ECE 269  Lecture #8

1/25/2019
Admin. notes

- Midterm #1 in a week (open book/notes)
- Practice midterm problems posted (from Win 18, Aut 18)
- Midterm #2 schedule change (M 2/25 ⇒ F 2/22)
- Any questions? Office hours, email, Piazza, etc.

Last

- Norm

- Inner product ⇒ norm ⇒ metric
Cauchy–Schwarz inequality

For any inner product space \( V \),

\[
|x, y| \leq \sqrt{|x||y|} = \sqrt{|<x, x>|}
\]

iff \( x = \alpha y \) or \( y = \alpha x \)

For \( x, y \in \mathbb{R}^n \) (\( x, y \) are nonzero),

\[-1 \leq \frac{<x, y>}{||x|| ||y||} \leq 1\]

\[
<x, y> = ||x|| ||y|| \cos \theta
\]

\[
\theta = \cos^{-1} \frac{<x, y>}{||x|| ||y||}
\]
In other words, $x$ and $y$ are orthogonal if $\langle x, y \rangle = 0$ (for $\mathbb{R}^n$, $x^\top y = 0$)

Note: Sometimes $x^\top y = 0$ is used loosely as "orthogonality".
Let \( x, y \in V \) (inner product space).

Problem: Find \( \alpha \in F \) and \( w \in V \) such that

\[ y = \alpha x + w \]

and

\[ x \perp w. \]

Since \( y = \alpha x + w \) and \( x \perp w \),

\[ \langle y, x \rangle = \langle \alpha x + w, x \rangle = \alpha \|x\|^2 + \langle w, x \rangle \]

\[ = \alpha \|x\|^2 \]

\[ \Rightarrow \alpha = \frac{\langle y, x \rangle}{\|x\|^2} \quad \text{and} \quad w = y - \alpha x = y - \frac{\langle y, x \rangle}{\|x\|^2} x. \]
Hyperplane
\[ \{ x \in \mathbb{R}^n : \langle v_0, x \rangle = v_0'x = 0 \} \]

Plane
\[ \{ x \in \mathbb{R}^n : \langle v_0, x \rangle < 0 \} \]

Half space
\[ \{ x \in \mathbb{R}^n : \langle v_0, x \rangle \geq 0 \} \]
We say that a set of vectors $u_1, u_2, \ldots, u_k \in \mathbb{R}^n$ are orthogonal if $u_i \perp u_j$, $\forall i \neq j$.

* Orthogonality $\Rightarrow$ independence

**Proof:** Consider $\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_k u_k = 0$

Then $\langle \alpha_1 u_1 + \cdots + \alpha_k u_k, u_1 \rangle = \alpha_1 \|u_1\|^2 = 0$

$\Rightarrow \alpha_1 = 0$

Similarly, $\alpha_2 = \cdots = \alpha_k = 0 \Rightarrow u_1, \ldots, u_k$ are independent.

* $u_1, \ldots, u_k$ are said to be unit vectors if $\|u_i\| = 1$.

* We say $u_1, \ldots, u_k$ are orthonormal if they are unit vectors and orthogonal.
If columns of \( U = \mathbb{R}^{n \times k} \) are orthonormal, then
\[
U'U = I \in \mathbb{R}^{k \times k} (= I_k)
\]

As a linear transformation, such a matrix “preserves” the norm and inner product.

Let \( y = Ux \). Then
\[
\|y\|^2 = \|Ux\|^2 = \langle Ux, Ux \rangle
\]
\[
= x'U'Ux
\]
\[
= x'x = \|x\|^2
\]

Let \( y = Ux \) and \( \tilde{y} = U\tilde{x} \). Then
\[
\langle y, \tilde{y} \rangle = \langle Ux, U\tilde{x} \rangle = x'U'U\tilde{x} = \langle x, \tilde{x} \rangle
\]

Note: A similar property (isometry) holds for \( \mathbb{C}^n \).
A square matrix $U \in \mathbb{R}^{n \times n}$ is said to be **orthogonal** if its columns $u_1, \ldots, u_n$ form an **orthonormal basis** for $\mathbb{R}^n$ (i.e., $u_i, \ldots, u_n$ are orthogonal unit vectors that span $\mathbb{R}^n$).

**Properties:**
1. $U'U = I$
2. $U^{-1} = U'$
3. $UU' = I$

**Note:** For $U \in \mathbb{C}^{n \times n}$, $U^*U = UU^* = I$ and $U^{-1} = U^*$ is said to be **unitary**.
Examples (of orthogonal matrices)

1) Identity matrix $I$

2) Permutation matrices

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

3) Rotation

$$Q = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

4) Reflection

$$R = \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix}$$