Your answer should be as clear as possible. There is no need to justify anything.

1. **Equivalence (45 points).** Group the following 10 statements by their equivalence. For example, let $\alpha, \beta \in \mathbb{R}$. The four statements

   (w) $\alpha \beta = 0$.
   (x) $\beta = -\alpha$.
   (y) $\alpha + \beta = 0$.
   (z) $\alpha = 0$ or $\beta = 0$.

   can be grouped into (w), (z) (x), (y) For each pair of statements, the correct identification of equivalence earns one point.

   Now the real problem begins. Throughout this problem, the matrix $A \in \mathbb{R}^{n \times n}$ is real and square.

   (a) $A = I$.
   (b) There exists a matrix $B$ such that $AB = I$.
   (c) Rows of $A$ form an orthonormal basis for $\mathbb{R}^n$.
   (d) $A^2 = 0$.
   (e) $A = 0$.
   (f) rank($A'A$) = 0.
   (g) $A^2$ is invertible.
   (h) rank($A$) = $n$.
   (i) $A^{-1} = A'$.
   (j) $A^2 = A$ and $\mathcal{R}(A) = \mathbb{R}^n$.

**Solution:** The correct grouping is

   (b), (g), (h) (a), (j) (e), (f) (c), (i) (d)

   We gave a point to each correct pairwise equivalence, even when the answers did not satisfy basic axioms of equivalence relations such as the transitivity. Each equivalence class is justified as follows.

   (b) $\implies$ (g). For any $b \in \mathbb{R}^n$, $Ab$ is a linear combination of the columns of $A$. Therefore, letting $b_1, \ldots, b_n$ be the columns of $B$, we observe that $AB = I$ implies that $Ab_i = e_i$ for $i = 1, \ldots, n$. This implies that $e_1, \ldots, e_n$ are in the column span of $A$. This shows that

   \[
   \text{rank}(A) = \text{dim}(\text{col. sp. } A) \geq \text{dim}(\text{span}\{e_1, \ldots, e_n\}) = n,
   \]
i.e., \(\text{rank}(A) = n\). This also follows immediately from the rank inequality \(\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}\). Since \(\text{rank}(A) = n\) and \(A\) is a square matrix, \(A\) is invertible. Let \(A^{-1}\) denote the inverse of \(A\). Then, \(A^2(A^{-1})^2 = I\) and \((A^{-1})^2A^2 = I\). By definition, \(A^2\) is invertible with the inverse \((A^{-1})^2\).

\((g) \implies (h)\). This follows from \(n = \text{rank}(A^2) \leq \text{rank}(A) \leq n\).

\((h) \implies (b)\). If \(A\) is full-rank, then it is invertible, which directly leads to (b).

\((a) \implies (j)\). \(I^2 = I\) and \(Ix = x\) for every \(x \in \mathbb{R}^n\); therefore \(\mathcal{R}(I) = \mathbb{R}^n\).

\((j) \implies (a)\). Since \(\mathcal{R}(A) = \mathbb{R}^n\), \(A\) is full rank and thus invertible. Let \(A^{-1}\) be the inverse of \(A\). We have

\[A^2 = A \implies A^{-1}A^2 = A^{-1}A \implies A = I.\]

\((e) \implies (f)\). \(A = 0 \implies A'A = 0 \implies \text{rank}(A'A) = 0.\)

\((f) \implies (e)\). From Problem 3, HW#2, \(\text{rank}(A'A) = \text{rank}(A)\) over \(\mathbb{R}\). Therefore

\[\text{rank}(A'A) = 0 \implies \text{rank}(A) = 0 \implies A = 0.\]

\((c) \implies (i)\). We have, for any two rows \(a'_i, a'_j\) of \(A\), \(a'_ia_j = 0\) if \(i \neq j\) and \(a'_ia_i = 1\). Therefore, we have \(AA' = I\). This shows that

\[\text{rank}(A) = \text{rank}(AA') = n = \text{rank}(A'A),\]

where the last equality follows from Problem 3, HW#2. Therefore, \(\mathcal{R}(A'A) = \mathbb{R}^n\), and moreover,

\[(A'A)^2 = A'A A'A = A'A,\]

so from the equivalence of (a) and (j), \(A'A = I\). Thus, \(AA' = A'A = I\), hence, \(A^{-1} = A'\).

