Homework Set #1
(Due: Wednesday, January 16, 2019)

1. **Linear functions over** $\mathbb{F}^n$.

   (a) Show that the function $f : \mathbb{F}^n \rightarrow \mathbb{F}^m$ defined by $f(x) = Ax$, where $A \in \mathbb{F}^{m \times n}$, is linear.

   (b) Show that any linear function $f : \mathbb{F}^n \rightarrow \mathbb{F}^m$ has a representation $f(x) = Ax$ for some $A \in \mathbb{F}^{m \times n}$.

   (c) Show that the representation in part (b) is unique by proving that $Ax = Bx$ for every $x$ implies that $A = B$.

2. **A linear function from convolution.** Suppose that real-valued sequences $\{u(n)\}_{n=-\infty}^{\infty}$ and $\{v(n)\}_{n=-\infty}^{\infty}$ represent the input and output signals of a discrete-time linear time-invariant system with impulse response $h(n) \in \mathbb{R}$, $n \in \mathbb{Z}$. Then, $\{u(n)\}$ and $\{v(n)\}$ are related via convolution as

   $$v(n) = \sum_{k=-\infty}^{\infty} h(k)u(n-k), \quad n \in \mathbb{Z}.$$  

   Suppose that $u(n) = 0$ for $n < 0$ or $n > N$, and define

   $$x = \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(N) \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} v(0) \\ v(1) \\ \vdots \\ v(N) \end{bmatrix}.$$  

   Thus $x$ and $y$ are vectors that capture $N + 1$ values of the input and output signals, respectively.

   (a) Find the matrix $T$ such that

   $$y = Tx$$

   in terms of $h(n)$.

   (b) Describe the structure of $T$. Matrices of this structure is said to be *Toeplitz*.

3. **Matrix multiplication.** Let $A, B \in \mathbb{R}^{n \times n}$. Prove or provide a counterexample to each of the following statements.

   (a) If $AB = 0$, then $A = 0$ or $B = 0$.

   (b) If $A^2 = 0$, then $A = 0$.

   (c) If $A' A = 0$, then $A = 0$. 
4. **Affine functions.** A function \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is said to be **affine** if for any \( x, y \in \mathbb{R}^n \) and any \( \alpha, \beta \in \mathbb{R} \) with \( \alpha + \beta = 1 \), we have

\[
f(\alpha x + \beta y) = \alpha f(x) + \beta f(y).
\]

Note that without the restriction \( \alpha + \beta = 1 \), this would be the definition of linearity.

(a) Suppose that \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \). Show that the function \( f(x) = Ax + b \) is affine.

(b) Prove the converse, namely, show that any affine function \( f \) can be represented uniquely as \( f(x) = Ax + b \) for some \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \).

(Hint: Consider the linearity of the function \( g(x) = f(x) - f(0) \).)

5. **Symmetric and Hermitian matrices.** A square matrix \( A \) is said to be **symmetric** if its transpose \( A' \) satisfies \( A' = A \), and a complex-valued square matrix \( A \) is said to be **Hermitian** if its conjugate transpose \( A^* = (A)' = \overline{A} \) satisfies \( A^* = A \). Thus, a real-valued square matrix \( A \) is symmetric if and only if it is Hermitian. Which of the following is a vector space?

(a) The set of all \( n \times n \) real-valued symmetric matrices over \( \mathbb{R} \).

(b) The set of all \( n \times n \) complex-valued symmetric matrices over \( \mathbb{C} \).

(c) The set of all \( n \times n \) complex-valued Hermitian matrices over \( \mathbb{R} \).

(d) The set of all \( n \times n \) complex-valued Hermitian matrices over \( \mathbb{C} \).

For each case, either verify that it is a vector space or prove otherwise.

6. **Subspaces.** Let \( V \) and \( W \) be subspaces of a vector space. Which of the following is also a subspace?

(a) **Minkowski sum** \( V + W = \{v + w : v \in V, w \in W\} \).

(b) \( V \cap W \).

(c) \( V \cup W \).

For each case, either verify that it is a subspace or prove otherwise.

7. **Bases.** Find a basis for each of the following subspaces of \( \mathbb{R}^4 \).

(a) All vectors whose components are equal.

(b) All vectors whose components sum to zero.

(c) All vectors orthogonal to both \( [1 \ 1 \ 0 \ 0]' \) and \( [0 \ 0 \ 1 \ 1]' \).

(d) All vectors spanned by \( [1 \ 1 \ 0 \ 0]' \), \( [0 \ 1 \ 1 \ 0]' \), \( [0 \ 0 \ 1 \ 1]' \), and \( [1 \ 0 \ 0 \ 1]' \).

Repeat parts (a)–(d) for \( \mathbb{F}_2^4 \) instead of \( \mathbb{R}^4 \).