1. *When is it true? (5 points for each correct answer, -3 point for each wrong answer, 0 point for each blank)* Fill in each blank with “always,” “sometimes,” or “never.” For example,

- A nonsingular matrix is **always** invertible.
- A square matrix is **sometimes** full-rank.
- A strictly tall matrix is **never** onto.

Here the matrix dimensions are such that each expression makes sense, but they are otherwise unspecified. Every vector and matrix has real entries.

(a) The union of two subspaces of \( \mathbb{R}^n \) is **sometimes** a subspace.

**Solution:** Consider two subspaces \( \mathcal{V} \) and \( \mathcal{W} \) of \( \mathbb{R}^n \) such that \( \mathcal{W} \subseteq \mathcal{V} \). In this case, \( \mathcal{V} \cup \mathcal{W} = \mathcal{V} \) which is a subspace. On the other hand, consider the two subspaces \( \mathcal{V}' \) and \( \mathcal{W}' \) of \( \mathbb{R}^n \) spanned by the vectors \( v' = [1 \ 1 \ \cdots \ 1] \) (vector of all ones) and \( w' = [1 \ 1 \ \cdots \ -1] \) (vector zero at all positions except the last) respectively. Then, \( v' + w' = [1 \ 1 \ \cdots \ 0] \) which doesn’t belong to either of \( \mathcal{V}' \) or \( \mathcal{W}' \). Thus, \( \mathcal{V}' \cup \mathcal{W}' \) is not a subspace. In fact, it can be proved that for two subspaces \( \mathcal{V} \) and \( \mathcal{W} \), \( \mathcal{V} \cup \mathcal{W} \) is a subspace iff either \( \mathcal{V} \subseteq \mathcal{W} \) or \( \mathcal{W} \subseteq \mathcal{V} \).

(b) If \( AB = 0 \), then \( BA \) is **sometimes** a zero matrix.

**Solution:** Take \( A = [1 \ 1] \) and \( B = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \). Then \( AB = 0 \), but \( BA \neq 0 \). On the other hand, if \( A = 0 \) and \( B = 0 \), then both \( AB \) and \( BA \) are zero matrices.

(c) If \( \mathcal{R}(A) \subseteq \mathcal{N}(B) \), then \( BA \) is **always** a zero matrix.

**Solution:** Assume \( B \in \mathbb{R}^{m \times n} \) and \( A \in \mathbb{R}^{n \times k} \). Then, \( C := BA \in \mathbb{R}^{m \times k} \). If the columns of \( C \) are denoted as \( c_1, c_2, \ldots, c_k \) and the columns of \( A \) as \( a_1, a_2, \ldots, a_k \), then by the rules of matrix multiplication, \( c_i = Ba_i = 0 \) \( \forall i = \{1, 2, 3, \ldots, k\} \) since \( a_i \in \mathcal{R}(A) \subseteq \mathcal{N}(B) \). Hence \( C = 0 \).

(d) If \( \mathcal{R}(A) \perp \mathcal{N}(B^T) \), then \( \text{rank}([A \ B]) \) is **always** equal to \( \text{rank}(B) \).

**Solution:** Recall that \( \mathcal{N}(B^T)^\perp = \mathcal{R}(B) \). Since \( \mathcal{R}(A) \perp \mathcal{N}(B^T) \), \( \mathcal{R}(A) \subseteq \mathcal{R}(B) \). Thus, \( \text{colspace}(A) \subseteq \text{colspace}(B) \). A basis for \( \mathcal{R}([A \ B]) \) is given by the union of the bases of \( \mathcal{R}(A) \) and \( \mathcal{R}(B) \), which by the previous observation is just \( \mathcal{R}(B) \). Thus, \( \text{rank}([A \ B]) = \dim(\mathcal{R}([A \ B])) = \text{rank}(B) \).

(e) The nullspace of \( \begin{bmatrix} A \\ A + B \end{bmatrix} \) is **always** equal to \( \mathcal{N}(A) \cap \mathcal{N}(B) \).

**Solution:** Let \( C = \begin{bmatrix} A \\ A + B \end{bmatrix} \). Then, if \( x \in \mathcal{N}(A) \cap \mathcal{N}(B) \), \( Cx = \begin{bmatrix} Ax \\ Ax + Bx \end{bmatrix} = 0 \), implying that \( x \in \mathcal{N}(C) \). Thus \( \mathcal{N}(A) \cap \mathcal{N}(B) \subseteq \mathcal{N}(C) \).
Also, if \( x \in \mathcal{N}, Cx = 0 \implies \begin{bmatrix} Ax \\ Ax + Bx \end{bmatrix} = 0 \). Clearly (by equating the respective blocks), this implies that \( Ax = 0 \) and \( Ax + Bx = 0 \), which in turn gives us \( Ax = 0 \) and \( Bx = 0 \). Thus, \( \mathcal{N}(C) \subseteq \mathcal{N}(A) \cap \mathcal{N}(B) \).