\((i) \implies (c)\). \(A^{-1} = A' \implies AA' = I\), and therefore, \(a'_ia_j = \delta_{ij}\) for any two rows \(a'_i, a'_j\) of \(A\). Thus, the rows of \(A\) form an orthonormal basis for \(\mathbb{R}^n\).

To conclude, we have to show that none of these equivalence classes can be merged to create a larger one. This is achieved pairwise between equivalence classes \(A\) and \(B\) by picking some statement \(P\) from class \(A\) and statement \(Q\) from class \(B\), and demonstrating through counterexamples that they are not equivalent. For the above five classes, this in principle involves disproving 10 possible equivalences, but in practice, several of them are trivial; for example, the first class and the last class can never be merged, since (g) is clearly contradicting (d). Similarly, (e) and (a) clearly contradict. Other nonequivalence relations can be similarly verified.
2. **Inequalities (55 points).** Fill in each blank with “≤”, “=”, or “≥”. For example,

\[
\sin^2 \theta + \cos^2 \theta = 1
\]

Each completely correct answer earns 5 points. A correct inequality in place of equality (such as \(1 + 1 \leq 2\)) earns 3 points. A wrong answer (such as \(\alpha^2 \leq 0\) for a real number \(\alpha\)) or a blank earns zero points.

In the following, \(A \in \mathbb{F}^{m \times n}\) for an arbitrary field \(\mathbb{F}\).

(a) \(\text{rank}(A) + \text{nullity}(A) = n\).

**Solution:** By the rank–nullity theorem.

(b) \(\text{rank}(A) = \text{rank}(A^\prime)\).

**Solution:** By the row-rank column-rank theorem.

(c) \(\text{rank}(A) \geq \text{rank}(A^A)\).

**Solution:** Since \(\mathcal{N}(A) \subseteq \mathcal{N}(A^A)\), \(\text{nullity}(A) \leq \text{nullity}(A^A)\). By the rank nullity theorem, \(\text{rank}(A) = n - \text{nullity}(A) \geq n - \text{nullity}(A^A) = \text{rank}(A^A)\). Note that there are some fields in which we can find examples where the inequality is strict. For example, consider

\[
A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \in \mathbb{F}_2^{2 \times 2}.
\]

Then, \(\text{rank}(A) = 1\) but \(\text{rank}(A^A) = 0\). (See Problem 3, HW#2.)

(d) If \(A^A\) is invertible, then \(m \geq n\).

**Solution:** If \(A^A\) is invertible, then \(\text{rank}(A^A) = n\). By (c), \(\text{rank}(A^A) \leq \text{rank}(A) \leq \min(m, n)\). Thus, \(\text{rank}(A) = n\) and \(n \leq m\).

To see that the inequality might be strict, let \(A = [1 \ 0]^\prime \in \mathbb{R}^{2 \times 1}\).

(e) If \(B \in \mathbb{F}^{n \times n}\) is invertible, then \(\text{rank}(AB) = \text{rank}(A)\).

**Solution:** Let \(B^{-1}\) denote the inverse of \(B\). Then, \(A = ABB^{-1}\). Using similar arguments to the solution of (c), we have \(\text{rank}(A) = \text{rank}(ABB^{-1}) \leq \text{rank}(AB) \leq \text{rank}(A)\), which implies \(\text{rank}(AB) = \text{rank}(A)\).

(f) If \(m = n\) (\(A\) is square) and \(A^2 = 0\), then \(\text{rank}(A) \leq \text{nullity}(A)\).

