Instructions

1. This is an open book, open notes exam. Use of calculators is permitted. Use of internet is strictly prohibited. You can use electronic devices provided you only refer to soft copies of textbook, notes, homework and solutions. Any violation will result in expulsion from the exam hall, and will be duly reported.

2. Although you may refer to the textbook, you may not directly use any result from the textbook that was not proved in class. If a question (or its sub-part) asks you to prove or justify something, you have to provide a proof, and not merely quote a result from the textbook.

3. Use your bluebook to write the exam. Clearly write your name and UCSD ID on the front page.

1. Problem 1 (40 Points):

   (a) Let $\mathcal{V}$ be the set of all continuous real-valued functions $f(x)$ on the interval $0 \leq x \leq 1$, such that

   $$f(0) = \alpha, \quad f(1) = \beta$$

   For what values of $\alpha$ and $\beta$ will $\mathcal{V}$ be a vector space over $\mathbb{R}$? Justify your answer.

   (b) Given a vector $x \in \mathbb{C}^m$, its $n$-point Discrete Fourier Transform (DFT) is given by the vector $y \in \mathbb{C}^n$ such that

   $$y_k = \sum_{i=1}^{m} x_i e^{-j2\pi(i-1)(k-1)/n}, \quad k = 1, 2, \ldots, n$$

   i. Represent DFT in matrix vector form, i.e. find the matrix $W_{m,n} \in \mathbb{C}^{m \times n}$ such that

   $$y = W_{m,n}x$$

   ii. Case 1: $m = n$. Show that $\frac{1}{\sqrt{n}} W_{n,n}$ is a unitary matrix. Can $x$ be uniquely recovered from $y$? Write down $x_k$ in terms of a linear combination of $y_1, y_2, \ldots, y_n$ in this case.
iii. Case 2: \( n > m \). How is the \( n \)-point IDFT of \( y \) related to \( x \)? Can you uniquely recover \( x \) from \( y \)? If yes, justify. If not, give a counterexample.

iv. Case 2: \( n < m \). Can you uniquely recover \( x \) from \( y \) in this case? If yes, justify. If not, give an example of two distinct vectors \( x_1, x_2 \in \mathbb{R}^m \) such that their \( n \)-point DFT are identical.

2. **Problem 2 (40 Points):**
Consider the vector space of all \( n \times n \) real valued matrices (over the field \( \mathbb{R} \))

\[
\mathcal{V} = \{ X \in \mathbb{R}^{n \times n} \}
\]

Define the function \( f : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R} \)

\[
f(A, B) = \text{Trace}(B^T A)
\]

(a) Verify \( f(.) \) represents an inner product. What is the norm induced by this inner product?

(b) Given \( Y \in \mathcal{V} \), find \( \hat{Y} \) satisfying

\[
\hat{Y} = \arg \min_{\hat{Y}} f(Y, \hat{Y}) \quad \text{such that } \hat{Y} \in S
\]

for each of the following cases

i. \( S = \{ X \in \mathcal{V}, X = X^T \} \).

ii. \( S = \{ X \in \mathcal{V}, X_{ij} = 0, \text{ whenever } j \neq i, i - 1, \text{ or } i + 1 \} \).

3. **Problem 3 (40 Points):**
Given \( A \in \mathbb{C}^{m \times n} \), \( y \in \mathbb{C}^m \), consider the normal equations

\[
A^H A x = A^H y
\]

(a) Recall that normal equations always have solutions. Solve the following problem

\[
\min_x \| x \|_2 \quad A^H A x = A^H y
\]

(b) Given \( \epsilon > 0 \), now consider the problem

\[
\min_x \| y - Ax \|_2
\]

\[
\| x \|_2 \leq \epsilon
\]

Instead of solving this problem directly, we will instead solve the following problem

\[
\| y - Ax \|_2^2 + \lambda \| x \|_2^2
\]

by choosing an appropriate \( \lambda \).

i. Given any \( \lambda > 0 \), find \( x^* \) that solves (2).
ii. Find a condition in terms of \( \lambda \) and \( \epsilon \) such that the minimum of (1) is no larger than \( \|y - Ax^*\|_2 \).

iii. What happens to \( x^* \) as \( \lambda \to 0 \). Find a condition on \( \epsilon \) in this case such that \( x^* \) also minimizes (1).

4. **Problem 4 (30 Points):**

Let \( A \) and \( B \) be square matrices of size \( N \times N \). Suppose \( A, B \) satisfy \( AB = BA \).

(a) Given that \( B \) has all distinct eigenvalues, show that \( A \) is diagonalizable.

(b) If \( B \) is invertible, does this mean \( A \) is also invertible? Prove it, or give a counterexample.

(c) Are the eigenvalues of \( A \) and \( B \) related? Justify.