Putting these two together, we get \( \mathcal{N}(C) = \mathcal{N}(A) \cap \mathcal{N}(B) \).

(f) The nullspace of \( \begin{bmatrix} A \\ AB \end{bmatrix} \) is sometimes equal to \( \mathcal{N}(A) \cap \mathcal{N}(B) \).

Solution: Let \( A \in \mathbb{R}^{n \times k} \) and \( B \in \mathbb{R}^{k \times k} \). If we take \( A = 0 \) and \( B = I \), then \( \begin{bmatrix} A \\ AB \end{bmatrix} = 0 \) and \( \mathcal{N}(\begin{bmatrix} A \\ AB \end{bmatrix}) = \mathbb{R}^k \neq \mathcal{N}(A) \cap \mathcal{N}(B) = \mathbb{R}^k \cap \{0\} = \{0\} \). However, if \( A \) is tall and full-rank and \( B = I \), then \( \mathcal{N}(\begin{bmatrix} A \\ AB \end{bmatrix}) = \mathcal{N}(\begin{bmatrix} A \\ A \end{bmatrix}) = \{0\} = \mathcal{N}(A) \cap \mathcal{N}(B) \), since \( \mathcal{N}(A) = \mathcal{N}(B) = \{0\} \).

(g) If \( A^T A \) is onto, then \( A \) is sometimes onto.

Solution: Let \( A \in \mathbb{R}^{m \times n} \). Using the Sylvester’s rank inequality and \( \text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\} \), it follows that \( \text{rank}(A) + \text{rank}(A^T) - m = 2\text{rank}(A) - m \leq \text{rank}(A^T) = n \leq \text{rank}(A) \), which implies that \( n \leq \text{rank}(A) \leq m \). Hence, if \( m = n \), \( \text{rank}(A) = n \), and \( A \) is onto. However, if \( m > n \), i.e., \( A \) is strictly tall, we have \( \mathcal{R}(A) \subseteq \mathbb{R}^m \), where the inclusion is strict.

(h) If the matrix \( \begin{bmatrix} A \\ B \end{bmatrix} \) is onto, then \( A \) and \( B \) are always onto.

Solution: Let \( A \in \mathbb{R}^{m \times k} \) and \( B \in \mathbb{R}^{n \times k} \). Consider \( \begin{bmatrix} A \\ B \end{bmatrix} x = \begin{bmatrix} Ax \\ Bx \end{bmatrix} \). For \( \begin{bmatrix} Ax \\ Bx \end{bmatrix} \) to span \( \mathbb{R}^{m+n} \), we need the columns of \( A \) to span \( \mathbb{R}^m \), i.e., we need \( \mathcal{R}(A) = \mathbb{R}^m \). This is because if for some vector \( v_1 \in \mathbb{R}^m, v_1 \notin \mathcal{R}(A), \) for any \( v_2 \in \mathbb{R}^n, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \) cannot belong to \( \mathcal{R}(\begin{bmatrix} A \\ B \end{bmatrix}) \), contradicting our assumption that \( \begin{bmatrix} A \\ B \end{bmatrix} \) is onto. Thus, \( A \) needs to be onto. The proof that \( B \) is onto is similar.

(i) If \( A \) and \( B \) are onto, then \( \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \) is always onto.

Solution: To get \( \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \) (where \( v_1 \) and \( v_2 \) are arbitrary and have dimensions as appropriate), first note that \( \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} Aa + Cb \\ Bb \end{bmatrix} \). Choose \( b \) such that \( Bb = v_2 \). Such a \( b \) exists since \( B \) is onto. Next, choose an \( a \) such that \( Aa = v_1 - Cb \). Again, since \( A \) is onto, such an \( a \) is guaranteed to exist. We have now shown that any arbitrary \( \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathcal{R}(\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}) \). Thus, \( \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \) is onto if \( A \) and \( B \) are onto.

(j) The rank of \( AB \) is never greater than \( \text{rank}(A) \).

Solution: cf. Homework #2, Problem 5(b).
(k) The rank of $A + B$ is **sometimes** greater than $\text{rank}(A)$.