**Solution:** Let \(a_i\) denote the \(i\)-th column of \(A\). Since \(A^2 = 0\), \(Aa_i = 0 \forall i \in \{1, 2, \ldots, n\}\), which implies \(a_i \subseteq \mathcal{N}(A) \forall i \in \{1, 2, \ldots, n\}\). Therefore, \(\mathcal{R}(A) \subseteq \mathcal{N}(A)\), from which the result follows. To see that the inequality can be strict, consider \(A = 0 \in \mathbb{R}^{2 \times 2}\).

For the next three parts, \(B \in \mathbb{F}^{m \times p}\).

(g) \(\text{rank}(A) + \text{rank}(B) \geq \text{rank}([A \ B])\).

**Solution:** Let \(r_A\) and \(r_B\) denote the rank of \(A\) and \(B\) respectively. Consider a basis \((u_1, u_2, \cdots, u_{r_A})\) that spans \(\mathcal{R}(A)\) and another basis \((v_1, v_2, \cdots, v_{r_B})\) that spans \(\mathcal{R}(B)\).
For the last two parts, let \( V \) be an arbitrary element in \( R([A \ B]) \), which can be written as \( z = Ax + By \) for some \( x \in \mathbb{F}^n \) and for some \( y \in \mathbb{F}^p \). Since \( Ax \in R(A) \) and \( By \in R(B) \), \( z \) can be written as a linear combination of \((u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_r)\).

Since the vectors \((u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_r)\) span the column space of \([A \ B]\), it immediately follows that \( \text{rank}([A \ B]) \leq r_A + r_B = \text{rank}(A) + \text{rank}(B) \). The equality holds if and only if the set of vectors \((u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_r)\) are independent. In particular, to see that the inequality can be strict, let \( A = B = I \in \mathbb{R}^{2 \times 2} \).

(h) If \( R(A) = R(B) \), then \( \text{rank}([A \ B]) = \text{rank}(A) \).

**Solution:** Let \( z \) be an arbitrary element in \( R([A \ B]) \), which can be written as \( z = Ax + By \) for some \( x \in \mathbb{F}^n \) and for some \( y \in \mathbb{F}^p \). By fixing \( x \) as all zero vector, it is easy to see that \( R(B) \subseteq R(A) \subseteq R([A \ B]) \). Similarly, \( R(A) \subseteq R([A \ B]) \), from which it follows that \( \text{rank}([A \ B]) \geq \max\{\text{rank}(A), \text{rank}(B)\} \). For the other direction, since \( Ax \in R(A) \), \( By \in R(B) \) and \( R(A) = R(B) \), we have \( z \in R(A) = R(B) \). Therefore, \( R([A \ B]) \subseteq R(A) = R(B) \). It follows that \( \text{rank}([A \ B]) \leq \text{rank}(A) = \text{rank}(B) \).

(i) If \( \text{nullity}(A'B) = 0 \), then \( \text{nullity}(A') \leq \text{nullity}(B') \).

**Solution:** This is perhaps the most difficult problem in this midterm exam (if done right). We first show that \( N(A') \cap R(B) = \{0\} \). Let \( z \in N(A') \cap R(B) \). Then, \( z = Bx \) for some \( x \in \mathbb{F}^p \) and \( A'z = 0 \). Plugging \( z \) in gives \( A'Bx = 0 \), which implies \( x \in N(A') \). Since \( \text{nullity}(A'B) = 0 \), \( x = 0 \). Therefore, \( z = Bx = 0 \).

Now, since \( N(A') \cap R(B) = \{0\} \), by part (k) below, we have \( \text{nullity}(A') + \text{rank}(B) \leq m \). By the rank nullity theorem and the row-rank column-rank theorem, \( \text{nullity}(A') + \text{rank}(B) = \text{nullity}(A') + \text{rank}(B') = \text{nullity}(A') + m - \text{nullity}(B') \). Therefore, \( \text{nullity}(A') \leq \text{nullity}(B') \).

