ECE 269 Lecture #18

2/25/2019
Admin. notes

- Midterm #2 has been graded.
- You must follow the honor codes!
  (do NOT copy somebody else's exams, homework, and programming assignments)

Last

- Linear diff. eqs
- Cayley - Hamilton theorem

Today

- Perron - Frobenius theorem
A matrix \( A \in \mathbb{R}^{m \times n} \) or a vector \( x \in \mathbb{R}^n \) is \underline{positive} (denoted \( A > 0 \) or \( x > 0 \)) if all of its entries are \( > 0 \). Similarly, \( A \geq 0 \) or \( x \geq 0 \) if all the entries are \( \geq 0 \).

- A nonnegative square matrix \( A \) is \underline{primitive} (regular) if \( A^k > 0 \) for some \( k \).

- A nonnegative square matrix \( A \) is \underline{irreducible} if for every \((i,j)\), \((A^k)_{ij} > 0\) for some \( k = k(i,j) \).

Examples

1. \( A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \) is \( \geq 0 \) (positive)
(2) \[ B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \geq 0 \text{ is primitive} \] (but not positive)

since

\[ B^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \geq 0 \]

(3) \[ P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \geq 0 \text{ is irreducible} \] (but not primitive)

since

\[ P_{12} = P_{21} > 0 \text{ and } P^2 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

satisfies \((P^2)_{11} = (P^2)_{22} > 0\)
positive $\iff$ primitive $\iff$ irreducible

- If $A$ is not irreducible, $A$ is called reducible.

- Let $A \in \mathbb{R}^{n \times n}$ be nonnegative and irreducible. Then the gcd of all $k$ such that $(A^k)_{ii} > 0$ is called the period of $A$.

**Examples**

(1) For $B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $(B')_{22} = 0$, $(B^2)_{22} > 0$

$(B^3)_{22} > 0$, $(B^4)_{22} > 0$, ...

$\Rightarrow \gcd \{2, 3, 4, \ldots \} = 1$

$(B')_{11} > 0$, $(B^2)_{11} > 0$, ...

$\Rightarrow \gcd \{1, 2, 3, \ldots \} = 1$
Fact. It is independent of which index \( i \) we consider.

(2) For \( P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \), \( (P^1)_n = (P^3)_n = (P^5)_n = \cdots = 0 \)

and \( (P^2)_n = (P^4)_n = (P^6)_n = \cdots = 1 > 0 \)

\( \Rightarrow \gcd \{ 2, 4, 6, \ldots \} = 2 \)

\( P \) has the period of 2.

* If \( A \) has the period of 1, then \( A \) is called aperiodic. Otherwise, \( A \) is periodic.

Fact. For square \( A > 0 \), \( A \) is irreducible and aperiodic if \( \gcd = 1 \nRightarrow A \) is primitive.
Let \( A \in \mathbb{R}^{n \times n} \) be nonnegative and irreducible with period \( p \). Then the "largest" eigenvalue \( \lambda_{PF} \) of \( A \) (Person - Frobenius eigenvalue) satisfies the following properties

1. \( \lambda_{PF} \) is real and positive
2. \( \lambda_{PF} \geq |\lambda| \) for every eigenvalue \( \lambda \) of \( A \) and every eigenvalue \( \lambda \) satisfying \( |\lambda| = \lambda_{PF} \) is simple (algebraic and geometric multiplicity of \( \lambda \)) and of the form \( \omega \lambda_{PF} \), where \( \omega \) is the \( p \)-th root of unity.
Example Consider \( P = [1 \ 0] \geq 0 \) (irreducible).

Then \( \lambda_{pf} = 1 \) (for eigenvector \((1\ i))\) and the other eigenvalue is \( \lambda = -\lambda_{pf} = -1 \).

(3) \( \lambda_{pf} \) is simple (algebraic multiplicity of \( 1 \)

\( = \lambda_{pf} \) is a simple root of the characteristic polynomial of \( A \); geometric multiplicity of \( 1 \)

\( = \) there is a unique \( 1 \times 1 \) Jordan block associated with \( \lambda_{pf} \).

(4) \( \lambda_{pf} \) has a positive eigenvector \( \nu_{pf} \) associated with it

and every nonnegative eigenvector of \( A \) must be a multiple of \( \nu_{pf} \).
In particular, if $A$ is aperiodic ($p = 1$), then

$$\lambda_p > |\lambda|$$

for every other eigenvalue.

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**Application**

Let $A \in \mathbb{R}^{n \times n}$ be primitive (= irreducible & aperiodic) with Perron-Frobenius eigenvalue $\lambda$. Then

$$\lim_{k \to \infty} \left( \frac{A}{\lambda} \right)^k = v w'$$

right and left

where $v$ and $w$ are eigenvectors of $A$ associated with $\lambda$ and normalized such that $w'v = 1$. 
If \( x > 0 \), then

\[
A^k x \approx \lambda^k (w' x) v
\]

\[\uparrow\]

approximately equal

\[= V w' x \]

projection