**Solution:** If $B = 0$, then $A + B = A$ and hence rank$(A + B) = \text{rank}(A)$. On the other hand, consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Since $A + B = I$, rank$(A + B) = 2 > \text{rank}(A) = 1$.

(l) If $A$ and $B$ are full-rank and tall, then $AB$ is **always** full-rank and tall.

**Solution:** Let $A \in \mathbb{R}^{n \times m}$, and $B \in \mathbb{R}^{m \times k}$. Since $n \geq m$ and $m \geq k$, $n \geq k$ which implies that $AB \in \mathbb{R}^{n \times k}$ is tall. Also, $m + k - m \leq \text{rank}(AB) \leq \min(m, k) \implies \text{rank}(AB) = k$. Thus $AB$ is tall and full-rank.

(m) If $AB$ is full-rank, then $A$ and $B$ are **sometimes** full-rank.

**Solution:** Consider $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then $AB = \begin{bmatrix} 1 \end{bmatrix}$. We observe that $AB$ is full-rank even though $A$ is not. On the other hand, if $A$ and $B$ are square matrices and $AB$ is full-rank, then $A$ and $B$ must be full-rank. Hence, $AB$ being full-rank doesn’t definitively indicate whether $A$ and $B$ are full-rank or not.

(n) If $A$ and $B$ are full-rank and $A^T B = 0$, then $[ A \ B ]$ is **always** full-rank.

**Solution:** Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times k}$. Since $A^T B = 0$, we have $\mathcal{R}(B) \subseteq \mathcal{N}(A^T) = \mathcal{R}(A)^\perp$, and thus the columns of $A$ and $B$ are orthogonal. This immediately implies that they are independent. Further, by the Sylvester’s rank inequality, rank$(A) + \text{rank}(B) - m \leq \text{rank}(A^T B) = 0$, which implies that dim$(\mathcal{R}(A)) + \text{dim}(\mathcal{R}(B)) \leq m$. Also, it is easy to show that rank$(A) = n$ and rank$(B) = k$, and thus both $A$ and $B$ are tall. Therefore, since columns of $A$ and columns of $B$ are linearly independent and the total number of columns is $n + k \leq m$, we conclude that $[ A \ B ] \in \mathbb{R}^{m \times (n+k)}$ is full-rank.

(o) If $A$ is full-rank and tall, then $\|Ax\| = 0$ **always** implies $x = 0$.

**Solution:** Since $A$ is tall and full-rank, by the rank-nullity theorem, we know that the dim$(\mathcal{N}(A)) = 0$. Thus, $\mathcal{N}(A) = \{0\}$. Since for $\|Ax\| = 0$ we need $Ax = 0$ by definiteness property of the norm, $x = 0$.

(p) If $A$ is strictly fat, then $A^T A$ is **never** invertible.

**Solution:** Let $A \in \mathbb{R}^{m \times n}$. Then $A^T A \in \mathbb{R}^{n \times n}$. Since $A$ is strictly fat, rank$(A) \leq \min(m, n) = m < n$. Now consider rank$(A^T A) \leq \text{rank}(A) < n$. Hence, $A^T A$ cannot be full-rank and is hence not invertible.

(q) If $A$ has a rank decomposition $A = BC$, then $CC^T$ is **always** invertible.

**Solution:** Any nonzero matrix $A \in \mathbb{R}^{m \times n}$ (rank$(A) = r \geq 1$) can be decomposed as $A = BC$, where $B \in \mathbb{R}^{m \times r}$ is a full-rank tall matrix and and $C \in \mathbb{R}^{r \times n}$ is a full-rank fat matrix. Now by Homework #3, Problem 7(f), $CC^T$ is invertible.

(r) If $A^2$ is invertible, then $A$ is **always** invertible.

**Solution:** We first note that for $A^2$ to be a valid matrix product, $A$ needs to be square. Assume $A \in \mathbb{R}^{n \times n}$. We then have rank$(A^2) \leq \text{rank}(A) \leq \min(n, n) = n$. Since rank$(A^2) = n$ for $A^2$ to be invertible, we must have rank$(A) = n$. Thus, $A$ is full-rank and hence invertible.
(s) If the linear equation \( y = Ax \) has a unique solution, then \( A \) is \underline{sometimes} square.

**Solution:** If \( A = I \), then \( Ax = y \) has a unique solution. However, \( A \) need not necessarily be square for \( Ax = y \) to have a solution. Consider \( A = \begin{bmatrix} I \\ I \end{bmatrix} \) and \( y = \begin{bmatrix} y \\ y \end{bmatrix} \) for any \( y \). The system of equations \( Ax = y \) has a unique solution even though \( A \) is not square.