As an alternative solution, we can use the rank inequality and observe that

\[
\text{nullity}(A'B) = 0 \iff \text{rank}(A'B) = p \iff \text{rank}(B) \geq p,
\]

but since \( B \in \mathbb{F}^{m \times p} \), we have \( \text{rank}(B) \leq p \), which shows that \( \text{rank}(B) = p = \text{rank}(B') \) and thus, \( \text{nullity}(B') = m - p \). Using the rank inequality again with \( A' \) instead of \( B \) in \([1]\), we can similarly show that \( \text{nullity}(A') \leq m - p \), completing the proof.

To see that the inequality can be strict, let \( A = I_3 \) and \( B = [1 \ 0 \ 0]' \) over \( \mathbb{R} \). Notice that \( \text{nullity}(A'B) = 0 \) and \( \text{nullity}(A') = 0 < \text{nullity}(B') = 2 \). In general, if \( A \) is a fat full rank matrix \((m \leq n \text{ and } \text{rank}(A) = n)\) and \( B \) is a strictly tall full rank matrix \((m > p \text{ and } \text{rank}(B) = p)\), it can be shown that \( \text{nullity}(A') = 0 < \text{nullity}(B') = m - p \).

For the last two parts, let \( V_1 \) and \( V_2 \) be two subspaces of \( \mathbb{F}^n \).

(j) If \( V_1 \subseteq V_2 \), then \( \dim(V_1) \leq \dim(V_2) \).

**Solution:** Let \( S_i, \ i = 1, 2 \), denotes the set of basis vectors for \( V_i \). Since \( V_1 \subseteq V_2 \), \( S_1 \subseteq S_2 \), from which it immediately follows that \( \dim(V_1) = |S_1| \leq |S_2| \), where equality holds if and only if \( V_1 = V_2 \). In particular, to see that the inequality can be strict, let \( n = 3 \), \( V_1 = \text{span}([1 \ 0 \ 0]') \) and \( V_2 = \text{span}([1 \ 0 \ 0]', [0 \ 1 \ 0]') \).
(k) If \( V_1 \cap V_2 = \{0\} \), then \( \dim(V_1) + \dim(V_2) \leq n \).

**Solution:** Let \( r_1 = \dim(V_1) \) and \( r_2 = \dim(V_2) \). Consider a basis \( \{v_1, v_2, \ldots, v_{r_1}\} \) that spans \( V_1 \) and another basis \( \{w_1, w_2, \ldots, w_{r_2}\} \) that spans \( V_2 \). We will show by contradiction that the set of vectors \( \{v_1, v_2, \ldots, v_{r_1}, w_1, w_2, \ldots, w_{r_2}\} \) are independent. Suppose without loss of generality that \( v_1 = \sum_{i=2}^{r_1} \alpha_i v_i + \sum_{j=1}^{r_2} \beta_j w_j \) can be written as a linear combination of the others. Then, \( v_1 - \sum_{i=2}^{r_1} \alpha_i v_i = \sum_{j=1}^{r_2} \beta_j w_j \in V_1 \cap V_2 \) by the closedness of subspaces under addition and scalar multiplication. Since \( V_1 \cap V_2 = \{0\} \), we have \( v_1 - \sum_{i=2}^{r_1} \alpha_i v_i = \sum_{j=1}^{r_2} \beta_j w_j = 0 \), which contradicts the fact that \( v_i \)'s are independent. Since the set of independent vectors \( \{v_1, v_2, \ldots, v_{r_1}, w_1, w_2, \ldots, w_{r_2}\} \subseteq \mathbb{F}^n \), it follows that \( r_1 + r_2 \leq n \). This inequality can be strict, for example, when \( n = 3 \), \( V_1 = \text{span}([1 0 0]) \) and \( V_2 = \text{span}([0 1 0]) \).