{x}) = (x - \hat{x})^2$ be the quadratic distortion measure.

(a) Show that the rate–distortion function is bounded as

$$h(X) - \frac{1}{2} \log(2\pi eD) \leq R(D) \leq \frac{1}{2} \log \left( \frac{P}{D} \right)$$

with equality iff $X$ is a WGN($P$) source. (Hint: For the upper bound, consider $\hat{X} = (P - D)X/P + Z$, where $Z \sim N(0, D(P - D)/P)$ is independent of $X$.)

Remark: The lower bound is referred to as the Shannon lower bound.

(b) Is the Gaussian source harder or easier to describe than other sources with the same variance?

3.19 **Lossy source coding from a noisy observation.** Let $X \sim p(x)$ be a DMS and $Y$ be another DMS obtained by passing $X$ through a DMC $p(y|x)$. Let $d(x, \hat{x})$ be a distortion measure and consider a lossy source coding problem in which $Y$ (instead of $X$) is encoded and sent to the decoder who wishes to reconstruct $X$ with a prescribed distortion $D$.

Unlike the regular lossy source coding setup, the encoder maps each $y^n$ sequence to an index $m \in [1 : 2^nR]$. Otherwise, the definitions of $(2^nR, n)$ codes, achievability, and rate–distortion function are the same as before.

Let $D_{\min} = \min_{\hat{x}(y)} \mathbb{E}[d(X, \hat{x}(Y))]$. Show that the rate–distortion function for this setting is

$$R(D) = \min_{p(\hat{x}|y): \mathbb{E}[d(X, \hat{x})] \leq D} I(Y; \hat{X}) \text{ for } D \geq D_{\min}. $$

(Hint: Define a new distortion measure $d'(y, \hat{x}) = \mathbb{E}(d(X, \hat{x})|Y = y)$, and show that

$$\mathbb{E}[d(X^n, \hat{x}^n(m(Y^n)))] = \mathbb{E}[d'(Y^n, \hat{x}^n(m(Y^n)))].$$ )

11.6 **Side information with occasional erasures.** Let $X$ be a Bern(1/2) source, $Y$ be the output of a BEC($p$) with input $X$, and $d$ be a Hamming distortion measure. Find a simple expression for the rate–distortion function for $X$ with side information $Y$.

15.7 **Triangular cyclic network.** Consider the 3-node graphical multiple-unicast network in Figure 1, where $C_{12} = C_{23} = C_{31} = 1$. Node $j = 1, 2, 3$ wishes to communicate a message $M_j \in [1 : 2^{nR_j}]$ to its predecessor node.

(a) Find the cutset bound.

(b) Show that the capacity region is the set of rate triples $(R_1, R_2, R_3)$ such that $R_j + R_k \leq 1$ for $j, k = 1, 2, 3$ with $j \neq k$.

16.13 **Properties of the Gaussian relay channel capacity.** Let $C(P)$ be the capacity of the Gaussian RC with average power constraint $P$ on each of $X_1$ and $X_2$.

(a) Show that $C(P) > 0$ if $P > 0$ and $C(P)$ tends to infinity as $P \to \infty$.

(b) Show that $C(P)$ tends to zero as $P \to 0$.

(c) Show that $C(P)$ is concave and strictly increasing in $P$. 

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17.11 Directed information. Prove the following properties of directed information and causally conditional probability distributions:

(a) Chain rule: \( p(y^n, x^n) = p(y^n | x^n)p(x^n | y^{n-1}) \).

(b) Nonnegativity: \( I(X^n \rightarrow Y^n) \geq 0 \) with equality iff \( p(y^n | x^n) = p(y^n) \).

(c) Conservation: \( I(X^n ; Y^n) = I(X^n \rightarrow Y^n) + I(Y_0, Y^{n-1} \rightarrow X^n) \), where \( Y_0 = \emptyset \).

(d) Comparison to mutual information: \( I(X^n \rightarrow Y^n) \leq I(X^n ; Y^n) \) with equality if there is no feedback, i.e., \( p(x_i | x_i^{i-1}, y_i^{i-1}) = p(x_i | x_i^{i-1}) \), \( i \in [1 : n] \).

17.14 Gaussian two-way channel. Consider the Gaussian two-way channel

\[
Y_1 = g_{12}X_2 + Z_1, \\
Y_2 = g_{21}X_1 + Z_2,
\]

where the noise pair \((Z_1, Z_2) \sim N(0, K)\). Assume average power constraint \( P \) on each of \( X_1 \) and \( X_2 \). Find the capacity region of this channel in terms of the power constraint \( P \), channel gains \( g_{12} \) and \( g_{21} \), and the noise covariance matrix \( K \).

17.15 Common-message feedback capacity of broadcast channels. Consider the DM-BC \( p(y_1, y_2 | x) \) with feedback from the receivers. Find the common-message capacity \( C_F \).

18.7 Broadcasting over a diamond network. Consider a DMN \( p(y_2, y_3 | x_1, x_2, x_3) = p(y_2 | x_1)p(y_3 | x_1)p(y_4 | x_2, x_3) \). Node 1 wishes to communicate a common message \( M \) to all other nodes. Find the capacity.