(t) If the linear equation \( y = Ax \) has multiple solutions, then \( A \) is \underline{sometimes} fat.

**Solution:** For the fat matrix \( A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \) and \( y = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \), the system of equations \( Ax = y \) has infinitely many solutions. However, \( A \) need not necessarily be fat for \( Ax = y \) to have multiple solutions. Consider the square matrix \( A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \) and \( y = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \). The system of equations \( Ax = y \) has infinitely many solutions of the form \( x = (1, x_2) \) even though \( A \) is square.

2. **Permutation (30 points).** Let \( x = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \). Define a transformation \( T : \mathbb{R}^5 \rightarrow \mathbb{R}^5 \) as

\[
T(x) = (x_3, x_5, x_4, x_1, x_2).
\]

(a) Find its inverse \( T^{-1} : \mathbb{R}^5 \rightarrow \mathbb{R}^5 \) that satisfies \( T^{-1}(T(x)) = T(T^{-1}(x)) = x \). For \( y = (y_1, y_2, y_3, y_4, y_5) \),

\[
T^{-1}(y) = (y_4, y_5, y_1, y_3, y_2).
\]

**Solution:** \( T^{-1}(y) \) is simply a permutation that moves the elements in \( T(y) \) back to their original positions. In this case, \( T^{-1}(y) = (y_4, y_5, y_1, y_3, y_2) \). We can verify that \( T^{-1}(T(y)) = T(T^{-1}(y)) = y \).

(b) Find matrix representations of \( T \) and \( T^{-1} \) as \( T(x) = Px \) and \( T^{-1}(y) = Qy \):

\[
P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}
\]

**Solution:** By inspection.

(c) How are \( P \) and \( Q \) related? Answer: \( P = Q^{-1} = Q^T \).

**Solution:** Since \( T^{-1}(T(y)) = T(T^{-1}(y)) = y \), this implies that \( PQy = QPy = y \) for all \( y \in \mathbb{R}^5 \). Hence, \( PQ = QP = I \), or equivalently, \( P \) and \( Q \) are inverses of each other, i.e. \( P = Q^{-1} \) and \( Q = P^{-1} \). But since \( P \) and \( Q \) are orthogonal, \( P = Q^T \) and \( Q = P^T \) as well.

3. **Kronecker product (30 points).** For two matrices \( A \) and \( B \), define their Kronecker product
$A \otimes B$ as the block matrix
\[
\begin{bmatrix}
a_{11}B & a_{12}B & \cdots & a_{1n}B \\
a_{21}B & a_{22}B & \cdots & a_{2n}B \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1}B & a_{m2}B & \cdots & a_{mn}B
\end{bmatrix}.
\]

For example, if
\[
A = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -2 & 1 \\ 3 & 1 & 2 \end{bmatrix},
\]
then
\[
A \otimes B = \begin{bmatrix} 2 & 1 & -2 & 1 \\ 3 & 1 & 2 \\ -1 & 1 & -2 & 1 \\ 3 & 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & -2 & 1 \\ 3 & 1 & 2 \\ 1 & -2 & 1 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -4 & 2 & 3 & -6 & 3 \\ 6 & 2 & 4 & 9 & 3 & 6 \\ -1 & 2 & -1 & 1 & -2 & 1 \\ -3 & 1 & 2 & 3 & 1 & 2 \end{bmatrix}.
\]

(a) $I_m \otimes I_n = I_{mn}$.  

**Solution:** Writing $I_m \otimes I_n$ as
\[
I_m \otimes I_n = \begin{bmatrix} I_n & 0 & \cdots & 0 \\ 0 & I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_n \end{bmatrix},
\]

We can see from this representation that $I_m \otimes I_n$ is a $mn \times mn$ matrix with diagonal elements 1 and non-diagonal elements 0. Thus, $I_m \otimes I_n = I_{mn}$.

(b) Express $\text{rank}(A \otimes B)$ in terms of $\text{rank}(A)$ and $\text{rank}(B)$:
\[
\text{rank}(A \otimes B) = \frac{\text{rank}(A) \text{ rank}(B)}{.}
\]

**Solution:** Let $\text{rank}(A) = m$, $\text{rank}(B) = n$, and the basis vectors for $\mathcal{R}(A)$ and $\mathcal{R}(B)$ be $v_1, v_2, v_3, \cdots, v_m$ and $w_1, w_2, w_3, \cdots, w_n$ respectively. We note that the set of vectors $v_i \otimes w_j, (i, j) \in [m] \times [n]$ form a basis set for $\mathcal{R}(A \otimes B)$. Since there are $mn$ of these vectors, $\text{rank}(A \otimes B) = mn = \text{rank}(A) \text{ rank}(B)$.

More details: Let $A \in \mathbb{F}^{p \times q}$, $B \in \mathbb{F}^{p' \times q'}$. From the definition of the Kronecker product, we see that for $s \in [1 : q]$, $t \in [1 : q']$, the $((s-1)q' + t)^{th}$ column of $A \otimes B$ is given by $a_s \otimes b_t$, where $a_s$ and $b_t$ are, respectively, the $s^{th}$ column of $A$ and the $t^{th}$ column of $B$. Letting
\[
a_s = \sum_{i=1}^{m} \alpha_i^s v_i \quad \text{and} \quad b_t = \sum_{j=1}^{n} \beta_j^t w_j,
\]
and noting that from the definition, Kronecker product distributes over matrix addition (in either argument), we see that the \((s-1)q' + t)^{th}\) column of \(A \otimes B\) is given by

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_i^s \beta_j^t (v_i \otimes w_j).
\]

Therefore, the set \(\{v_i \otimes w_j, i \in [1 : m], j \in [1 : n]\}\) spans the column space of \(A \otimes B\), and the only thing left to show is that this set is independent.

For some \(\gamma_{ij} \in \mathbb{F}, i \in [1 : m], j \in [1 : n]\), let \(\sum_{i=1}^{m} \sum_{j=1}^{n} \gamma_{ij} (v_i \otimes w_j) = 0\). We then have

\[
\sum_{j} \left( \sum_{i} \gamma_{ij} v_i \right) \otimes w_j = 0.
\]

By independence of \(\{w_1, \ldots, w_n\}\) and the definition of Kronecker product, this implies that \(\sum_i \gamma_{ij} v_i = 0\) for every \(j\), which in turn implies, from the independence of \(\{v_1, \ldots, v_m\}\), that \(\gamma_{ij} = 0\) for every \(i \) and \(j\).

(c) Assume that \(A\) and \(B\) are square and nonsingular. Express \((A \otimes B)^{-1}\) in terms of \(A^{-1}\) and \(B^{-1}\):

\[
(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}.
\]

Solution: Let \(A \in \mathbb{R}^{m \times m}\) and \(B \in \mathbb{R}^{n \times n}\). First of all, we note that \((A \otimes B)(C \otimes D) = AC \otimes BD\). Choosing \(C = A^{-1}\) and \(D = B^{-1}\), we get \((A \otimes B)(A^{-1} \otimes B^{-1}) = I_m \otimes I_n = I_{mn}\). Thus, \((A \otimes B)^{-1} = A^{-1} \otimes B^{-1}\).

More details: Let \(A \in \mathbb{R}^{m \times k}, C \in \mathbb{R}^{k \times n}, B \in \mathbb{R}^{p \times r}, D \in \mathbb{R}^{r \times q}\). Then, by the block multiplication of matrices, we have

\[
(A \otimes B)(C \otimes D) = \begin{bmatrix}
a_{11}B & a_{12}B & \cdots & a_{1k}B \\
a_{21}B & a_{22}B & \cdots & a_{2k}B \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1}B & a_{m2}B & \cdots & a_{mk}B \\
\end{bmatrix} \begin{bmatrix}
c_{11}D & c_{12}D & \cdots & c_{1j}D & \cdots & c_{1n}D \\
c_{21}D & c_{22}D & \cdots & c_{2j}D & \cdots & c_{2n}D \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
c_{k1}D & c_{k2}D & \cdots & c_{kj}D & \cdots & c_{kn}D \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\left( \sum_{l=1}^{k} a_{1l} c_{1l} \right) BD & \left( \sum_{l=1}^{k} a_{1l} c_{2l} \right) BD & \cdots & \left( \sum_{l=1}^{k} a_{1l} c_{ij} \right) BD & \cdots \\
\left( \sum_{l=1}^{k} a_{2l} c_{1l} \right) BD & \left( \sum_{l=1}^{k} a_{2l} c_{2l} \right) BD & \cdots & \left( \sum_{l=1}^{k} a_{2l} c_{ij} \right) BD & \cdots \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots \\
\left( \sum_{l=1}^{k} a_{ml} c_{1l} \right) BD & \left( \sum_{l=1}^{k} a_{ml} c_{2l} \right) BD & \cdots & \left( \sum_{l=1}^{k} a_{ml} c_{ij} \right) BD & \cdots \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots \\
\end{bmatrix}
= (AC) \otimes (BD).
\]

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